A note on transitive and irreducible action on the Stiefel manifold $V_{n,n-2}$

By

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1. Introduction

The purpose of this note is to prove the following

THEOREM. *Let G be a compact connected Lie group. If G acts on $X=SO(n)/SO(2)$ continuously, transitively and irreducibly, then G is locally isomorphic to $SO(n)$ and isotropy subgroup is $SO(2)$.*

This work was stimulated by the work of W. Y. Hsiang and J. C. Su [1]. They have proved that if (n, k) satisfies some conditions, which exclude the case $k=2$, the standard action $(SO(n), SO(k), SO(n)/SO(k))$ is the only effective and irreducible transitive action on $SO(n)/SO(k)$. We do not know whether in our case isotropy subgroup is conjugate to the standardly embedded $SO(2)$.

2. Statement of results

We consider a transitive action (continuous) of a compact connected Lie group G on $X=SO(n)/SO(2)$ with isotropy subgroup H. A transitive action of G is said to be irreducible when no proper normal subgroup of G acts transitively. Recall that given a compact connected Lie group G, one can always write $G=T^r\times G_1\times\ldots\ldots\times G_s/N$, where T^r is r-dimensitional torus, $G_i's$ are simply connected simple Lie groups and N is a finite normal subgroup of $G^*=T^r\times G_1\times\ldots\ldots\times G_s$. Let $p: G^*\longrightarrow G$ the canonical projection. Then G^* acts transitively on X in a natural way with isotropy subgroup $H^-=p^{-1}(H)$. Since X is simply connected, H^- is connected, and hence $H^-=H^*/M$, where $H^*=T^t\times H_1\times\ldots\ldots\times H_u$ and M is a finite normal subgroup of H^*.

Considering the following parts of the homotopy exact sequences of the fibering (G^*, H^-, X);

$$0\longrightarrow\pi_2(X)\longrightarrow\pi_1(H^-)\longrightarrow\pi_1(G^*)\longrightarrow\pi_1(X)$$

and

$$\pi_4(X)\longrightarrow\pi_3(H^-)\longrightarrow\pi_3(G^*)\longrightarrow\pi_3(X)\longrightarrow 0,$$

we have that $u=s-1$ and $t=r+1$.

We will prove the following

PROPOSITION 1. *If G^* and H^* contain a factor G_k such that the composition $G_k \longrightarrow H^* \longrightarrow \overline{H} \longrightarrow G^* \longrightarrow G_k$ is epimorphic, then the normal subgroup G^*/G_k of G^* acts transitively on X.*

This follows immediately from the result in [2] which says; the restriction of the natural action $G\times G/H \longrightarrow G/H$ to a subgroup K of G is transitive if and only if $KH=G$.

According to proposition 1, it is easy to show that $G^*/T^r=G_1\times \ldots\ldots \times G_s$ acts transitively on X. Hence by irreducibility we may assume that $G^*=G_1\times \ldots\ldots \times G_s$ and $H^*=T \times H_1 \times \ldots\ldots \times H_{s-1}$.

Recall that the rational cohomology ring of a compact connected Lie group G is an exterior algebra $\Lambda_Q(x_1, x_2, \ldots\ldots, x_r)$, where $x_i' s$ are of odd degree and r is the rank of G. Let $n(G)$ be the highest degree of generators $x_1, \ldots\ldots, x_r$. For reference, we list below the degree and number of primitive generotors of simple Lie groups;

A_n: 3, 5, 7, ..., $2n+1$
B_n: 3, 7, 11,......................., $4n-1$
C_n: 3, 7, 11,......................., $4n-1$
D_n: 3, 7, 11,......................., $4n-5$, $2n-1$
G_2: 3, 11
F_4: 3, 11, 15, 23
E_6: 3, 9, 11, 15, 17, 23
E_7: 3, 11, 15. 19. 23, 27, 35
E_8: 3, 11, 23, 27, 35, 39, 47, 59

It is well known that $\pi_i(G)\otimes Q \neq 0$ if i is equal to some degree of x_i and $\pi_i(G)\otimes Q=0$ otherwise, where Q is the field of rationals [2].

By the following proposition, we may assume that $n(G^*)$ and $n(H^*)$ are not greater than $n(Spin(n))$.

PROPOSITION 2. *G^* and H^* are factored as follows;*

$$G^*=G_0\times Spin(2i_1)^{\varepsilon_1}\times Spin(2i_2)^{\varepsilon_2}\times \ldots\ldots\ldots\ldots \times Spin(2i_k)^{\varepsilon_k}$$
$$H^*=H_0\times Spin\ (2i-1)^{\varepsilon_1}\times \ldots\ldots\ldots\ldots \times Spin(2i_k\times 1)^{\varepsilon_k}$$

where ε_j are some integer, $n<2i_1-1<\ldots\ldots<2i_k-1$ and $n(G_0)$, $n(H_0)\leqq n(Spin(n))$. Moreover let f_{ij} denote the composition $H_j \longrightarrow H^* \longrightarrow G^* \longrightarrow G_i$; then the matrix (f_{ij}) is diagonal.

Let G_0 and H_0 be factored as follows;

$G_0=G_1\times G_2\times \ldots\ldots\ldots\ldots \times G_s$ and $H_0=T\times H_1\times \ldots\ldots\ldots\ldots \times H_{s-1}$, respectively.

We will prove the following

PROPOSITION 3. If a factor $H_j\neq Spin(n)$ is mapped non-trivially into a factor G_i of G_0, then we have $n(H_j) \lneqq n(G_i)$.

To prove these propositions, we need the following lemma which is useful in the sequel. Let H_1 and G_1 be one of factors of H^* and G^* respectively, f the composition $H_1 \longrightarrow H^* \longrightarrow G^* \longrightarrow G_1$ and p'_1, p_1 the projection $H^* \longrightarrow H_1$, $G^* \longrightarrow G_1$ resp. Then we have the following diagram;

$$\begin{array}{ccc} H^* & \longrightarrow & G^* \\ {\scriptstyle p'_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ H_1 & \underset{f}{\longrightarrow} & G_1 . \end{array}$$

This diagram is commutative if the composition; $H^*/H_1 \longrightarrow H^* \longrightarrow G^* \longrightarrow G_1$ is trivial. We will define an map h: $G^*/H^- \longrightarrow G_1/f(H_1)$ such that the diagram;

$$\begin{array}{ccc} G^* & \longrightarrow & G^*/H^- \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle h} \\ G_1 & \longrightarrow & G_1/f(H_1) \end{array}$$

is commutative and $h_\sharp$: $\pi_i(G^*/H^-) \longrightarrow \pi_i(G_1/f(H_1))$ is surjective into the image of $(\pi_1)_\sharp$ for $i \geqq 3$. Define $h(gH^-) = p_1(g) f(H_1)$. In fact h is well defined; let $g_1H^- = g_2H^-$. Then we have $g_1^{-1} g_2 \in H^-$, and hence we have $p_1(g_1)^{-1} p_1(g_2) \in p_1(H^-) \subseteq f(H_1)$. Consider the following diagram of homotopy groups;

$$\begin{array}{ccccccc} \pi_i(H^-) & \longrightarrow & \pi_i(G^*) & \longrightarrow & \pi_i(G^*(H^-) & \longrightarrow & \pi_{i-1}(H^-) \\ \wr\wr & & & & & & \wr\wr \\ \pi_i(H^*) & & \Big| & & \Big| & & \pi_{i-1}(H^*) \\ \downarrow & & {\scriptstyle p_{1\sharp}} & & {\scriptstyle h_\sharp} & & \downarrow \\ \pi_i(H_1) & & \Big| & & \Big| & & \pi_{i-1}(H_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_i(f(H_1)) & \longrightarrow & \pi_i(G_1) & \longrightarrow & \pi_i(G_1/f(H_1)) & \longrightarrow & \pi_{i-1}(f(H_1)) \end{array}$$

Let x be in image of $\pi_{1\sharp}$. Choose y in $\pi_i(G_1)$ such that $x = \pi_{1\sharp}(y)$. Since $p_{1\sharp}$ is sujective, we find z in $\pi_i(G^*)$ such that $p_{1\sharp}(z) = y$. Put $w = \pi_\sharp(z)$, and we have $h_\sharp\,(x) = x$. This implies that $h_\sharp$ is surjective onto the image of $\pi_{1\sharp}$.

Thus we have proved the following

LEMMA 1. *Let G^* and H^* be factored into product $G_1 \times G_2$ and $H_1 \times H_2$ resp., where H_1 and G_1 are semi-simple. Then if the composition; $H_2 \longrightarrow H^* \longrightarrow G^* \longrightarrow G_1$ is trivial, there is defined a map h: $G^*/H^- \longrightarrow G_1/f(H_1)$ such that $h_\sharp$: $\pi_i(G^*/H^-) \longrightarrow \pi_i(G_1/f(H_1))$ is surjective onto the image of the homomorphism $\pi_i(G_1) \longrightarrow \pi_i(G_1/f(H_1))$.*

REMARK. To complete the proof of the above lemma, we need the following fact; let G_1 and G_2 be simple Lie groups and f a homomorphism of $G_1 \times G_2$ into a compact connected Lie group G. Then if f is non-trivial, f is locally isomorphic. The proof of this fact is not difficult.

We shall consider the factor H_0 and G_0 of H^* and G^* in proposition 2. Let a_i, a_i'; b_i, b_i'; c_i, c_i'; d_i, d_i'; g_2, g_2'; f_4, f_4'; e_6, e_6'; e_7, e_7' and e_8, e_8' be the number of factors $SU(i)$, $Spin(2i+1)$, $Sp(i)$, $Spin(2i+2)$, G_2, F_4, E_6, E_7 and E_8 in G_0 and H_0 respectively. We will prove the following.

LEMMA 2. *We have the following equalities*;

(1) $a_i=a_i'=0$ *for* $i=2, 3, 4, 5, 6$

(2) $a_{2i+1}=0$

(3) $g_2=g_2'=0$

(4) $b_i=b_i'=0$ *for* $i=2, 3, 4, 5$

(5) $d_i=d_i'=0$ for $i=4, 5$

(6) $f_4=f_4'=0$

(7) $e_6=e_6'=0$

(8) $2d_6+b_6+e_7=2d_6'+b_6'+e_7'$

(9) $e_7+e_8=e_7'+e_8'$.

Modulo verifications of the above propositions and lemmas, we will prove the theorem mentioned in introduction.

Let G_0 and H_0 be factored as follows; $G_0=G_1\times\ldots\ldots\times G_s$ and $H^*=T\times H_1\times\ldots\ldots\times H_{s-1}$ We assume that $n(H_1)\leqq n(H_2)\leqq\ldots\ldots\leqq n(H_{s-1})$. Let $G_{i1},\ldots\ldots G_{it}$ be the factors of G_0 into which H_{s-1} is mapped non-trivially. We will prove that t is at most 1. Assume that $n(G_{i1})\leqq\ldots\ldots\leqq n(G_{it})$. Put $m=n(G_{i1})$. By proposition 3, $n(H_{s-1})$ is smaller than m. Let H_{j1}, H_{ju} be the simple factors of H_0 which are mapped non-trivially into $G_{i1}\times\ldots\ldots\times G_{it}$.

We are concerned only in the case when n is odd.

CASE 1. *G_{ij} is a classical group for all ij.*

Combining the facts $\pi_m(H_{j1}\times\ldots\ldots\times H_{s-1})\otimes Q=0$, $\pi_m(G_{i1}\times\ldots\ldots\times G_{it})\otimes Q\supseteqq tQ$ and $\pi_m(X)\otimes Q=Q$ and Lemma 1, we have that t is at most 1

CASE 2. *G_{i1} is exeptional and all other G_{ij} are classical.*

This case is similar to the case 1.

CASE 3. *G_{i1} is classical and some G_{ij} is exeptional.*

We may assume that $n(H_{s-1})\geqq 19$. In fact, $n(H_{s-1})<19$, then H_0 does not contain $Spin(12)$, $Spin(13)$ and E_7, E_8, and hence $d_6=b_6=e_7=e_8=0$.

CASE 3–1. *Some G_{ij} is E_7.*

Since $n(E_7)=35$, the possible value of m is 31, 27 and 23. When $m=23$, or 27, we have $\pi_m(G_{ij})\otimes Q\neq 0$ for all G_{ij}. The argument similar to the case 1 shows that t is at most 1. Consider the case $m=31$. If $n(H_{s-1})\leqq 23$, $\pi_{27}(G_{ij})\otimes Q\neq 0$, and hence we have $t\leqq 1$. If $n(H_{s-1})=27$, H_{s-1} is one of $Spin(15)$, $Spin(16)$, $Sp(7)$ and $SU(14)$. These cannot be mapped non-trivially into E_7.

CASE 3–2. *Some G_{ij} is E_8.*

Since $n(E_8)=59$, we have $m\leqq 55$. We may assume $m\geqq 23$. The similar arguments show that t is at most 1.

Thus we have proved that the factor H_{s-1} is mapped non-trivially into at most one factor, say $G_{\tau(s-1)}$. Next consider the second factor H_{s-2}. If $n(H_{s-2})=n(H_{s-1})$, it is similarly proved that H_{s-2} is mapped only one factor, say $G_{\tau(s-2)}$ non-trivially. Assume that $n(H_{s-2})<n(H_{s-1})$. Let $G_{i1},\ldots\ldots, G_{it}$ be the factors of G_0 into which H_{s-2} is mapped

non-trivially, and $H_{j1}\times \ldots\ldots \times H_{ju}\times H_{s-2}$ the factor of H_0 which is mapped into $G_{i1}\times\ldots\ldots \times G_{jt}$. It is easy to show that t is at most 2. Let H_{s-2} be mapped into $G_1\times G_2$ non-trivially. Then H_{s-1} is mapped into G_1, or G_2. For if H_{s-1} is not mapped into neither G_1 nor G_2, we have that $n(H_i)\leqq n(H_{s-2})$ for every H_i which is mapped $G_1\times G_2$. This is impossible.

Summing up, we have obtained a correspondence (not neccessarily injective) τ: $[1, 2,\ldots\ldots, s-1]\longrightarrow[1, 2,\ldots\ldots s]$ such that H_{s-1} is mapped inte $G_{\tau(s-1)}$ non-trivially, H_{s-2} is mapped into $G_{\tau(s-1)}$, $G_{\tau(s-2)}$ non-trivially and so on. In other words. the matrix (f_{ij}) is a triangular matrix, where f_{ij} is the composition $H_j\longrightarrow H^*\longrightarrow G^*\longrightarrow G_i$. Therefore there is at least one factor of G_0 such that no factor of $H_0\neq T$ is mapped non-trivially. The following proposition completes the proof of the theorem.

PROPOSITION 4. *Let G_1 be one of simple factor of G (different from $Spin(n)$). Then there exists one simple factor H_1 of H_0 such that H_1 is mapped into G_1 non-trivially.*

2. The proof of Proposition 2

In this sections, we will restict ourselves in the case when n is odd: $n=2k+1$. Assume that $n(G^*)=4m+1$. If $4m+1>4k-1$, then we have that rank $\pi_{4m+1}(G^*)=$rank $\pi_{4m+1}(H^*)$, and hence $a_{2m+1}=a_2'{}_{m+1}$. Since the factor $SU(2m+1)$ of H^* is mapped only into the factor $SU(2m+1)$ of G^*, proposition 1 and the irreducibility concludes that $a_{2m+1}=a_{2m+1}'=0$. Therefore we may assume that $n(G^*)\leqq 4m-1$. It is easy to show that rank $\pi_{4m-1}(G^*)$ is given by the following formula;

$$
\begin{aligned}
\text{rank } \pi_{4m-1}(G^*) &= a_{2m}+b_m+c_m+d_{m+1} && \text{for } m>15\\
&= a_{2m}+b_m+c_m+d_{m+1}+e_8 && \text{for } m=15\\
&= a_{2m}+b_m+c_m+d_{m+1} && \text{for } 9<m<15\\
&= a_{2m}+b_m+c_m+d_{m+1}+e_7 && \text{for } m=9\\
&= a_{2m}+b_m+c_m+d_{m+1} && \text{for } 6<m<9\\
&= a_{2m}+b_m+c_m+d_{m+1}+f_4+e_6 && \text{for } m=6.
\end{aligned}
$$

The formula of rank π_{4m-1} (H^*) is the same as above, but it is primed. If $4m-1>4k-1$, the homotopy exact sequence of the fibering (G^*, H^-, X) shows that rank π_{4m-1} $(G^*)=$ rank π_{4m-1} (H^*). Consider the case when n is is greater than 15. Then we have $a_{2m}+b_m+c_m+d_{m+1}=a_{2m}'+b_m'+c_m'+d_{m+1}'$. Since the factor $SU(2n)$ of H^* can be mapped non-trivially only $SU(2n)$ of G^*, proposition 1 and Irreducibility imply that $a_{2n}=0$. Similar argument shows that $d_{m+1}'=0$. It is not difficult to show that $a_{2m}=c_n$ and $d_{m+1}=b_m'$.

Next we consider the case when n is 15. Then we have $a_{30}+b_{15}+c_{15}+d_{16}+e_8=a_{30}'+b_{15}'+c_{15}'+d_{16}'+e'_8$. Since E_8 is not mapped non-trivially into $SU(30)$, $Spin(31)$, $Sp(15)$ and $Spin(32)$, we have $e_8=e_8'=0$, $a_{30}=c_{15}'$ and $d_{16}=b_{15}'$. By the same arguments as above, it is shown that

$$a_{2m}=c_m' \text{ and } d_{m+1}=b_m' \text{ for } m>6$$

and

$$a_{2m}=c_m',\ d_{m+1}=b_m' \text{ and } e_6=f_4' \qquad \text{for } m=6.$$

We will prove that $a_{2m}=c_m'$ and $e_6=f_4'=0$ for both cases. Firstly consider the case when n is greater than 6; let a factor $Sp(m)$ of H^* be mapped non-trivially into an $SU(2m)$. It is not difficult to show that no factor H_1 of H^* different from T can be mapped into $SU(2m)$ in the way that $H_1\times Sp(m)$ is mapped non-trivially into $SU(2m)$ (cf. remark below lemma 1). Consider the following sequence;

$$\pi_5(Sp(m)\times T^\varepsilon)\longrightarrow\pi_5(SU(2m))\longrightarrow\pi_5(SU(2m)/Sp(m)\times T^\varepsilon)\longrightarrow\pi_4(Sp(m)\times T^\varepsilon),$$

where ε is 0 or 1. Since $\pi_5(Sp(m)\times T^\varepsilon)$ and $\pi_4(Sp(m)\times T^\varepsilon)$ are finite groups, the image of the homomorphism $\pi_5(SU(2m)\longrightarrow\pi_5(SU(2m)/Sp(m)\times T^\varepsilon)$ is of rank 1. Combining the fact that $\pi_5(X)=0$, lemma 1 and irreducibility, it is concluded that $a_{2m}=c_m'=0$. By the similar argument, it is easy to show that e_6 and f_4' vanish when m is 6. When m is at least 6. When m is at least 6, there is no non-tritial homomorphism of $Spin(2m+1)$ into $Spin(2m+2)$ other than the standard one (because of representation theory). Hence we have obtained the factorization of G^* and H^* such that

$$G^*=G_1\times Spin(2m+2) \quad \text{and} \quad H^*=H_1\times Spin(2m\times 1),$$

where the compositions $H_1\longrightarrow H^*\longrightarrow G^*\longrightarrow Spin(2m+2)$ and $Spin(2m+1)\longrightarrow H^*\longrightarrow G^*\longrightarrow G_1$ are trivial. Repeating this procedure, we shall obtain the factorization of G^* and H^* as stated in Proposition 2. This completes the proof of proposition 2.

3. The proof of Proposition 3

In this section, we will prove Proposition 3 only when n is greater than 9. Let H_1 and G_1 be factor of H_0 and G_0 respectively and the composition φ_1: $H_1\longrightarrow H^*\longrightarrow G^*\longrightarrow G_1$ non-trivial. It is known that $n(H_1)$ is at least $n(G_1)$ and $n(H_1)$ is equal to $n(G_1)$ only when (H_1, G_1, φ_1) is the following triples;

case	(1)	(2)	(3)	(4)	(5)	(6)
H	$Sp(l)$	G_2	$Spin(7)$	G_2	F_4	$Spin(2p+1)$
G	$SU(2l)$	$Spin(7)$	$Spin(8)$	$Spin(8)$	E_6	$Spin(2p+2)$
φ_1	φ_1	φ_2	φ_3	φ_2+N	φ_4+N	φ_1+N

CASE 1. Since $\pi_{2l}(Sp(l))=Z_2$ or $\pi_{2l}(SU(2l))=Z_{(2l)!}$ and $\pi_{2l}(X)=Z_2$ or 0, lemma 1 shows that this case is impossible.

By similar arguments, it is easy to show that case 3, case 4, case 2 and case 5 are all impossible.

CASE 6. Considering the following table of homotopy groups, one can conclude that this case is also impossible.

	$\pi_{2l+4}(Spin(2l+1))$	$\pi_{2l+4}(Spin(2l+2))$	$\pi_{2l+4}(X)$ $n=2l+3$	$2l+4$	$2l+5$
$l\equiv 0(4)$	0	Z_{12}	Z_2	$Z_8\oplus Z_2$	Z_8
$l\equiv 1(4)$	$Z_{8d}(d=1, 2)$	$Z_4\oplus Z_{24d}$	Z_2	Z_4	$Z_2\oplus Z_2$
$l\equiv 2(4)$	Z_2	$Z_{12}\oplus Z_2$	Z_2	Z_2^3	Z_8
$l\equiv 3(4)$	Z_8	$Z_{24}\oplus Z_8$	$Z_2\oplus Z_2$	Z_4	Z_2.

4. The proof of Lemma 2

In this section, we restrict ourself in the case when n is greater than 16. Let $n_p(q)$ and $n_p'(q)$ denote the number of the factors Z_q in π_p (G^*) and π_p (H^*) respectively.

From the homotopy exact sequence of the fibering (G^*, H^-, X), we have

i) $n_8(2)=n'_8(2)$, ii) $n_8(8)\geqq n'_8(8)$, iii) $n_{13}(2)=n'_{13}(2)$ and iv) $n_9(3)=n_9'(3)$.

These imply that

i) $a_3+a_4=a_3'+a_4'$, ii) $a_4\geqq a_4'$, iii) $2a_2+a_3=2a_2'+a_3'$ and iv) $a_2+a_3+g_2$ $=a_2'+a_3'+g_2'$.

Hence we have that $a_2=a_2'$, $a_3=a_3'$, $a_4=a_4'$ and $g_2=g_2'$. It is not difficult to show that a_2, a_3 and g_2 vanish. It is also easy to show that $a_5+a_6=a_5'+a_6'$ and hence $a_5=a_6=a_5'=a_6'$ $=0$. This completes the proof of (1).

We omit the proof of equalities (2), (3), ..., (9), since they are tedious, but not difficult.

5. The proof of Proposition 4

Let G_1 be one of factor of G_0 different from *Spin* (n). Then if no factor of H_0 is mapped non-trivially into G_1, there is a surjective homomorphism of $\pi_i(X)$ onto $\pi_i(G_1)$ for every $i\geqq 3$. This follows from lemma 1. Therefore, to prove proposition 4, it is sufficient to show that there exists an integer i satisfying the condition (#) *$\pi_i(X)$ is not mapped surjectively onto $\pi_i(G_1)$.*

If G_1 is $SU(2k)$, or $Sp(k)$, then $i=5$ satifies the condition (#). When G_1 is $Spin(k)$ we can also find an integer i satisfiying the condition (#). When G_i is E_7, $i=12$ satisfies the condition (#).

Consider the case when G_1 is E_8. First we will prove that $e_8=0$ if n is smaller than 64. In fact, by representation theory, it is clear that E_7 and E_8 can not be mapped non-trivially in $Spin(k)$, where $k\leqq 64$. Considering homotopy groups $\pi_{12}(E_7)$ and $\pi_{12}(E_8)$, it is impossible that E_7 is mapped non-trivially into E_8. Hence we have $e_7=e_8=e_7'=e_8'=0$.

Therefore we may consider only the case when n is greater than 64, By a result in [3], it is known that for prime $p\geqq 31$, $\pi_q(E_8)\otimes Z_p=\pi_q(S^3\times S^5\times \dots\dots\dots \times S^{59})\otimes Z_p$. It is proved in [4] that for odd prime p, $k\geqq 1$ and $i\geqq 3$, $\pi_{2k(p-1)-1+1}(S^i)\otimes Z_p=0$ and in particular, $\pi_{62}(S^3)\otimes Z_{31}\neq 0$. Since n is greater than 64, $\pi_{62}(X)\otimes Z_{31}=0$. Hence $i=62$ satifies the condition (#). This completes the proof of Proposition 4.

6. Concluding remarks

1. Suppose that G is any compact connected Lie group such that $SO(n)\subset G\subset SO(n)\times SO(n-k)$, then it is easy to see that G acts on $SO(n)/SO(k)$ transitively. Hsiang and Su have proved that for many values of n and k $(k\neq 2)$ every transitive and effective action on $SO(n)/SO(k)$ is differentiably equivalent to the above example. However we do not know whether the same result holds in the case $k=2$.

2. We have ommited the proof of the theorem when the rank of $Spin(n)$ is smaller than 8. When n is small, we can prove the theorem more directly by counting the factors G^* and H^*.

3. We have omitted the proof of the theorem when n is even. When n is even, we can prove the theorem in the similar method.

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References

[1] HSIANG, W. Y. and SU, J.: *On the classification of transitive effective action on Stiefel manifolds. Trans. Amer. Math. Soc.* (1968) 322–336.

[2] ONISCIK, A. L.: *Transitive compact transformation groups. Amer. Math. Soc. Translations, Ser.* 2, *vol.* 55, 153–194.

[3] SERRE, J. P.: *Groupes d'homotopie et classes de groupes abéliens. Ann. of Math.* 58(1953) 258–294.

[4] TODA, H.: *On unstable homotopy groups of sphere and classical groups. Proc. Nat. Acad. Sci. U.S.A.* 46 (1960) 1102–1105.