On holomorphic sectional curvature and metric in 4-dimensional Kählerian manifolds

By

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1. Introduction

Let M be a 2n-dimensional Kählerian manifold and R its Riemannian curvaturn tensor. At each point p of M, R is a quadrilinear mapping $T_p(M)\times T_p(M)\times T_p(M)\times T_p(M)\longrightarrow R^1$ with well known properties.

Let σ be a plane in $T_p(M)$, i.e, a 2-dimensional subspace of $T_p(M)$. Choosing an orthonormal basis $\{X, Y\}$ for σ, we define the sectional curvature $K(\sigma)$ by

$$K(\sigma)=R(X, Y, X, Y).$$

We shall occasionally write $K(X, Y)$ for $K(\sigma)$. The right hand side depends only on σ, not on the choice of an orthonormal basis $\{X, Y\}$. The sectional curvature K is a defined function on the Grassmann bundle of planes in the tangent space of M. A plane σ is said to be holomorphic if it is invariant by the (almost) complex structure tensor J. The set of J-invariant planes σ is a holomorphic bundle over M with fibre $P_{n-1}(M)$ (complex projective space of complex dimension $n-1$). The restriction of the sectional curvature K to this complex projective bundle is called the holomorphic sectional curvature and will denote by H. In other words, $H(\sigma)$ is defined only when σ is invariant by J and $H(\sigma)=K(\sigma)$. If X is a vecor in σ, we shall also write $H(X)$ for $H(\sigma)$.

Let (M, g), $(\overline{M}, \overline{g})$ be two Riemannian manifolds. Denoting the corresponding sectional curvature by K respectively $\overline{K}$, we say that M, $\overline{M}$ are isocurved if there exists a sectional curvature preserving diffeomorphism $f; M\longrightarrow\overline{M}$, i.e, for every $p\in M$ and for every σ, a plane section of tangent space $T_p(M)$, we have

$$K(\sigma)=\overline{K}(f_* \sigma)$$

where f_* is the differential at $p\in M$ of f.

R. S. Kulkarni [4] showed the following

THEOREM. Suppose that (M, g), $(\overline{M}, \overline{g})$ are isocurved, dim $M\geqq 4$, g analytic and $K\not\equiv$ constant. Then (M, g), $(\overline{M}, \overline{g})$ are isometric.

Let (M, g, J), $(\overline{M}, \overline{g}, \overline{J})$ be two Kählerian manifolds. If there exists a holomorphic mapping f ; $M \longrightarrow \overline{M}$, then $\overline{J} \circ f_* = f_* \circ J$. Therefore we say that M, $\overline{M}$ are h-isocurved if there exists a holomorphic sectional curvature preserving holomorphic isomorphism f ; $M \longrightarrow \overline{M}$, i.e, for every $p \in M$ and for every σ, a holomorphic plane section of the tangent space $T_p(M)$, we have

$$H(X) = \overline{H}(f_* X)$$

where H, $\overline{H}$ are the corresponding holomorphic sectional curvatures.

In this paper, we write *h. s. c. p. h. i* for a holomorphic sectional curvature preserving holomorphic isomorphism.

The main purpose of the present paper is to prove a Kählerian analogue of the above theorem in the case of dimension 4.

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2. Holomorphic sectional curvature and the conformal class of a metric

Let M be an even dimensional Riemannian manifold and J an almost complex structure on M, that is, a tensor field of type (1, 1) on M satisfying $J^2 = -I$ where I denotes the field of identity endomorphisms. A Riemannian metric g on M is called a Hermitian metric if the almost complex structure J on M is an isometry with respect to g. A triplet (M, g, J) is said to be Kählerian if it is satisfies

$$\nabla J = 0$$

where ∇ denotes the covariant differentiation with respect to the Riemannian connection determined by g.

It is well known that a Riemannian curvature tensor R of a Kählerian manifold satisfies the four conditions,

(a) $R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z)$,

(b) $R(X, Y, Z, W) = R(Z, W, X, Y)$,

(c) $R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0$

and

(d) $R(JX, JY, Z, W) = R(X, Y, JZ, JW) = R(X, Y, Z, W)$

where X, Y, Z and W are any vector fields.

Let M be a 2n-dimensional Kählerian manifold and let $\{X_1, \ldots\ldots, X_n, JX_1, \ldots\ldots, JX_n\}$ be an orthonormal basis for $T_p(M)$ at each point p of M. We also write $X_{1*}, \ldots\ldots, X_{n*}$ for $JX_1, \ldots\ldots, JX_n$. Set $f_* X_\alpha = \overline{X_\alpha}$ and $\overline{g}(\overline{X_\alpha}, \overline{X_\beta}) = A_{\alpha\beta}$. We use the convention that the indices $\alpha, \beta, \gamma, \delta, \ldots\ldots$ run through $1, \ldots\ldots n, 1^*, \ldots\ldots, n^*$, while indices $i, j, k, \ldots\ldots$ run from 1 to n. We set $R_{\alpha\beta\gamma\delta} = R(X_\alpha, X_\beta, X_\gamma, X_\delta)$ and $\overline{R}_{\alpha\beta\gamma\delta} = \overline{R}(\overline{X_\alpha}, \overline{X_\beta}, \overline{X_\gamma}, X_\delta)$.

The auther has ever proved the following ([5])

LEMMA 2. 1. Let M be a 2n-dimensional Kählerian manifold. $H(X)$ is constant for any vector $X \in T_p(M)$ if and only if there exists an orthonormal basis $\{X_1, \ldots\ldots, X_n, X_{1*}, \ldots\ldots, X_{n*}\}$ for the tangent space $T_p(M)$ with the following properties;

(2. 1) $\quad R_{ii*ii*} = R_{jj*jj*} = 4R_{ijij} = 4R_{ij*ij*} \quad$ for each $i \neq j$

and

(2. 2) $\quad$ all the other $\quad R_{\alpha\beta\gamma\delta} = 0 \quad$ except $R_{ii^*jj^*}$.

On the other hand, to prove Proposition 2. 3, we need the following

LEMMA 2. 2 (M. Berger [1]). *Let M be a 4-dimensional Kählerian manifold. Then there exists an orthonormal basis $\{X_1, X_{1*}, X_2, X_{2*}\}$ for tangent space $T_p(M)$ at each point $p \in M$ with the following properties*

(2. 3) $\quad R_{11*12} = R_{11*12} = R_{1212*} = 0.$

Now we shall prove

PROPOSITION 2. 3. *Let f: $M \longrightarrow \overline{M}$ be an h. s. c. p. h. i of two Kählerian manifolds and the set of points of M such that $H(X) \not\equiv$ constant for $X \in T_p(M)$ dense in M. If dimension M is 4, then f is conformal.*

PROOF. Let $\{X_1, X_2, X_{1*}, X_{2*}\}$ be an orthonormal basis for $T_p(M)$ at $p \in M$ and x, y real numbers, not both zero. By the algebraic relations ((a), (b), (c) and (d)) of the curvature tensor of the Kählerian manifold, we obtain the following:

$$H(xX_1 + yX_2) = \{R_{11*11*}x^4 + 4R_{11*12*}x^3y + 2(R_{11*22*} + 2R_{12*12*})x^2y^2 + 4R_{12*22*}xy^3 + R_{22*22*}y^4\} / (x^2+y^2)^2$$

and

$$\overline{H}(x\overline{X}_1 + y\overline{X}_2) = \{\overline{R}_{11*11*}x^4 + 4\overline{R}_{11*12*}x^3y + 2(\overline{R}_{11*22*} + 2\overline{R}_{12*12*})x^2y^2 + 4\overline{R}_{12*22*}xy^3 + \overline{R}_{22*22*}y^4\} / (x^2A_{11} + 2xyA_{12} + y^2A_{22})^2.$$

Cross multipling, we get an identity in x, y under the assumption $H(X) = \overline{H}(f_* X)$. Set $a = R_{11*11*}$, $b = 4R_{11*12*}$, $c = 2(R_{11*22*} + 2R_{12*12*})$ $d = 4R_{12*22*}$, $e = R_{22*22*}$, $a' = \overline{R}_{11*11*}$, $b' = \overline{R}_{11*12*}$, etc.
Comparing coefficient we get

(A) Coefficient of x^8:
$a' = aA_{11}^2$.

(B) Coefficient of x^7y:
$b' = 2aA_{11}A_{22} + bA_{11}^2$.

(C) Coefficient of x^6y^2:

$$2a'+c'=2a(2A_{12}^2+A_{11}A_{22})+8bA_{11}A_{22}+cA_{11}^2.$$

(D) Coefficient of x^5y^3:

$$d'+2b'=2aA_{11}A_{22}+2b(2A_{12}^2+A_{11}A_{22})+2cA_{11}A_{22}+dA_{11}^2.$$

(E) Coefficient of x^4y^4:

$$e'+2c'+a'=aA_{22}^2+8bA_{11}A_{22}+2c(2A_{12}^2+A_{11}A_{22})+8da_{11}A_{22}+eA_{11}^2.$$

(F) Coefficient of x^3y^5:

$$b'+2d'=bA_{22}^2+2cA_{11}A_{22}+2d(2A_{12}^2+A_{11}A_{22})+2eA_{11}A_{22}.$$

(G) Coefficient of x^2y^6:

$$c'+2e'=cA_{22}^2+8dA_{11}A_{22}+2e(2A_{12}^2+A_{11}A_{22}).$$

(H) Coefficient of xy^7:

$$d'=dA_{22}^2+2eA_{11}A_{22}.$$

(I) Coefficient of y^8:

$$e'=eA_{22}^2.$$

Put $X=A_{11}$, $Y=A_{22}$ and $Z=A_{12}$. Then by the equations from (A) to (I), we get the following

(J) $b'=2aXY+bX^2$.

(K) $b'=bY^2+2cXY+2d(2Z^2+XY)-2eXY-2dY^2$.

(L) $2b'=2aXY+2b(2Z^2+XY)+2cXY+dX^2-dY^2-2eXY$.

(M) $c'=2a(2Z^2+XY)+cX^2-2aX^2$.

(N) $2c'=aY^2+8dXY+2c(2Z^2+XY)+8bXY+eX^2-aX^2-eY^2$.

(O) $c'=cY^2+8dXY+2e(2Z^2+XY)-2eY^2$.

By (J) and (K) we obtain

(2.4) $$bX^2+2(a-c-d+e)X+(2d-b)Y^2-4dZ=0.$$

By (J) and (L) we obtain

(2.5) $$(d-2b)X^2+2(-a+b-c-e)XY-dY^2+4bZ^2=0.$$

By (M) and (N) we obtain

(2.6) $$(2c-3a-e)X^2+2(2a+4b-c-4d)XY+4(2a-c)Z^2+(e-a)Y^2=0.$$

By (N) and (O) we obtain

(2.7) $$(c-2a)X^2+2(a+4b-4d-e)XY+(2e-c)Z^2+4(a-e)Y^2=0.$$

REMARK. With various combinations of the equations from (A) to (I), we can certainly get many equations, but in fact the above four equations should be essential.

Similary we obtain the following from $H(yX_1+yJX_2)=\overline{H}(x\overline{X_1}+y\overline{JX_2})$

(2.4)′ $$\overline{b}X^2+2(a-\overline{c}-\overline{d}+e)XY+(2\overline{d}-\overline{b})Y^2-4\overline{dZ^2}=0,$$

(2.5)′ $$(\overline{d}-\overline{2b})X^2+2(-a+\overline{b}+\overline{c}-e)XY-\overline{d}Y^2+4\overline{bZ^2}=0,$$

(2.6)′ $$(\overline{2c}-3a-e)X^2+2(2a+\overline{4b}-\overline{c}-\overline{4d})XY+4(2a-\overline{c})\overline{Z^2}+(e-a)Y^2=0$$

and

$$(2.7)' \qquad (\bar{c}-2a)X^2+2(a+4\bar{b}-\bar{c}-4\bar{d}-e)XY+(2e-\bar{c})Y^2+4(a-e)\bar{Z}^2=0$$

where $\bar{b}=4R_{1*112*}$, $\bar{c}=2(R_{11*22*}+2R_{1212})$, $\bar{d}=4R_{122*2}$ and $\bar{Z}=A_{12*}$.

(2. 6) minus (2. 7) gives

$$(2.8) \qquad (c-a-e)\{(X-Y)^2-4Z^2\}=0.$$

(2. 4) plus (2. 5) gives

$$(2.9) \qquad (d-b)\{(X-Y)^2-4Z^2\}=0.$$

Especially we choose an orthonormal basis $\{X_1, X_2, X_{1*}, X_{2*}\}$ in Lemma 2. 2. Then we have from $b=0$

$$(2.9)' \qquad d\{(X-Y)^2-4Z\}=0$$

For (2. 8) and (2. 9)′, we have to consider only the following four cases:

Case I.	$(X-Y)^2=4Z^2$	if $d\neq 0$ and $c=a+e$.
Case II.	$(X-Y)^2=4Z^2$	if $d=0$ and $c\neq a+e$.
Case III.	$(X-Y)^2=4Z^2$	if $d\neq 0$ and $c\neq a+e$.
Case IV.	$d=0$ and $c=a+e$	if $(X-Y)^2=4Z^2$.

Case I. Substituting $(X-Y)^2=4Z^2$ and $c=a+e$ into (2. 4), or (2. 5), we get $d(X-Y)(X+Y)=0$. Since $d\neq 0$ and $X+Y\neq 0$, we have $X=Y$ and $Z=0$, i.e, $A_{11}=A_{22}$ and $A_{12}=0$.

Case II. Similarly we have $A_{11}=A_{22}$ and $A_{12}=0$.

Case III. Substituting $(X-Y)^2=4Z^2$ into (2. 5), we have

$$(2.10) \qquad d(X-Y)(X+Y)=2(a+e-c)XY.$$

Similarly we get from (2. 7)

$$(2.11) \qquad (c-a-e)(X+Y)(X-Y)=8dXY.$$

By (2. 10) and (2. 11), we have

$$(X-Y)(X+Y)\{4d^2+(c-a-e)^2\}=0.$$

Hence we obtain $A_{11}=A_{22}$ and $A_{12}=0$.

Case IV. Suppose that $c=a+e$, $d=0$ and $(X-Y)^2\neq 4Z^2$. Substituting $c=a+e$ and $b=d=0$ into (2. 6), or (2. 7), we have

$$(2.12) \qquad a=e.$$

On the other hand, suppose that $H(X)\not\equiv$constant for $X\in T_p(M)$ at a point $p\in M$. Then by Lemma 2. 1, we cannot take any orthonormal basis with the properties (2. 1) and (2. 2). So if our basis for $T_p(M)$ have the condition $a\neq e$, then from (2. 12) we can see that Case IV is impossible to occur. If $d\neq 0$, we have either Case I, or Case II. Put

$u=a=e$. If $u\neq 4R_{12*12*}$ and $u=4R_{1212}$, we can see that $c-a-e\neq 0$ and have either Case I or Case III. In the case $u=4R_{12*12*}$ and $u\neq 4R_{1212}$, we also have either Case II or Case III. Moreover, in any other case, we have either Case II or Case III if $H(X)\neq$constant on $T_p(M)$. Thus we have

$$(2.13)\qquad A_{11}=A_{22},\ A_{12}=0.$$

In the same way, from (2. 4)′, (2. 5)′, (2. 6)′ and (2. 7)′, we get

$$(2.14)\qquad A_{11}=A_{22},\ A_{12*}=0.$$

Since $\overline{g}$ is a Hermitian metric, from (2. 13) and (2. 14) we can see that together with $\overline{J}\circ f_*=f_*\circ J$, $\{\overline{X_1}, \overline{X_2}, \overline{X_{1*}}, \overline{X_{2*}}\}$ is orthogonal basis for $T_{f(p)}(\overline{M})$ at $f(p)$, whose vectors are of the same length. In other words, f_* is a homothety. Q. E. D.

3. Proof of main theorem

The following beautiful conformal invariant

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z \\ &+\frac{1}{n-1}\{Ric(Y, Z)X-Ric(X, Z)Y+g(Y, Z)Ric_0X-g(X, Z)Ric_0Y\} \\ &-\frac{Sc}{(n-1)(n-2)}\{g(Y, Z)X-g(X, Z)Y\} \end{aligned}$$

was first written down by Weyl, and is sometimes called the Weyl's conformal curvature tensor where $Ric(X, Z)=Trace\ (Y\longrightarrow R(X, Y)Z)$, $g(Ric_0\ X, Y)=Ric(X, Y)$ and $Sc=Trace\ Ric_0$ for $X, Y, Z\in T_p(M)$.

We note the following theorem which was also proved by Weyl.

PROPOSITION 3. 1 (L. P. Eisenhalt [2]). *Let (M, g) be a Riemannian manifold. Suppose dimension of $M\geqq 4$. Then M is conformally flat if and only if $C\equiv 0$.*

Next we denote the well known fact in Kählerian manifolds.

PROPOSITION 3. 2 (K. Yano [6]). *A conformally flat Kählerian space is flat.*

Now to prove Proposition 3. 4, we need the following

LEMMA 3. 3 (S. Kobayashi and K. Nomizu [3]). *Let V be a $2n$-dimensional real vector space with a Hermitian inner product g and a complex structure J. Let R and T be two quadrilinear mappings satisfying the conditions (a), (b), (c) and (d).* If

$$R(X, JX, X, JX)=T(X, JX, X, JX)\qquad \textit{for all } X\in V,$$

then $R=T$.

The following proposition summarizes the basic relations among second order tensors (related to metric $g, \overline{g}$ respectively) under the "holomorphic sectional curvature preserving" hypothesis.

PROPOSITION 3. 4. *Let V be a $2n$ dimensional real vector space equipped with two Hermitian inner products g, $\bar{g}$, two curvature tensors R, $\bar{R}$ respectively and a Kählerian structure J such that*

(e) $\bar{g}=\lambda g$ *for some* $\lambda\in R>0$,

(f) $\bar{H}=H$ *(equality of corresponding holomorphic sectional curvature).*

Then (1) $\bar{R}=\lambda R$, (2) $\overline{Ric}=\lambda Ric$, (3) $\overline{Ric}_0=Ric_0$, (4) $\overline{Sc}=Sc$, (5) $\bar{C}=\lambda C$.

PROOF. The condition (f) means for any vector X,

$$\frac{\bar{g}(\bar{R}(X, JX)X, JX)}{\bar{g}(X, X)\,\bar{g}(JX, JX)}=\frac{g(R(X, JX)X, JX)}{g(X, X)\,g(JX, JX)}.$$

From (e), we have $g(\bar{R}(X, JX)X, JX)=\lambda g(R(X, JX)X, JX)$. Here put $T=\lambda R$, then it is trivial that T satisfies the conditions (a), (b), (c) and (d). Therefore by Lemma 3. 3, we have $R=T$. The rest follows immediately. Q. E. D.

We shall prove the following

PROPOSITION 3. 5. *Let (M, g, J), $(\bar{M}, \bar{g}, \bar{J})$ be h-isocurved Kählerian manifolds. Suppose dimension of* $M\geqq 4$ and M *non flat. Then an h. s. c. p. h. i which is conformal is an isometry.*

PROOF. Let f: $M\longrightarrow\bar{M}$ be an *h. s. c. p. i.* We identify M with $\bar{M}$ via f and so reduce to the following situation. M has two metric g, $f^*\bar{g}$ (which for convenience we write simply as $\bar{g}$) such that the identity mapping of

$$(M, g, J)\xrightarrow{IM}(\bar{M}, \bar{g}, \bar{J})$$

is a holomorphic sectional curvature preserving. Since M is assumed to be non flat and dimension of $M\geqq 4$, we obtain $C\not\equiv 0$ from Proposition 3. 1 and Proposition 3. 2. Moreover since f is assumed to be conformal, we can see that there exists a positive real valued function ϕ on M such that $g=\phi\bar{g}$. By Proposition 3. 4, we have $\bar{C}=\phi C$. But since C is conformal invariant, we also have $\bar{C}=C$. Since $C\neq 0$, $\phi=1$. Q. E. D.

Thus from Proposition 2. 3 and Proposition 3. 5, we have the following

THEOREM 3. 6. *Let (M, g, J), $(\bar{M}, \bar{g}, \bar{J})$ be h-isocurved Kählerian manifold. Suppose dimension of $M=4$ and M non flat. If the set of points $p\in M$ such that $H(X)\not\equiv$*constant for $X\in T_p(M)$ is dense in M, then (M, g, J), $(\bar{M}, \bar{g}, \bar{J})$ *are isometric.*

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