On 4-dimensional connected Einstein spaces satisfying the condition R(X, Y)·R=0

By

Kouei SEKIGAWA

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1. Introduction

Let M be a 4-dimensional connected Einstein space with the Ricci tensor $S=\lambda g$, where g is the Riemannian metric of M and λ is a constant.

In this paper, we show the following theorem

THEOREM 1. 1 *Let M be a 4-dimensional connected Einstein space. Assume that*

(1. 1) $\quad R(X, Y)\cdot R=0$ *for all tangent vectors X and Y.*

Then, $\nabla R=0$, *that is, M is locally symmetric.*

Now, we can see that there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ at each tangent space of M such that

(1. 2)
$$\begin{aligned} &R_{1212}=a, && R_{1313}=b, && R_{1414}=c, \\ &R_{2323}=c, && R_{2424}=b, && R_{3434}=a, \\ &R_{1234}=f, && R_{1342}=h, && R_{1423}=-(f+h), \end{aligned}$$

otherwise zero. Where, $R_{ijkl}=g(R(e_i, e_j)e_k, e_l)$, $1\leqq i, j, k, l,\leqq 4$.

And, as M is an Einstein space with the Ricci curvature λ, the relation

(1. 3) $\quad a+b+c=-\lambda,\quad$ holds good.

As the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M, (1. 1) implies

(1. 4)
$$[R(e_i, e_j), R(e_k, e_l)]=R(R(e_i, e_j)e_k, e_l)+R(e_k, R(e_i, e_j)e_l)$$

2. Proof of theorem

First we state a lemma

LEMMA 2.1 (*Lichnerowicz* [4]) *In a Riemannian manifold we have*

$$\Delta(R_{ijkl}R^{ijkl})=2(\nabla_h R_{ijkl}\nabla^h R^{ijkl})-4R^{ijkl}\nabla_i(\nabla_k S_{jl}-\nabla_l S_{jk})$$
$$-4R^{ijkl}H^h{}_{jkl,hi}$$

where $H_{ijk}{}^l{}_{,st}X^sY^t$ *are components of* $R(X,Y)\cdot R$.

Now, from (1.2), we have

$$\begin{aligned}
&R(e_1,e_2)=ae_2\wedge e_1+fe_4\wedge e_3,\\
&R(e_1,e_3)=be_3\wedge e_1+he_2\wedge e_4,\\
&R(e_1,e_4)=ce_4\wedge e_1+(f+h)e_2\wedge e_3,\\
&R(e_2,e_3)=ce_3\wedge e_2+(f+h)e_1\wedge e_4,\\
&R(e_2,e_4)=be_4\wedge e_2+he_1\wedge e_3,\\
&R(e_3,e_4)=ae_4\wedge e_3+fe_2\wedge e_1,
\end{aligned}\tag{2.1}$$

where, in general, $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Z,Y)X-g(Z,X)Y$.

Thus, from (1.4), by using (2.1) we have

(2.2) $a(b-c)+f(f+2h)=0,$

(2.3) $f(b-c)+a(f+2h)=0,$

(2.4) $h(a-c)+b(h+2f)=0,$

(2.5) $b(a-c)+h(h+2f)=0,$

(2.6) $c(a-b)+(f+h)(h-f)=0,$

(2.7) $c(f-h)+(f+h)(b-a)=0.$

Thus, from (2.2) and (2.3), we have

(2.8) $(a^2-f^2)(b-c)=0,$

and similarly we have

(2.9) $(b^2-h^2)(a-c)=0,$

(2.10) $c((a-b)^2-(f-h)^2)=0.$

Therefore, we can see that following four cases are possible and essential. That is,

I. $a^2\neq f^2$, $b^2\neq h^2$, and $c\neq 0$.

Then, by (2.8), (2.9) and (2.10), we have $a=b=c$, and $f=h$.
Thus, by (2.2), we have $f=h=0$.

Therefore, from (1. 3), we have $a=b=c=-\frac{\lambda}{3}\neq 0$, $f=h=0$.

II. $a^2\neq f^2$, $b^2=h^2$, and $c\neq 0$.

Then, by (2. 8), we have $b=c$. Thus, by (2. 2) and (2. 3), we have $f=-2h$. And, then from (2. 10), we have $(a+2b)(a-4b)=0$.

Then, we have $a=4b$.

Therefore, from (1. 3), we have $a=-\frac{2\lambda}{3}$, $b=c=-\frac{\lambda}{6}$, $f=\frac{\lambda}{3}$, $h=-\frac{\lambda}{6}$, or $f=-\frac{\lambda}{3}$, $h=\frac{\lambda}{6}$. Where $\lambda\neq 0$.

III. $a^2\neq f^2$, $b^2=h^2$, and $c=0$.

Then, by (2. 8), we have $b=c=0$. And, from (2. 6), we have $f=h=0$.

Therefore, from (1. 3), we have $a=-\lambda$, $b=c=0$, $f=h=0$. Where $\lambda\neq 0$.

IV. $a^2=f^2$, $b^2=h^2$, and $c=0$.

Then, by (2. 2) and (2. 7), we have $a=b=c=0$, $f=h=0$.

Therefore, we can see that the length of the curvature tensor in each case is constant.

Thus, from lemma 2. 1., we have $\nabla R=0$.

NIIGATA UNIVERSITY

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