Note on positive linear maps of Banach algebras with an involution

By

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1. Introduction

Positive linear maps of function algebras are studied by many authors. In this note, we consider positive linear maps between Banach*-algebras. We first investigate the boundedness of positive linear maps. In § 2, we consider the following problem: Extreme points of the convex set of all positive linear maps which carry the identity element into the identity element are multiplicative? And the converse problem is true?

Let A be a complex Banach*-algebra. Denoting by A^+ the subset of H_A consisting of all finite sums of elements of the form $x^*x(x \in A)$ where H_A is the set of all self-adjoint elements of A, we shall those elements positive elements of A.

Let B also be complex Banach*-algebra. Then the linear map $T: A \to B$ is called positive if $T(A^+) \subset B^+$. We can show the following hermitian property for the positive linear maps, by using the similar argument in functional case.

(H) $\qquad T(x^*y) = (T(y^*x))^*$ for $x, y \in A$.

Therefore positive linear maps are *-preserving when A has an identity element. We remark that *-homomorphism of A into B is positive. In the following, the norm of the identity is assumed to be one.

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2. Boundedness of positive linear maps

The boundedness for the special case is obtained by **Yood** [5].

We have the following theorem.

THEOREM 2. 1 *Let A, B be two complex Banach*-algebras. Suppose that A has an identity element and the involution of A is continous. Suppose that B is *-semisimple* ([2], *p.* 210), *and that $T: A \to B$ is a positive linear map. Then T is bounded.*

PROOF. Let B_1 be the complex Banach*-algebra obtained by adjunction of an identity element to B. Since B^+ is contained in $B_1{}^+$,we may assume that B has an identity, say e_B. We define for $x \in B$ $|x|_0$ in the following manner:

$$|x|_0 = \sup\{|f(x)|; f \text{ is positive linear functional on } B \text{ such that } f(e_B) \leq 1\}$$

Then $|x|_0$ is a pseud-norm on B and $\{x \in B; |x|_0 = 0\}$ is *-radical. Since B is *-semi-simple, $|x|_0$ is a normed space norm on B.

For any positive linear functional f on B, $f \circ T$ is a positive functional on A. Hence, for every $a \in A$,

$$|f(T(a))| \leqq (f \circ T)(e_A)(\nu(a^* a))^{\frac{1}{2}}$$
$$\leqq C \| T(e_A) \| \| a \|,$$

where $\nu(a^*a)$ is the spectral radius of element a^*a in A and e_A is an identity element of A.

Therefore,

$$|T(a)|_0 = \sup\{|f(T(a))|; f \text{ is positive linear functional on } B \text{ such that } f(e_B) \leqq 1\}$$
$$\leqq C \| T(e_A) \| \| a \| \qquad (*)$$

Now, let $a_n \to x$ in A. Suppose that $T(a_n) \to y$ in B with respect to the norm $\| \ \|$. From the definition of the norm $| \ |_0$, it is clear that $|T(a_n) - y|_0 \leqq \| T(a_n) - y \|$. It follows that $T(a_n)$ converge to y with respect to the norm $| \ |_0$. On the other hand, from the above inequality (*) it follows that $|T(a_n) - T(x)|_0 \leqq C \| T(e_A) \| \| a_n - x \|$. Therefore, $T(a_n)$ converge to $T(x)$ with respect to $| \ |_0$. Consequently, we have $T(x) = y$. By the closed graph theorem T is bounded. We complete the proof.

The above theorem is also true in the slight general situation. Indeed we have the following corollary.

COROLLARY 2. 2 *Let A, B be complex Banach*-algebras. Suppose that A has a left (or right) approximate identity and B is *-semi-simple. Suppose that the involution of A is continous. Then any positive linear map T from A to B is bounded.*

PROOF. Denote by A_1 the complex Banach*-algebra obtained by adjunction of an identity element to A. We define for a, $b \in A$, $T_{a,b}(x) = T(axb)$, $(x \in A_1)$. Then $T_{a,b}$ is a linear map from A_1 to B. Moreover, since $aA_1{}^+a^*$ is contained in A^+, T_{a,a^*} is a positive linear map from A_1 to B. Hence by Theorem 2. 1, T_{a,a^*} is bounded.

Now, $$axb = \frac{1}{4}\{(a+b^*)x(a+b^*)^* - (a-b^*)x(a-b^*)^* + i(a+ib^*)x(a+ib^*)^* - i(a-ib^*)x(a-ib^*)^*\}.$$

Therefore, $T_{a\,b}$ is linear combination of maps of the form T_{a,a^*}. Consequently, $T_{a,b}$ is bounded.

Let $a_n \to 0$ in A. Since A has an approximate identity, there exist b, c and d_n in A such that $a_n = bd_nc$, $d_n \to 0$ (See [4]). Thus $T(a_n) = T(bd_nc) = T_{b,c}(d_n) \to 0$. It follows that

T is bounded.

REMARK 1. The continuity of the involution of A in Theorem 2.1 is dropped.

Indeed, even if the involution of A is discontinous, we can show that $|T(a)|_0 \leqq C\|T(e_A)\|\|(a)\|$ for $a \in H_A$. Let R be the radical of A. Since R is *-ideal, T vanishes on the radical R, *i. e.* $R \subset Ker T$. Since the Banach algebra A/R is semi-simple, it has the unique complete norm topology (See [1]). Therefore the involution is continous and A/R has an identity.

Let π is the canonical map: $A \rightarrow A/R$. For each $x \in A$, we define $\tilde{T}(\pi(x)) = T(x)$. Then T is a positive linear map from A/R to B and thus bounded map. Consequently, T is bounded.

REMARK 2. If A and B are C*-algebras and A has an identity element e_A, then $\|T(e_A)\| = \|T\|$.

3. Extreme positive linear map

In the following, for the sake of simplicity we assume that involutions are continous. Moreover, we assumed that A and B have identity elements (norm one) e_A, e_B respectively. We define the convex sets of positive linear maps $P(A, B)$, $P_1(A, B)$ and $P_2(A, B)$ as follows:

$$P(A, B) = \{T: A \rightarrow B;\ positive\ linear\ map\}$$

$$P_1(A, B) = \{T \in P(A, B)\ ;\ T(e_A) = e_B\}$$

$$P_2(A, B) = \{T \in P(A, B)\ ; \|T(e_A)\| \leqq 1\}$$

Then, we consider the following problem; Is every extreme point (if exist) of $P_j(A, B)$ (j=1, 2) multiplicative, that is, *-homomorphism? Conversely, is every multiplicative element of $P_j(A, B)$ extreme?

Many authors have treated the above problem when A, B are algebras of functions. For $h \in H_A$, we write $h \geqq 0$ when $h \in A^+$. Moreover, we write $h \geqq k$ provided that $h - k \geqq 0$ for h, $k \in H_A$. Similary, we write $T \geqq S$ when $T - S \in P(A, B)$ for T, $S \in P(A, B)$.

THEOREM 3.1 *For $T \in P_1(A, B)$, followings are equivalent.*

(1) *T is an extreme point of $P_1(A, B)$.*

(2) *Suppose S be an element of $P(A, B)$ such that $S(e_A)$ is in the center of B and $T \geqq S$. Then S is the form $S(e_A) \cdot T$.*

PROOF (1)→(2). Multiplying T by a scalar, we can assume $\|S(e_A)\| < 1$. Then $e_B - S(e_A)$ has an inverse and $(e_B - S(e_A))^{-1} \in B^+$. Moreover, $(e_B - S(e_A))^{-1}$ is in the center of B. On the other hand, there exists a positive constant k such that $k < min(1, 1/\|e_B - S(e_A))^{-1}\|)$ and $k(e_B - S(e_A))^{-1} \leqq e_B$. Now, we define two positive linear maps S_1, S_2 as follows;

$$S_1=(e_B-S(e_A))^{-1}(T-S)$$
$$S_2=(1-k)^{-1}(T-k(e_B-S(e_A))^{-1}(T-S))$$

then $S_1, S_2\in P_1(A, B)$ and $T=kS_1+(1-k)S_2$. Since T is extreme, $T=S_1=S_2$. Therefore, $S=S(e_A)\cdot T$.

(2)→(1). It is clear.

COROLLARY 3. 2 *Let T be an extreme point in $P_1(A, B)$. Suppose that a and $T(a)$ are in the center of A, B respectively. Then $T(ax)=T(a)T(x)$ for every $x\in A$.*

PROOF. We have only to prove this corollary in the case when $a\in A$ is a hermitian element. There exists positive constant k such that $0<k<1/\|a\|$. Moreover there exists $h>0$ such that $0<h<min(1, 1/\|e_A-ka\|)$. Then $\|e_A-h(e_A-ka)\|<1$ is valid and $h(e_A-ka)$ is in the center of A. We define $S(x)=T(h(e_A-ka)x)$ for $x\in A$. Then $S(e_A)$ is in the center of B and $S\in P(A, B)$. Moreover $T\geqq S$. Since T is extreme point, $S=S(e_A)\cdot T$. Consequently, $T(ax)=T(a)\ T(x)$ for $x\in A$. (q. e. d)

REMARK. If A, B be commutative, every extreme point in $P_1(A, B)$ is *-homomorphism.

We next consider the converse problem. We need the following lemma.

LEMMA 3. 3. *Suppose that B is symmetric Banach*-algebra. Let T be in $P_2(A, B)$ and a be in H_B. Then if $T(a^2)$ and $T(a)$ are commutative, $T(a^2)-(T(a))^2\in\overline{B^+}$ (norm closure of B^+).*

PROOF. Since B is symmetric Banach*-algebra, $H_B{}^+=\overline{B^+}$, where $H_B{}^+=\{h\in H_B;\ SP_B(h)\geqq 0\}$. Because of the hermitian property of T, $T(a)$ is hermitian element and thus $T(a^2)-(T(a))^2$ is hermitian in B. We shall show $SP_B(T(a^2)-T(a))^2)\geqq 0$. Consider a maximal commutative *-subalgebra D of B which contains e_B, $T(a)$ and $T(a^2)$. Then D is a commutative symmetric Banach*-algebra.

$$SP_B(T(a^2)-(T(a))^2)=SP_D(T(a^2)-(T(a))^2)$$
$$=\{\varphi(T(a)^2-(T(a))^2);\ \varphi\in\Phi_D\}$$

where Φ_D is the carrier space of D. Since D is symmetric, Φ_D consists of all *-homomorphism of D to complex numbers. For $\varphi\in\Phi_D$, $\varphi(T(a^2)-(T(a))^2)=(\varphi\circ T)(a^2)-((\varphi\circ T)(a))^2\geqq 0$. Therefore, $T(a^2)-(T(a))^2$ has non-negative real spectrum. Thus $T(a^2)-(T(a))^2\in H_B{}^+=\overline{B^+}$. (q. e. d)

Using the well known argument, we obtain the following.

THEROEM 3. 4. Suppose B be commutative semi-simple symmetric Banach*-algebra. Then any multiplicative element T of $P_1(A, B)$ is extreme point of $P_1(A, B)$.

PROOF. Suppose that there exist T_1, $T_2\in P_1(A, B)$ such that $T=\frac{1}{2}(T_1+T_2)$. For any $a\in H_A$ and positive real ε,

$$0=\frac{1}{2}(T_1(a^2)+T_2(a^2))-\frac{1}{4}((T_1(a))^2+(T_2(a))^2+T_1(a)T_2(a)+T_2(a)T_1(a))$$
$$\geqq\frac{1}{2}((T_1(a))^2+(T_2(a))^2-2\varepsilon e_B)-\frac{1}{4}((T_1(a))^2+2T_1(a)T_2(a)+(T_2(a)^2)$$
$$=\frac{1}{4}(T_1(a)-T_2(a))^2-\varepsilon e_B.$$

Thus $\varepsilon e_B\geqq\frac{1}{4}(T_1(a)-T_2(a))^2\geqq 0$. Since ε is arbitrary and B is semi-simple, we have $T_1(a)=T_2(a)$. Moreover a is arbitrary, hence $T=T_1=T_2$. Consequently it follows that T is extreme point of $P_1(A, B)$.

REMARK 1. Theorem 3. 4 holds even if is an element of $P_1(A, B)$ such that $T(a^2)=(T(a))^2$ for every $a\in H_A$.

REMARK 2. The semi-simplicity of B in the above theorem is necessary, that is, if B is non-semi-simple, we can find a non-extreme, multiplicative element of $P_1(A, B)$.

REMARK 3. We can discuss a similar argument for $P_2(A, B)$ and $P(A, B)$. In the latter case we need consider the characterization (2) in Theorem 3. 1.

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