On the abstract quasi-linear differential equation

By

Michiaki WATANABE

(Received October 31, 1969)

1. Introduction

The present paper is concerned with the solution of the Cauchy problem for the abstract quasi-linear evolution equation

$$(1) \qquad \frac{d}{dt}u(t)+A(t, u(t))u(t)=F(t, u(t)), \qquad 0<t\leq T,$$

$$(2) \qquad u(0)=\varphi$$

in a Banach space X, where $A(t, p)$ is, for every $t\in[0, T]$ and $p\in X$, not necessarily a bounded linear operator acting in X and $F(t, p)$ is a non-linear perturbation which takes values in X.

We have tried to integrate the above equation in the case when $A(t, p)$ does not contain $p: A(t, p)=A(t)$ ([5]). In this paper, however, we assume neither this nor the condition that the domain $D(t)=D(A(t, p))$ of $A(t, p)$ is independent of t.

To this end we shall consider the following integral equation associated with (1)-(2):

$$(3) \qquad v(t)=V(t, 0; v)\varphi+\int_0^t V(t, s; v)F(s, v(s))\,ds, \qquad 0\leq t\leq T,$$

wher $V(t, s; v)$ is, for a certain function $v(\cdot)$ on $[0, T]$ to X, a bounded-operator valued function on $0\leq s\leq t\leq T$ satisfying among other things (Definition 1)

$$\frac{\partial}{\partial s}V(t, s; v)g=V(t, s; v)A(s, v(s))g$$

for any $g\in D(s)$.

Here we call $v(t)$ a mild solution of (1)-(2) in $[0, T]$ if $v(t)$ is strongly continuous on $[0, T]$ and satisfies (3). As is verified in Proposition, a mild solution $v(t)$ in $[0, T]$ is also a strict solution (Definition 2) in $[0, T']\subset[0, T]$ as long as $F(t, v(t))$ is strongly continuous on $[0, T']$ if $v(t)$ belongs to $D(t)$ and is strongly differentiable for $0<t\leq T'$.

It is our main object of this article to construct a solution of (1)-(2) as the strong limit of the sequence $u_m(t)$, $m=0, 1, \cdots\cdots$ defined by

$$\frac{d}{dt}u_m(t)+A(t,\ u_{m-1}(t))u_m(t)=F(t,\ u_{m-1}(t)), \qquad 0<t\leq T,$$

$$u_m(0)=\varphi, \qquad m=1,\ 2,\ \cdots\cdots$$

and

$$u_0(t)=\varphi.$$

This argument would be admitted to be very natural. To make this sketch possible, we suppose that $A(t,\ p)$ and $F(t,\ p)$ satisfy for example the assumptions (A), (B) and (C), and use the semi-group method, above all, T. Kato and H. Tanabe's theory on abstract linear evolution equations ([1], [3]). Namely $A(t,\ \varphi)$ are, for a fixed $\varphi\in X$, assumed to fulfil the hypothesis on $A(t)$ in H. Tanabe [3] and to be densely defined uniformly on $[0,\ T]$ in the sense that for any $x\in X$ $\exp(-hA(t,\ \varphi))x$ converges to x uniformly on $[0,\ T]$ as $h\downarrow 0$.

But the solution obtained is merely a mild one although it is strongly Lipshitz continuous in t. The unique existence of the mild solution $v(t)$ of (1)-(2) in $[0,\ T_0]\subset[0,\ T]$ with an arbitrary initial value $\varphi\in D(0)$ is established in Theorem 1. While Theorem 2 shows that if this mild solution $v(t)$ is strongly continuously differentiable on $[0,\ T_0]\subset[0,\ T]$, then it becomes a strict one there and further the associated operator $V(t,\ s;v)$ coincides with the evolution operator $U(t,\ s;v)$ to the linear equation

$$\frac{d}{dt}u(t)+A(t,\ v(t))u(t)=0,\ 0<t\leq T_0,$$

that is,

$$V(t,s;v)=\exp(-(t-s)A(t,v(t)))+\int_s^t\exp(-(t-r)A(t,v(t)))R(r,\ s;v)dr,$$

where $R(t,\ s;v)$ is the solution of the integral equation

$$R(t,\ s;v)=-(\partial/\partial t+\partial/\partial s)\exp(-(t-s)A(t,\ v(t)))$$

$$-\int_s^t(\partial/\partial t+\partial/\partial r)\exp(-(t-r)A(t,\ v(t)))R(r,\ s;v)dr.$$

2. Definitions and assumptions

We begin this section with the following definitions:

Definition 1. *We call $v(t)$ a mild solution of* (1)-(2) in $[0,\ T]$ if

(i) *$v(t)$ is strongly continuous on $[0,\ T]$,*

(ii) *$F(t,\ v(t))$ is strongly integrable on $[0,\ T]$ and $v(t)$ satisfies*

$$v(t)=V(t,\ 0;v)\varphi+\int_0^t V(t,\ s;v)F(s,\ v(s))ds, \qquad 0\leq t\leq T,$$

where $V(t,\ s;v)$, $0\leq s\leq t\leq T$ is a family of bounded operators on X to *X and has the* *properties*:

(1°) $V(t, s; v)$ *is strongly continuous for* $0 \leq s \leq t \leq T$;

(2°) $V(t, r; v)V(r, s; v) = V(t, s; v)$, $V(r, r; v) = I$;

(3°) *for any* $g \in D(s)$, $\text{s-}\lim_{h \to 0} h^{-1}\{V(t, s+h; v) - V(t, s; v)g$ *exists and is equal to* $V(t, s; v)A(s, v(s))g$.

Definition 2. *We call* $v(t)$ *a strict solution of* (1)-(2) *in* $[0, T]$ *if*

(i) $v(t)$ *is strongly continuous on* $[0, T]$ *and strongly differentiable in* $t \in (0, T]$,

(ii) *for each* $t \in (0, T]$, $v(t)$ *belongs to* $D(t)$,

(iii) $v(t)$ *satisfies* (1) *and* (2).

From the above definitions we can prove

Proposition. *Let* $v(t)$ *be a mild solution of* (1)-(2) *in* $[0, T]$. *Suppose that* $v(t)$ *belongs to* $D(t)$ *and is strongly differentiable or* $0 < t \leq T'(\leq T)$ *and that* $F(t, v(t))$ *is strongly continuous on* $[0, T']$.

Then $v(t)$ *is a strict solution of* (1)-(2) *in* $[0, T']$.

Proof. For any $t \in (0, T']$

$$V(\tau, t; v)\, v(t) = V(\tau, 0; v)\varphi + \int_0^t V(\tau, s; v)F(s, v(s))ds, \qquad 0 < t \leq \tau \leq T,$$

which implies

$$\frac{\partial}{\partial t}\{V(\tau, t; v)v(t)\} = V(\tau, t; v)F(t, v(t)).$$

Hence, thanks to (3°), we have

$$V(\tau, t; v)\frac{d}{dt}v(t) + V(\tau, t; v)A(t, v(t))v(t) = V(\tau, t; v)F(t, v(t)).$$

Putting $\tau = t$, we can conclude

$$\frac{d}{dt}v(t) + A(t, v(t))v(t) = F(t, v(t)), \qquad 0 < t \leq T'.$$

Throughout this paper we shall make the following assumptions for an arbitrarily fixed $\varphi \in X$.

(A) For each $t \in [0, T]$, $A_0(t) = A(t, \varphi)$ is a closed linear operator whose domain is $D(t)$ and satisfies:

(A. 1) The resolvent set of $A_0(t)$ contains a fied closed sector

$$\Sigma = \{z;\ \theta \leq \arg z \leq 2\pi - \theta\} \qquad (0 < \theta < \frac{\pi}{2}).$$

For any $t \in [0, T]$ and $z \in \Sigma$ it holds that

$$\|z(z - A_0(t))^{-1}\| \leq M \ (M > 0);$$

(A. 2) $A_0(t)^{-1}$ is continuously differentiable in t in the uniform operator topology.

The range $R\left(\frac{d}{dt}A_0(t)^{-1}\right)$ of $\frac{d}{dt}A_0(t)^{-1}$ is contained by $D(A_0(t)^\rho)$ *and* $A_0(t)^\rho\frac{d}{dt}A_0(t)^{-1}$ is continuous on $[0, T]$ with $\|A_0(t)^\rho\frac{d}{dt}A_0(t)^{-1}\|\le N$ $(0<\rho\le 1,\ N>0)$;

(A. 3) For any $x\in X$, $x_h(t)=\exp(-hA_0(t))x$ converges uniformly on $[0, T]$ to x in the strong topology as $h\downarrow 0$ (See [4], Theorem 1).

(B) For each $t\in[0, T]$ and $p\in X$, $A(t, p)$ is a closed linear operator with the domain $D(t)$ and fulfils:

$$\|\{A(t, p)-A(t, q)\}A_0(t)^{-1}\|\le a(\|p\|+\|q\|)\|p-q\|$$

for any p, $q\in X$ and $t\in[0, T]$;

If $u(t)$ is continuously differentiable on $[0, T]$ in the strong topology, so is $A(t, u(t))A_0(t)^{-1}$ in the uniform operator topology with

$$\left\|\frac{d}{dt}\{A(t, u(t))A_0(t)^{-1}\}\right\|\le b\left(\|u(t)\|+\left\|\frac{d}{dt}u(t)\right\|\right)$$

for any $t\in[0, T]$ and $u(t)\in X$.

(C) $F(t, p)$ is a function defined on $[0, T]\times X$ to X satisfying

$$\|F(t, p)-F(t, q)\|\le c(\|p\|+\|q\|)\|p-q\|,$$

$$\|F(t, p)-F(s, p)\|\le d(\|p\|)|t-s|^{\rho'}\quad(0<\rho'\le 1)$$

for any p, $q\in X$ and t, $s\in[0, T]$.

Here θ, M, N, ρ and ρ' are some constants dependent only on φ at the most and a, b, c and d are non-decreasing continuous functions on $[0, \infty)$ to $[0, \infty)$ which generally depend on φ. For the sake of simplicity we assume $\rho=\rho'$ with N or d a little changed.

3. Preparatory lemmas

Under the assumptions (A. 1), (A. 2) and (C), the fundamental solution (evolution operator) $U_1(t, s)$, $0\le s\le t\le T$ of

$$(1)_1\qquad \frac{d}{dt}u(t)+A_0(t)u(t)=F(t, \varphi),\qquad 0<t\le T$$

can be constructed in the following manner:

$$(4)_1\qquad \begin{cases} U_1(t, s)=\exp(-(t-s)A_0(t))+\int_s^t\exp(-(t-r)A_0(t))R_1(r, s)dr,\\ \exp(-(t-s)A_0(t))=\frac{1}{2\pi i}\int_\Gamma e^{-(t-s)z}(z-A_0(t))^{-1}dz,\\ R_1(t, s)=\sum_{k=1}^{\infty}R^k{}_1(t, s),\quad R^1{}_1(t, s)=-\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)\exp(-(t-s)A_0(t)),\\ R^k{}_1(t, s)=\int_s^t R_1{}^1(t, r)R_1{}^{k-1}(r, s)dr,\qquad k=2, 3, \cdots\cdots, \end{cases}$$

Whrer Γ is a smooth path running in Σ from $\infty e^{-\theta i}$ to $\infty e^{\theta i}$. And $(1)_1$–(2) admitts a unique solution in $[0, T]$ which is given by

$$u(t)=U_1(t, 0)\varphi+\int_0^t U_1(t, s)F(s, \varphi)ds.$$

For the details, see T. KATO-H. TANABE [1] and H. TANABE [3].

More generally we have

Lemma 1. *Let $v(t)$ be strongly continuously differentiable in $t\in[0, T]$ and satisfy $v(0)=\varphi\in X$. Then there exists a positive number T_1 with $T_1\leq T$ and the fundamental solution $U(t, s; v)$, $0\leq s\leq t\leq T_1$ of*

$$(1)_v \qquad \frac{d}{dt}u(t)+A(t, v(t))u(t)=F(t, v(t)), \qquad 0<t\leq T_1$$

can be constructed by the formula

$$(4_v)\qquad \begin{cases} U(t, s; v)=\exp(-(t-s)A(t, v(t)))+\int_s^t \exp(-(t-r)A(t, v(t)))R(r, s; v)dr, \\ R(t, s; v)=\sum_{k=1}^{\infty}R^k(t, s; v), \\ R^1(t, s; v)=-\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)\exp(-(t-s)A(t, v(t))), \\ R^k(t, s; v)=\int_s^t R^1(t, r; v)R^{k-1}(r, s; v)dr, \qquad k=2, 3, \cdots\cdots\cdots \end{cases}$$

The unique solution $u(t)$ of $(1)_v$–(2) in $[0, T_1]$ is given by

$$u(t)=U(t, 0; v)\varphi+\int_0^t U(t, s; v)F(s, v(s))ds.$$

Proof. Let K_1 and K_2 be the maximum values of $\|v(t)\|$, $\|\frac{d}{dt}v(t)\|$ respectively: $\|v(t)\|\leq K_1$, $\|\frac{d}{dt}v(t)\|\leq K_2$, $t\in[0, T]$. Next let T_1 be the maximum value of positive t satisfying

$$t\leq T, \quad a(K_1+\|\varphi\|)K_2(M+1)\,t\leq\frac{1}{2}.$$

Then the resolvent of $A(t, v(t))=A_1(t)$ is given by the Neumann series

$$(z-A_1(t))^{-1}=\sum_{k=0}^{\infty}(z-A_0(t))^{-1}\{(A_1(t)-A_0(t))(z-A_0(t))^{-1}\}^k$$

because of the inequality

$$\|(A_1(t)-A_0(t))(z-A_0(t))^{-1}\|\leq a(K_1+\|\varphi\|)K_2(M+1)\,t\leq\frac{1}{2},$$

which implies

$$(3.1)\qquad \|z(z-A_1(t))^{-1}\|\leq 2M, \qquad t\in[0, T_1], \qquad z\in\Sigma.$$

Moreover we have

$$(3.2)\qquad \|A_0(t)^\alpha A_1(t)(z-A_1(t))^{-1}A_0(t)^{-\beta}\|$$

$$\leq |z|\,\|A_0(t)^\alpha(z-A_1(t))^{-1}(A_1(t)-A_0(t))(z-A_0(t))^{-1}A_0(t)^{-\beta}\|$$

$$+\|A_0(t)^{1-\beta+\alpha}(z-A_0(t))^{-1}\|$$

$$\leq C|z|^{\alpha-\beta},\qquad z\in\Sigma\qquad (0\leq\alpha\leq\beta\leq 1)$$

if we note

$$\|A_0(t)^\alpha(z-A_1(t))^{-1}\|\leq 2\|A_0(t)^\alpha(z-A_0(t))^{-1}\|.$$

In what follows, we use C, C_0, C_1, …… to denote constants depending only on θ, M, N, ρ and T.

From the differentiability of $A_1(t)A_0(t)^{-1}$ and $A_0(t)^{-1}$ it follows that

$$A_1(t)^{-1}=\sum_{k=0}^{\infty}A_0(t)^{-1}\{(A_0(t)-A_1(t))A_0(t)^{-1}\}^k$$

is continuously differentiable in $t\in[0, T_1]$ in the uniform operator topology and that

$$dA_1(t)^{-1}/dt=\left\{dA_0(t)^{-1}/dt-A_1(t)^{-1}\frac{d}{dt}(A_1(t)A_0(t)^{-1})\right\}$$

$$\times\sum_{k=0}^{\infty}\{A_0(t)-A_1(t))A_0(t)^{-1}\}^k,\qquad t\in[0, T_1].$$

For the proof of the above formula, we have only to show that

$$\sum_{k=0}^{n}A_0(t)^{-1}\frac{d}{dt}\{(A_0(t)-A_1(t))A_0(t)^{-1}\}^k$$

is continuous on $[0, T_1]$ and converges uniformly on $[0, T_1]$ to

$$-A_1(t)^{-1}\frac{d}{dt}(A_1(t)A_0(t)^{-1})\sum_{k=0}^{\infty}\{(A_0(t)-A_1(t))A_0(t)^{-1}\}^k$$

in the uniform operator topology as $n\to\infty$.

From the above argument it follows that $R\left(\frac{d}{dt}A_1(t)^{-1}\right)\subset D(A_0(t)^\rho)$ holds and $A_0(t)^\rho\frac{d}{dt}A_1(t)^{-1}$ is continuous in t and that

$$(3.3)\qquad \|A_0(t)^\rho\frac{d}{dt}A_1(t)^{-1}\|\leq 2\{N+2\|A_0(t)^{\rho-1}\|b(K_1+{}_2K)\}\leq D,\qquad t\in[0, T_1],$$

$$D=2N+C_0b(K_1+K_2).$$

Hence, by (3. 2) and (3. 3) we have

$$(3.4)\qquad \|\frac{d}{dt}(z-A_1(t))^{-1}\|=\|A_1(t)(z-A_1(t))^{-1}A_0(t)^{-\rho}\cdot A_0(t)^\rho\frac{d}{dt}A_1(t)^{-1}$$

$$\times A_1(t)(z-A_1(t))^{-1}\|\leq CD|z|^{-\rho},\qquad z\in\Sigma$$

and

$$\|A_0(t)^\alpha \frac{d}{dt}(z-A_1(t))^{-1}\| \leq CD|z|^{\alpha-\rho}, \qquad z\in\Sigma \qquad (0\leq\alpha<\rho). \tag{3.5}$$

Consequently making use of (3. 1)-(3. 5) we can construct the fundamental solution $U(t, s; v)$ by the formula $(4)_v$ and the unique solution

$$u(t)=U(t, 0; v)\varphi+\int_0^t U(t, s; v)_0 F(s, v(s))ds.$$

Lemma 2. *If φ belongs to $D(0)$, then the solution $u(t)$ in $[0, T_1]$ of (1) with $u(0)=\varphi$ is strongly differentiable on $[0, T_1]$ and satisfies*

i) $\|u(t)-\varphi\|\leq E_1 t^\rho$,

ii) $\|\frac{d}{dt}u(t)+A_0(0)\varphi-F(0, \varphi)\|\leq\|\{I-\exp(-tA_0(0))\}(A_0(0)\varphi-F(0, \varphi))\|+E_2 t^\rho$ *for all $t\in[0, T_1]$, where E_1 and E_2 are constants depending only on K_1, K_2 and φ, and K_1 and K_2 are the maximum values on $[0, T]$ of $\|v(t)\|$, $\|dv(t)/dt\|$ respectively.*

Proof. Writing

$$A_1(t)U(t, 0; v)\varphi=\exp(-tA_0(0))A_0(0)\varphi+\{A_1(t)\exp(-A_1(t)) -A_0(0)\exp(-tA_0(0))\}\varphi+\int_0^t A_1(t)\exp(-(t-s)A_1(t))R(s, 0; v)\varphi ds$$

and marking use of the inequalities

$$\|A_1(t)\exp(-(t-s)A_1(t))-A_1(s)\exp(-(t-s)A_1(s))\|\leq CD(t-s)^{\rho-1}, \tag{3.6}$$

$$\|R(t, s; v)\|\leq E(t-s)^{\rho-1},\quad E=\sum_{k=1}^{\infty}(C_1 D\Gamma(\rho))^k T^{(k-1)\rho}/\Gamma(k\rho), \tag{3.7}$$

$$\|A_0(t)^\alpha R(t, s; v)\|\leq\|A_0(t)^\alpha R^1(t, s; v)\|+\|\int_s^t A_0(t)^\alpha R^1(t, r; v)R(r, s; v)dr\|$$
$$\leq CD\{(t-s)^{\rho-\alpha-1}+E(t-s)^{2\rho-\alpha-1}\} \quad (0<\alpha<\rho),$$

we obtain

$$\|\int_0^t A_1(s)U(s, 0; v)\varphi ds\|\leq C_2(\|A_0(0)\varphi\|t+D\|\varphi\|t^\rho+DE\|\varphi\|t^{2\rho}). \tag{3.8}$$

From

$$\|F(t, v(t))\|\leq c(K_1+\|\varphi\|)(K_1+\|\varphi\|)+d(\|\varphi\|)T^\rho+\|F(0, \varphi)\|=H,$$

we get

$$\|\int_0^t U(t, s; v)F(s, v(s))ds\|\leq C_3(t+Et^{\rho-1})H. \tag{3.9}$$

In view of (3. 8), (3. 9) and the expession

$$u(t)-\varphi=-\int_0^t A_1(s)U(s, 0; v)\varphi ds+\int_0^t U(t, s; v)F(s, v(s))ds,$$

we can conclude i) if we put

$$E_1=C_2(\|A_0(0)\varphi\|T^{1-\rho}+D\|\varphi\|+DE\|\varphi\|T^\rho)+C_3(T^{1-\rho}+TE)H. \tag{3.10}$$

Next, by means of (3. 2) and (3. 4), we have

(3.11)
$$\|\{(z-A_1(t))^{-1}-(z-A_1(s))^{-1}\}A_0(s)^{-1}\|$$
$$\leq\int_s^t\|\frac{d}{dr}(z-A_1(r))^{-1}\{A_0(r)^{-1}-A_0(s)^{-1}\}\|dr$$
$$+\int_s^t\|\frac{d}{dr}(z-A_1(r))^{-1}\cdot A_0(r)^{-1}\|dr\leq CD\{(t-s)^2|z|^{-\rho}+(t-s)|z|^{-\rho-1}\}$$

and hence

(3.12)
$$\|\{A_1(t)\exp(-tA_1(t))-A_0(0)\exp(-tA_0(0))\}\varphi\|\leq C_4D\|A_0(0)\varphi\|t^{\rho}.$$

Writing
$$-\int_0^t A_1(t)U(t,s;v)F(s,v(s))ds+\{I-\exp(-tA_0(0))\}A_0(0)\varphi-F(0,\varphi)+F(t,v(t))$$
$$=\{I-\exp(-tA_0(0))\}(A_0(0)\varphi-F(0,\varphi))+\exp(-tA_0(0))\{F(t,v(t))-F(0,\varphi)\}$$
$$-\int_0^t A_1(t)\exp(-(t-s)A_1(t))\{F(s,v(s))-F(t,v(t))\}ds$$
$$+\{\exp(-tA_1(t))-\exp(-tA_0(0))\}F(t,v(t))$$
$$-\int_0^t dr\int_r^t A_1(t)\exp(-(t-s)A_1(t))R(s,r;v)F(r,v(r))ds$$

and marking use of (3. 6) and (3. 7), we obtain

(3.13)
$$\|\{I-\exp(-tA_0(0))\}\cdot A_0(0)\varphi-F(0,\varphi)+F(t,v(t))$$
$$-\int_0^t A_1(t)U(t,s;v)F(s,v(s))ds\|$$
$$\leq\|\{I-\exp(-tA_0(0))\}(A_0(0)\varphi-F(0,\varphi))\|$$
$$+C_5c(K_1+\|\varphi\|)K_2t+c(2K_1)K_2t+d(K_1)t^{\rho}+DHt^{\rho}+DEHt^{2\rho}.$$

Noting
$$\|\frac{d}{dt}(z-A_1(t))^{-1}\cdot A_0(t)^{-1}\|\leq CD|z|^{-\rho-1},\qquad z\in\Sigma$$

and taking $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$, where
$$\Gamma_1=\{re^{-i\theta};\ (t-s)^{-1}\leq r<\infty\},\ \Gamma_3=\{re^{i\theta};\ (t-s)^{-1}\leq r<\infty\}$$

and
$$\Gamma_2=\{(t-s)^{-1}e^{i\varphi};\ \theta\leq\varphi\leq 2\pi-\theta\},$$

we have
$$\|R^1(t,s;v)A_0(t)^{-1}\|=\|\frac{1}{2\pi i}\int_\Gamma e^{-z(t-s)}\frac{d}{dt}(z-A_1(t))^{-1}\cdot A_0(t)^{-1}dz\|$$
$$\leq CD(t-s)^{\rho}$$

and hence
$$\|R^1(t,s;v)A_0(s)^{-1}\|\leq\|R^1(t,s;v)\{A_0(s)^{-1}-A_0(t)\}^{-1}\|+\|R^1(t,s;v)A_0(t)^{-1}\|$$
$$\leq CD(t-s)^{\rho}.$$

In view of $(4)_v$, we can prove

$$R^k(t,\ s;\ v)=\int_s^t R^{k-1}(t,\ r;\ v)R^1(r,\ s;\ v)dr,\ k=3,\ 2,\ \cdots\cdots\cdots$$

by induction and hence obtain

$$R(t,\ s;\ v)=R^1(t,\ s;\ v)+\int_s^t R(t,\ r;\ v)R^1(r,\ s;\ v)dr.$$

From this formula and (3. 7) we get

$$\|R(t,\ s;\ v)A_0(s)^{-1}\|\leq CD\{(t-s)^{\rho}+(t-s)^{2\rho}E\}. \tag{3.14}$$

From

$$\|A_0(\,t\,)^{\alpha}\frac{d}{dt}(z-A_1(t))^{-1}\cdot A_0(t)^{-1}\|\leq CD|z|^{\alpha-\rho-1},\qquad z\in\Sigma\quad(0<\alpha<\rho)$$

which is true because of (3. 2) and (3. 3), we have similarly

$$\|A_0(t)^{\alpha}R^1(t,\ s;\ v)A_0(s)^{-1}\|\leq CD(t-s)^{\rho-\alpha}$$

and by (3. 14)

$$\begin{aligned}&\|A_0(t)^{\alpha}R(t,\ s;\ v)A_0(s)^{-1}\|\\&\leq\|A_0(t)^{\alpha}R^1(t,\ s;\ v)A_0(s)^{-1}\|+\int_s^t\|A_0(t)^{\alpha}R^1(t,\ r;\ v)\cdot R(r,\ s;\ v)\ A_0(s)^{-1}\|dr\\&\leq CD\{(t-s)^{\rho-\alpha}+D(t-s)^{2\rho-\alpha}+DE(t-s)^{3\rho-\alpha}\}.\end{aligned} \tag{3.15}$$

Thus we have by (3. 6), (3. 14) and (3. 15)

$$\|\int_0^t A_1(t)\exp(-(t-s)A_1(t))R(s,0;v)\varphi ds\|\leq C_6\|A_0(0)\varphi\|\ \{Dt^{\rho}+D^2t^{2\rho}+D^2Et^{3\rho}\} \tag{3.16}$$

Collecting (3. 12), (3. 13) and (3. 16) and writing

$$\begin{aligned}\frac{d}{dt}u(\,t\,)=&-\exp(-tA_0(0))A_0(0)\varphi-\int_0^t A_1(t)\,U(t,\ s;\ v)F(s,\ v(s))ds\\&+F(t,\ v(\,t\,))-\{A_1(\,t\,)\exp(-tA_1(\,t\,))-A_0(0)\exp(-tA_0(0))\}\,\varphi\\&-\int_0^t A_1(\,t\,)\exp(-(t-s)A_1(\,t\,))R(s,\ 0;\ v)\varphi ds,\end{aligned}$$

we can conclude ii) if we put

$$\begin{aligned}E_2=&C_4\|A_0(0)\varphi\|+C_5\{c(K_1+\|\varphi\|)K_2T^{1-\rho}+c(2K_1)K_2T^{1-\rho}+d(K_1)+DH+DEHT^{\rho}\}\\&+C_6\|A_0(0)\varphi\|D(1+DT^{\rho}+DET^{2\rho}).\end{aligned} \tag{3.17}$$

From the argument above it will be clear that $\frac{d}{dt}u(\,t\,)$ is strongly continuous on $[0,\ T_1]$.

4. Construction of the solution

By virtue of the lemmas established in the previous section, we can construct a sequence $u_m(t)$, $m=0, 1, 2, \cdots\cdots\cdots$ defined in the following manner:

$$(1)_m \qquad du_m(t)/dt+A(t, u_{m-1}(t))u_m(t)=F(t, u_{m-1}(t)), \qquad 0\leq t\leq T_1,$$

$$(2)_m \qquad u_m(0)=\varphi, \ m=1, 2, \cdots\cdots\cdots;$$

$$u_0(t)=\varphi, \qquad \varphi\in D(0).$$

In this section we investigate the strong limit of this sequence and that of the sequence $U_m(t, s)$, $m=1, 2, \cdots\cdots\cdots$ of the fundamental solutions to $(1)_m$, $m=1, 2, \cdots\cdots\cdots$ constructed by $(4)_v$, $v=u_m$.

Lemma 3. *There are positive constants L_1, L_2 and T_0 dependent only on φ such that the sequence $u_m(t)$, $m=0, 1, 2, \cdots\cdots\cdots$ satisfies*

$$\|u_m(t)\|\leq L_1 \text{ and } \|\frac{d}{dt}u_m(t)\|\leq L_2$$

for all $t\in[0, T_0]$ and $m=0, 1, 2, \cdots\cdots\cdots$.

Proof. Let L_1 and L_2 be some constants such that

$$L_1>\|\varphi\|, \qquad L_2>\|A_0(0)\varphi-F(0, \varphi)\|$$

and T_0 be the maximum value of positive t satisfying

$$t\leq T, \quad a(L_1+\|\varphi\|)L_2(M+1)t\leq\frac{1}{2}, \qquad E_1(L_1, L_2)t^\rho\leq L_1-\|\varphi\|$$

and

$$\|\{I-\exp(-tA_0(0))\}(A_0(0)\varphi-F(0, \varphi))\|+E_2(L_1, L_2)t^\rho$$
$$\leq L_2-\|A_0(0)\varphi-F(0, \varphi)\|,$$

where the functions E_1 and E_2 of L_1 and L_2 are dependent only on φ and are defined by (3. 10) and (3. 17) together with

$$D=2N+C_0b\, L_1+L_2), \qquad E=\sum_{k=0}^{\infty}(c_1D\Gamma(\rho))^k T^{(k-1)\rho}/\Gamma(k\rho),$$
$$H=c(L_1+\|\varphi\|)(L_1+\|\varphi\|)+d(\|\varphi\|)T^\rho+\|F(0, \varphi)\|.$$

Arguing $(1)_1$ and $(4)_4$ and noting, for instance, that $D>N$ and $H>\|F(t, \varphi)\|$, we have easily

$$\|u_1(t)\|\leq L_1, \quad \|\frac{d}{dt}u_1(t)\|\leq L_2, \qquad t\in[0, T_0].$$

If we suppose that

$$\|u_k(t)\|\leq L_1, \quad \|\frac{d}{dt}u_k(t)\|\leq L_2, \qquad t\in[0, T_0],$$

then with aid of Lemma 2 we have

$$\|u_{k+1}(t)\| \le E_1(L_1, L_2)t^\rho + \|\varphi\| \le L_1,$$

$$\|\frac{d}{dt}u_{k+1}(t)\| \le \|\{I-\exp(-tA_0(0))\}(A_0(0)\varphi - F(0, \varphi))\| + E_2(L_1, L_2)t^\rho$$
$$+\|A_0(0)\varphi - F(0, \varphi)\| \le L_2, \qquad t\in[0, T_0].$$

Thus we have established this lemma by induction.

Lemma 4. *$u_m(t)$ converges uniformly on $[0, T_0]$ in the strong topology as $m\to\infty$ and $v(t)=\text{s-}\lim_{m\to\infty} u_m(t)$ satisfies*

$$\|v(t)\| \le L_1,\ \|v(t)-v(s)\| \le L_2|t-s|$$

for all $t, s\in[0, T_0]$ with $v(0)=\varphi$.

Proof. From the definition of u_m, it follows that

$$\|u_{m+1}(t)-u_m(t(\| \le \int_0^t \|U_m(t, s)\{F(s, u_m(s))-F(s, u_{m-1}(s))\}\|ds$$
$$+\int_0^t \|U_m(t, s)\{A(s, u_m(s))-A(s, u_{m-1}(s))\}u_m(s)\|ds, \qquad m=1, 2, \cdots\cdots$$

Estimating as

$$\|U_m(t, s)\| \le C_7(1+ET_0^\rho)=R_1, \tag{4.1}$$

$$\|A_0(t)u_m(t)\| \le \|\{A(t, u_{m-1}(t))-A_0(t)\}u_m(t)\| + \|A(t, u_{m-1}(t))u_m(t)\| \tag{4.2}$$
$$\le \frac{1}{2}\|A_0(t)u_m(t)\| + L_2 + H,$$

we can show that

$$\|u_{m+1}(t)-u_m(t)\| \le G\int_0^t \|u_m(s)-u_{m-1}(s)\|ds, \ m=1, 2, \cdots\cdots$$

where $G=R_1c(2L_1)+2R_1a(2L_1)(L_2+H)$ and hence that

$$\|u_{m+1}(t)-u_m(t)\| \le L_2G^m t^{m+1}/(m+1)\,!$$

for all $t\in[0, T_0]$ and $m=0, 1, \cdots\cdots$ because of $\|u_1(t)-u_0(t)\| \le L_2t$.
This implies that $u_m(t)$ onverges uniformly on $[0, T_0]$. The remaining part of this lemma will be clear from

$$\|u_m(t)\| \le L_1,\ \|u_m(t)-u_m(s)\| \le L_2|t-s| \ \text{ and } \ u_m(0)=\varphi.$$

Lemma 5. *If $m\to\infty$, $U_m(t, s)$ converges uniformly for $0\le t\le T_0$ in the strong topology. Put $V(t, s; v)\, x=\text{s-}\lim_{m\to\infty} U_m(t, s)x$ for an arbitrary $x\in X$. Then the bounded operator $V(t, s; v)$ on X to X satisfies* (1^0), (2^0) *and* (3^0) *of* Definition 1.

Proof. First we shall prove that

$$\|\frac{d}{dt}U_m(t, s)\cdot A_0(s)^{-1}\| \le R_2 \tag{4.3}$$

for all t, s with $0\leq s\leq t\leq T_0$ and some positive constant R_2 depending only on φ. Writing

$$\|A(t, u_m(t))U_m(t, s)A_0(s)^{-1}\|\leq\|A(s, u_m(s))\exp(-(t-s)A(s, u_m(s))A_0(s)^{-1}\|$$
$$+\|\{A(t, u_m(t))\exp(-(t-s)A(t, u_m(t))-A(s, u_m(s))\exp(-(t-s)A(s, u_m(s))\}A_0(s)^{-1}\|$$
$$+\int_s^t\|A(t, u_m(t))\exp(-(t-r)A(t, u_m(t))R(r, s; u_m)A_0(s)^{-1}\|dr,$$

we have easily

$$\|\frac{d}{dt}U_m(t, s)\cdot A_0(s)^{-1}\|\leq C_8\{1+D(t-s)^{\rho}+D^2(t-s)^{2\rho}+D^2E(t-s)^{3\rho}\}$$
$$\leq C_8(1+DT^{\rho}{}_0+D^2T^{2\rho}{}_0+D^2ET^{3\rho}{}_0)=R_2$$

by replacing v with u_m in (3. 2), (3. 11), (3. 14) and (3. 15).

From

$$U_m(t, s)x-U_n(t, s)x=(U_m(t, s)-(t, s))(x-x_h(s))+$$
$$+\int_s^t U_n(t, r)(A(r, u_n(r))-A(r, u_m(r)))U_m(r, s)x_h(s)dr$$

and

$$A_0(t)U_m(t, s)x_h(s)=-\{A(t, u_m(t))-A_0(t)\}U_m(t, s)x_h(s)$$
$$-\frac{d}{dt}U_m(t, s)\cdot x_h(s)$$

together with (4. 1) and (4. 3), we get

$$\|U_m(t, s)x-U_n(t, s)x\|\leq 2R_1\|x-x_h(s)\|+2R_1R_2a(2L_1)\int_s^t\|u_m(r)-u_n(r)\|\|A_0(s)x_h(s)\|dr$$
$$\leq 2R_1\|x-x_h(s)\|+2C(M+1)R_1R_2a(2L_1)\cdot h^{-1}\int_s^t\|u_m(s)-u_n(r)\|dr\|x\|.$$

Thus in view of the previous lemma and (A. 3) we can conclude the uniform convergence of $U_m(t, s)$ in the strong topology.

(1^0) is obvious and so is (2^0) from

$$V(t, r; v)V(r, s; v)x-V(t, s; v)x=(V(t, r; v)-U_m(t, r))V(r, s; v)x$$
$$+U_m(t, r)(V(r, s; v)-U_m(r, s))x+(U_m(t, s)-V(t, s; v))x$$

and

$$V(r, r; v)x-x=(V(r, r; v)-U_m(r, r))x.$$

To complete this lemma we have only to show that (3^0) is valid.

Clearly

$$A_0(t)^{-1}x-U_m(t, s)A_0(s)^{-1}x=\int_s^t\{U_m(t, r)A(r, u_m(r))A_0(r)^{-1}$$
$$+U_m(t, r)\frac{d}{dr}A_0(r)^{-1}\}xdr$$

for $0\leq s\leq t\leq T_0$ and $x\in X$.

Hence, from (4. 1) and the compactness in X of the sets

$$\{A(t, v(t))A_0(t)^{-1} : t\in[0, T_0]\} \text{ and } \left\{\frac{d}{dt}A_0(t)^{-1} : t\in[0, T_0]\right\}$$

we get

$$A_0(t)^{-1}x - V(t, s; v)A_0(s)^{-1}x = \int_s^t \{V(t, r; v)A(r, v(r))A_0(r)^{-1} + V(t, r; v)\frac{d}{dr}A_0(r)^{-1}\} x\,dr.$$

Here we have only to note

$$\begin{aligned} &h^{-1}\{V(t, s+h; v) - V(t, s; v)\}g - V(t, s; v)A(s, v(s))g \\ &= h^{-1}\int_s^{s+h}\{V(t, r; v)A(r, v(r))A_0(r)^{-1} \\ &\qquad - V(t, s; v)A(s, v(s))A_0(s)^{-1}\}A_0(s)g\,dr \\ &+ h^{-1}\int_s^{s+h}\{V(t, r; v) - V(t, s+h; v)\}\frac{d}{dr}A_0(r)^{-1}A_0(s)g\,dr, \qquad g\in D(s). \end{aligned}$$

5. Results and their proofs

Now we shall state main result.

Theorem 1. *Under the assumptions* (A), (B) *and* (C) *for an arbitrary* $\varphi\in D(0)$, *there exist positive numbers* T_0, L_1 *and* L_2 *dependent only on* φ *such that* (1)–(2) *admitts a unique mild solution* $v(t)$ *in* $[0, T_0]\subset[0, T]$ *satisfying*

$$\|v(t)\|\leq L_1 \text{ and } \|v(t)-v(s)\|\leq L_2|t-s|$$

for all $t, s \in[0, T_0]$.

Proof. In view of Lemma 4 and Lemma 5, it remains to show that $v(t)$ satisfies (3) in $[0, T_0]$ and is unique for the initial value φ.

From

$$\begin{aligned} &v(t) - V(t, 0; v)\varphi - \int_0^t V(t, s; v)F(s, v(s))ds \\ &= \{v(t) - u_{m+1}(t)\} + \{U_m(t, 0)\varphi - V(t, 0; v)\varphi\} \\ &+ \int_0^t U_m(t, s)\{F(s, u_m(s)) - F(s, v(s))\}ds \\ &+ \int_0^t \{U_m(t, s) - V(t, s; v)\}F(s, v(s))ds, \quad m=0, 1, \cdots\cdots\cdots \end{aligned}$$

and the compactness of the set $\{F(t, v(t)); t\in[0, T_0]\}$, we get

$$v(t) = V(t, 0; v)\varphi + \int_0^t V(t, s; v)F(s, v(s))ds, \quad 0\leq t\leq T_0.$$

Next, let $u(t)$ be any mild solution of (1)–(2) in $[0, T']$:

$$u(t) = V'(t, 0; u)\varphi + \int_0^t V'(t, s; u)F(s, u(s))ds, \quad 0\leq t\leq T',$$

where $V'(t, s; u)$, $0 \leq s \leq t \leq T'$ is a family of bounded operators satisfying the conditions (1°), (2°) and (3°) of Definition 1. Then as is easily seen, $\|u(t)\| \leq L^1$ and $\|V'(t, s; u)\| \leq R'$ with positive constants L' and R'.

Recalling the definition of u_m and noting

$$u(t)-u_{m+1}(t)=\{V'(t, 0; u)-U_m(t, 0)\}\varphi$$
$$+\int_0^t V'(t, s; u)\{F(s, u(s))-F(s, u_m(s))\}ds$$
$$+\int_0^t \{V'(t, s; u)-U_m(t, s)\}F(s, u_m(s))ds$$
$$=\int_0^t V'(t, s; u)\{F(s, u(s))-F(s, u_m(s))\}ds$$
$$+\int_0^t V'(t, s; u)\{A(s, u(s))-A(s, u_m(s))\}u_{m+1}(s)ds$$

and (4. 2), we have

$$\|u(t)-u_{m+1}(t)\| \leq \int_0^t K\|u(s)-u_m(s)\|ds, \qquad t\in[0, T_0]\cap[0, T'], \quad m=1, 2, \cdots,$$

where

$$K=R'c(L_1+L')+2R'a(L_1+L')(L_2+H).$$

Hence, passing to the limit in the above inequality we obtain

$$\|u(t)-v(t)\| \leq \int_0^t K\|u(s)-v(s)\|ds$$

and conclude $u(t)=v(t)$ on $[0, T_0]\cap[0, T']$. Q. E. D.

Remark. In the above argument it holds that

$$V(t, s; v)=V'(t, s; v)$$

for all t, s with $0 \leq s \leq t \leq \text{Min}(T_0, T')$. In fact, from

$$V(t, s; v)x-V'(t, s; v)x=\{V(t, s; v)-U_m(t, s)\}x-\{V'(t, s; v)-U_m(t, s)\}(x-x_h(s))$$
$$+\int_s^t V'(t, r; v)\{A(r, v(r))-A(r, u_m(r))\}U_m(r, s)x_h(s)dr \quad ,$$

we get

$$\|V(t, s; v)x-V'(t, s; x)x\| \leq \|V(t, s; v)x-U_m(t, s)x\|$$
$$+(R'+R)\|x-x_h(s)\|+2R'R_2a(2L_1)C(M+1)\cdot h^{-1}\int_s^t\|v(r)-u_m(r)\|dr\cdot\|x\|$$

for $0 \leq s \leq t \leq \text{Min}(T_0, T')$ and $x\in X$. Thus letting $m\to\infty$ we obtain the desired relation.

Finally we shall give a condition for a mild solution to be a strict one.

Theorem 2. *Let $v(t)$ be an X-valued function continuously differentiable on $[0, T]$ in the strong topology such that $v(0)=\varphi$. Then $v(t)$ is a strict solution of (1)–(2) in some interval $[0, T_0]\subset[0, T]$ if and only if $v(t)$ is a mild solution in $[0, T_0]$.*

Proof. Putting $M_1=\underset{0\leqq t\leqq T}{\text{Max}}\ \|v(t)\|$, $M_2=\underset{0\leqq t\leqq T}{\text{Max}}\ \|\frac{d}{dt}v(t)\|$ and letting T_0 be a positive number such that $T_0\leq T$ and $a(M_1+\|\varphi\|)M_2(M+1)T_0<1$, we can construct the fundamental solution $U(t, s; v)$, $0\leq s\leq t\leq T_0$ of

$$\frac{d}{dt}u(t)+A(t, v(t))u(t)=0, \qquad 0<t\leq T_0$$

through the formula $(4)_v$ in Lemma 1.

If $v(t)$ is a mild solution in $[0, T_0]$, then we have

$$\begin{aligned} &V(t, s+\varepsilon; v)U(s+\varepsilon, s; v)x-U(t, s; v)x \\ &\quad=-\int_{s+\varepsilon}^{t}\frac{\partial}{\partial r}\{V(t, r; v)U(r, s; v)\}xdr \\ &\quad=\int_{s+\varepsilon}^{t}V(t, r; v)\{A(r, v(r))-A(r, v(r))\}U(r, s; v)xdr=0 \end{aligned}$$

for any $\varepsilon>0$ and $x\in X$ and hence

$$V(t, s; v)=U(t, s; v), \quad 0\leq s\leq t\leq T_0,$$

where $V(t, s; v)$, $0\leq s\leq t\leq T_0$ is a family of bounded operators satisfying (1^0), (2^0) and (3^0) of Definition 1. Thus $v(t)$ belongs to $D(t)$ for $t\in(0, T_0]$ and consequently $v(t)$ is a strict solution in $[0, T_0]$ (See Proposition).

"Only if" part. Let $v(t)$ be a strict solution in $[0, T_0]$. Then for $0<s\leq t\leq T_0$

$$\begin{aligned} \frac{\partial}{\partial s}U(t, s; v)v(s)&=U(t, s; v)\Big(\frac{d}{ds}v(s)+A(s, v(s))v(s)\Big) \\ &=U(t, s; v)F(s, v(s)). \end{aligned}$$

Integrating this on $[\varepsilon, t]$ $(\varepsilon>0)$ we have

$$v(t)-U(t, \varepsilon; v)v(\varepsilon)=\int_{\varepsilon}^{t}U(t, s; v)F(s, v(s))ds$$

and by letting $\varepsilon\downarrow 0$ we conclude

$$v(t)=U(t, 0; v)\varphi+\int_0^t U(t, s; v)F(s, v(s))ds, \qquad 0\leq t\leq T_0.$$

Remark. Obviously the mild solution $v(t)$ of (1) with $v(0)=\varphi\in D(0)$ in $[0, T_0]$ whose unique existence has been established in Theorem 1, is a unique strict solution under the condition that $v(t)$ is continuously differetiable on $[0, T_0]$ in the strong topology.

NIIGATA UNIVERSITY

Bibliography

1. T. KATO and H. TANABE, *On the abstract evolution equation*, Osaka Math. J., 14 (1962), 101–133.
2. P. E. SOBOLEVSKI, *On equations of parabolic type in Banach space*, Trudy Moskov Mat. Ovsc., 10 (1961), 297–250.
3. H. TANABE, *Note on singular perturbation for abstract differential equations*, Osaka J. Math., 1

(1964), 239–252.

4. H. Tanabe and M. Watanabe, *Note on perturbation and degeneration of abstract differential equations in Banach space*, Funkcial. Ekvac., 9 (1966), 163–170.
5. M. Watanabe, *On the abstract semi-linear differential equations*, Sci. Rep. Niigata Univ., 5 (1967), 9–16.