On complex hypersurfaces of spaces of constant holomorphic sectional curvature satisfying a certain condition on the curvature tensor

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1. Introduction

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

(*) $\qquad R(X, Y)\cdot R=0 \qquad$ for all tangent vectors X and Y

where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. Conversely, does this algebraic condition on the curvature tensor field R imply that M is locally symmetric?
We conjecture that the answer is affirmative in the case where M is a complete and irreducible and dim $M\geqq 3$.

The main purpose of the present paper is to consider the complex hypersurfaces in spaces of constant holomorphic sectional curvature satisfying the condition (*) on the curvature tensor.

2. Complex space forms

A Riemannian manifold M with Riemannian metric g is called an Einstein manifold if its Ricci tensor S satisfies $S=\rho g$, where ρ is a constant. We call ρ the Ricci curvature of the Einstein manifold.

Let M be a complex analytic manifold of complex dimension n. By means of charts we may transfer the complex structure of complex n-dimensional Euclidean space C^n to M to obtain an almost complex structure J on M, i. e., a tensor field J on M of type (1.1) such that $J^2=-I$, where I is the tensor field which is the identity transformation on each tangent space of M.

A Riemannian metric g on M is a Hermitian metric if $g(JX, JY)=g(X, Y)$ for any vector fields X and Y on M; M is called a Hermitian manifold. If in addition the almost complex J is parallel with respect to the Riemannian connection of g, then J (resp. g) is called a Kähler structure (resp. Kähler metric); M is then called a Kähler manifold.

A plane which is tangent to M and is invariant by J will be called a holomorphic plane. If M is a Kähler manifold we denote by $K(p)$ the sectional curvature of a plane p tangent to M and by $K(X)$ the sectional curvature of the holomorphic plane generated by a unit tangent vector X. M is said to be of constant holomorphic sectional curvature c if the sectional curvature of every holomorphic tangent plane is equal to c. If M is of constant holomorphic sectional curvature c, then M is Einstein and, in the above notaion $\rho=(n+1)c/2$.

By a complex space form we will mean a complete Kähler manifold of constant holomorphic sectional curvature.

We now introduce some special Kähler manifolds which will occur in the course of our work. Let C^{n+2} dente complex $(n+2)$-dimensional Euclidean space with the natural complex coordinate system $z^0,\ldots\ldots, z^{n+1}$. $P^{n+1}(C)$ will denote complex $(n+1)$-dimensional projective space, $P^{n+1}(C)$ is a complex analytic mainifold which, when endowed with the Fubini-Study metric, is a Kähler manifold of constant holomorphic sectional couvature 1. There is a natural holomorphic mapping $f: C^{n+2}-\{0\}\longrightarrow P^{n+1}(C)$.

The variety in $P^{n+1}(C)$ determined by $z^{n+1}=0$ is merely $P^n(C)$, the induced metric being the Fubini-Study metric of $P^n(C)$.

The variety Q^n in $P^{n+1}(C)$ determined by $(z^0)^2+\ldots+(z^{n+1})^2=0$ is called the n-dimensional quadric; Q^n is a compact Kähler manifold with the metric and complex structure induced from $P^{n+1}(C)$.

The group $SO(n+2)$, as a subgroup of the group $U(n+2)$ of all holomorphic isometries of C^{n+2}, act on Q^n as a transitive group of holomorphic isometries. The isotropy group of this action at $(1, i, 0,\ldots, 0)\in Q^n$ is $SO(2)\times SO(n)$. It is easily checked that $SO(n+2)/SO(2)\times SO(n)$ is a symmetric space. Thus, if, $n>2$, Q^n is irreducible and hence it is an Einstein manifold. However, Q^2 is holomorphically isometric to $P^1(C)\times P^1(C)$, where $P^1(C)$ is endowed with the Fubini-Study metric. Hence Q^n is a compact Einstein manifold if $n\geqq 2$.

D^{n+1} will denote the open unit ball in C^{n+1} endowed with the natural complex structure and the Bergman metric. This is then a Kähler manifold of constant holomorphic sectional curvature -1. The submanifold of D^{n+1} determined by $z^n=0$ is merely D^n, the induced metric being the Bergman metric of D^n.

3. Complex hypersurfaces

Hence forth $\tilde{M}$ will be a connected Kähler manifold of complex dimension $n+1$, the Kähler structure and the Kähler metric of M being denoted by J and g respectively; moreover, M will be a connected complex manifold of complex dimension n which is a complex hypersurface of $\tilde{M}$, i.e., there exists a complex analytic mapping $\varphi: M \longrightarrow \tilde{M}$ whose differential φ_* is 1–1 at each point of M.

All metric properties on M will refer to the Hermitian metric g_0 induced on M by the immersion φ.

Then g_0 becomes to be a Kähler metric on M. Moreover it is well known that this is ture for arbitrary complex submanifolds of $\tilde{M}$.

In order to simplify the presentation, we identify, for each $x \in M$, tangent space $T_x(M)$ with $\varphi_*(T_x(M)) \subset T_{\varphi(x)}(\tilde{M})$ by means of φ_*. A vector in $T_{\varphi(x)}(\tilde{M})$ which is orthogonal, with respect to g, to the subspace $\varphi_*(T_x(M))$ is said to be normal to M at x. Since $\varphi^* g = g_0$ and $J\varphi_* = \varphi_* J_0$, where J_0 is the almost complex structure of M, the structures g_0 and J_0 on $T(M)$ are respectively identified with the restrictions of the structures g and J to the subspace $\varphi_*(T_x(M))$. With this identification in mind we drop the symbols g_0 and J_0, using instead the symbols g and J.

The following is a purely local argument. Let $U(x)$ be a neighborhood of a point $x \in M$ on which we choose a unit vector field ξ normal to M. $\tilde{\nabla}$ denotes the Riemannian covariant differentiation on the Kähler manifold $\tilde{M}$. Throughout, X, Y, Z and W will be either vector fields on one of the special neighborhoods $U(x)$ of x, or vectors tangent to M at a point of $U(x)$, unless otherwise specified.

If X and Y are vector fields on $U(x)$ we may write

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi + k(X, Y) J\xi, \tag{3.1}$$

where $\nabla_X Y$ denotes the component of $\tilde{\nabla}_X Y$ tangent to M.

Then we have

LEMMA 3.1. (i) *∇ is the covariant differentiation of the Hermitian manifold M; furthermore M is a Kähler manifold, that is $\nabla J = 0$.*
(ii) *h and k are symmetric covariant tensor fields of degree 2 on $U(x)$ satisfying*

$$h(X, JY) = -k(X, Y),$$

$$k(X, JY) = h(X, Y).$$

The identity $g(\xi, \xi) = 1$ implies $g(\tilde{\nabla}_X \xi, \xi) = 0$ on $U(x)$ for any vector field X on $U(x)$. We may therefore write

$$\tilde{\nabla}_X \xi = -A(X) + s(X) J\xi, \tag{3.2}$$

where $A(X)$ is tangent to M.

LEMMA 3.2. *A and s are tensor fields on $U(x)$ of type (1.1) and (0.1) respectively. Furthermore A and JA are symmetric with respect to g, $AJ = -JA$ and A satisfies*

$$h(X, Y)=g(AX, Y),$$

$$k(X, Y)=g(JAX, Y),$$

for any pair of vectors X and Y tangent to M at a point of U(x).

The following lemma will be used frequently in our work.

LEMMA 3.3.[1]) *Let V be a 2n-dimensional real vector space with a complex structure J and a positive definite inner product g which is Hermitian, i.e.,* $g(JX, JY)=g(X, Y)$ *for all* $X, Y\in V$. *If A is symmetric (with respect to g) and* $AJ=-JA$, *there exists an orthonormal basis* $\{e_1,\dots, e_n, Je_1,\dots, Je_n\}$ *of V with respect to which the matrix of A is diagonal of the form*

$$\begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ 0 & & & & & -\lambda_n \end{pmatrix}$$

In particular Trace A=Trace AJ=0.

And morever we have

LEMMA 3.4. *If* $\tilde{M}$ *is of constant holomorphic sectional curvature* $\tilde{c}$, *then for any pair of vectors X and Y tangent to M at a point of U(x), we have the equations*

$$(\nabla_X A)Y-(\nabla_Y A)X-s(X)JAY+s(Y)JAX=0 \quad \text{(Codazzi's equation)}, \tag{3.3}$$

$$S(X, Y)=-2g(A^2X, Y)+(n+1)\tilde{c}/2\; g(X, Y), \tag{3.4}$$

where S is the Ricci tensor of M.

4. Reduction of condition (*) and some results

In the section, we shall assume that $\tilde{M}$ is a space of constant holomorphic sectional curvature $\tilde{c}$ and M is a complex hypersurface of $\tilde{M}$ of complex dimension n. Then the equation of Gauss expresses the curvature tensor R of M in the form

$$R(X, Y)=AX\wedge AY+JAX\wedge JAY+\tilde{c}/4\{X\wedge Y+JA\wedge JY+2g(X, JY)\}, \tag{4.1}$$

where, in general, $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Z, Y)X-g(Z, X)Y$.

The type number $k(x)$ at x is, by definition, the rank of A at x.

Let $\{e_1,\dots\dots, e_n, Je_1,\dots\dots, Je_n\}$ be the orthonormal basis which is constructed in Lemma 3.3., then we have

1) See for example [4].

(4.2) $$R(e_i, e_j)=(\lambda_i\lambda_j+\tilde{c}/4)(e_i\wedge\bar{e}_j+\bar{e}_i\wedge e_j)$$

(4.3) $$R(e_i, \bar{e}_j)=(\lambda_i\lambda_j-\tilde{c}/4)(\bar{e}_i\wedge e_j-e_i\wedge\bar{e}_j)-\tilde{c}/2\delta_{ij}J.$$

where we put $\bar{e}_i=Je_i$. $\quad i, j=1,\ldots\ldots,n.$

As the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M, we get

(4.4) $$(R(X, Y)\cdot R)(Z, W)=[R(X, Y), R(Z, W)]-R(R(X, Y)Z, W) -R(Z, R(X, Y)W).$$

For reduction of the condition (*), we have only to consider the following cases.

I. $\quad X=e_i,\ Y=e_j,\ Z=e_k,\ W=e_l$

II. $\quad X=e_i,\ Y=\bar{e}_j,\ Z=e_k,\ W=\bar{e}_l$

III. $\quad X=e_i,\ Y=e_j,\ Z=e_k,\ W=\bar{e}_l$

IV. $\quad X=e_i,\ Y=\bar{e}_j,\ Z=e_k,\ W=e_l.$

Case I., then by making use of (4.2), from (4.4) we find that it is zero except possibly in the case where $k=i$ and $l\neq i, j (i\neq j)$.
Then we have

(4.5) $$(R(e_i, e_j)\cdot R)(e_i, e_l)=(\lambda_i\lambda_j+\tilde{c}/4)\lambda_l(\lambda_j-\lambda_i)(e_j\wedge e_l+\bar{e}_j\wedge\bar{e}_l).$$

Case II., then, similarly by making use of (4.3), from (4.4) we find that it is zero except possibly in the case where $k=i$ and $l\neq i, j (i\neq j)$.
Then we have

(4.6) $$(R(e_i, \bar{e}_j)\cdot R)(e_i, \bar{e}_l)=-(\lambda_i\lambda_j-\tilde{c}/4)\lambda_l(\lambda_j+\lambda_i)(e_j\wedge e_l+\bar{e}_j\wedge\bar{e}_l).$$

Case III., then by making use of (4.2) and (4.3), from (4.4) we find that it is zero except possibly in the following two cases, that is, for $k=i$ and $l\neq i, j (i\neq j)$, we get

(4.7) $$(R(e_i, e_j)\cdot R)(e_i, \bar{e}_l)=(\lambda_i\lambda_j+\tilde{c}/4)\lambda_l(\lambda_j-\lambda_i)(\bar{e}_j\wedge e_l-e_j\wedge\bar{e}_l).$$

and for $k=i$ and $l=j (i\neq j)$, we get

(4.8) $$(R(e_i, e_j)\cdot R)(e_i, \bar{e}_j)=2(\lambda_i\lambda_j+\tilde{c}/4)\lambda_i(\lambda_j-\lambda_i)\bar{e}_i\wedge e_i +2(\lambda_i\lambda_j+\tilde{c}/4)\lambda_j(\lambda_j-\lambda_i)\bar{e}_j\wedge e_j.$$

Case IV., then, similarly, we find that it is zero except possibly in the following cases, that is, for $k=i$ and $l\neq i, j (i\neq j)$, we get

(4.9) $$(R(e_i, \bar{e}_j)\cdot R)(e_i, e_l)=(\lambda_i\lambda_j-\tilde{c}/4)\lambda_l(\lambda_j+\lambda_i)(\bar{e}_j\wedge e_l-e_j\wedge\bar{e}_l).$$

and for $k=i$ and $l=j (i\neq j)$, we get

$$(R(e_i,\ \bar{e}_j)\cdot R)(e_i,\ e_j)=2(\lambda_i\lambda_j-\tilde{c}/4)\lambda_j(\lambda_j+\lambda_i)\bar{e}_j\wedge e_i -2(\lambda_i\lambda_j-\tilde{c}/4)\lambda_i(\lambda_j+\lambda_i)e_j\wedge\bar{e}_i. \tag{4.10}$$

Therefore, from (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10), we see that the condition (*) is equivalent to

$$\begin{cases} (\lambda_i\lambda_j+\tilde{c}/4)\lambda_l(\lambda_j-\lambda_i)=0 & \text{for } l\neq i,\ j(i\neq j) \\ (\lambda_i\lambda_j-\tilde{c}/4)\lambda_l(\lambda_j+\lambda_i)=0 & \text{for } l\neq i,\ j(i\neq j) \\ (\lambda_i\lambda_j+\tilde{c}/4)\lambda_j(\lambda_j-\lambda_i)=0 & \text{for } i\neq j \\ (\lambda_i\lambda_j-\tilde{c}/4)\lambda_i(\lambda_j+\lambda_i)=0 & \text{for } i\neq j \\ (\lambda_i\lambda_j+\tilde{c}/4)\lambda_i(\lambda_j-\lambda_i)=0 & \text{for } i\neq j \\ (\lambda_i\lambda_j-\tilde{c}/4)\lambda_j(\lambda_j+\lambda_i)=0 & \text{for } i\neq j.\quad i,\ j,\ l=1,\ldots\ldots,n. \end{cases} \tag{4.11}$$

However, if M is of complex 2-dimensional, then the condition (*) is equivalent to $(4.11)_3$, $(4.11)_4$, $(4.11)_5$ and $(4.11)_6$.
Thus, from $(4.11)_3$ and $(4.11)_6$, we have

$$\lambda_j^2(\lambda_i^2-\tilde{c}/4)=0 \qquad \text{for } i\neq j. \tag{4.12}$$

and moreover, from $(4.11)_4$ and $(4.11)_6$, we have

$$\lambda_i^2(\lambda_j^2-\tilde{c}/4)=0 \qquad \text{for } i\neq j. \tag{4.13}$$

Thus, we have the following

THEOREM 4.1. *Let M be a complex hypersurface satisfying the condition (*) in a space $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$ of complex dimension $n+1$.*

Then, the following statesments are valid. Where $n\geqq 2$.

(i) If $\tilde{c}>0$, then $k(x)=0$, or $2n$ at each point $x\in M$, that is, M is totally geodesic in $\tilde{M}$, or an Einstein space of Ricci curvature $\rho=n\tilde{c}/2$.
Hence, M is a locally symmetric space.

(ii) If $\tilde{c}<0$, then $k(x)=0$ at each point $x\in M$, that is, M is totally geodesic in $\tilde{M}$, hence also is a locally symmetric space.

(iii) If $\tilde{c}=0$, then $k(x)=0$, or 2 at each point $x\in M$.

PROOF. (i) From (4.12), we see that $k(x)$ is constant on M. If $k(x)\neq 0$, 2n, then, there exist zero characteristic root and nonzero characteristic root of A. Now, let λ_j be a zero characteristic root and λ_i be a nonzero one.

Then, from (4.12), we get $\lambda_i^2=\tilde{c}/4$.
However, then from (4.13), we have

$$\lambda_i^2(\lambda_j^2-\tilde{c}/4)=-(\tilde{c}/4)^2\neq 0,$$

This is a contradiction.

Thus, we see that $k(x)=0$, or $2n$ at each point $x \in M$. If $k(x)=2n$, then, from (3.4.), we have

$$S(X, Y)=-\bar{c}/2g(X, Y)+(n+1)\bar{c}/2g(X, Y)=n\bar{c}/2g(X, Y).$$

That is, $S(X, Y)=n\bar{c}/2g(X, Y)$, for all tangent vectors X and Y to M.
Therefore, M is an Einstein space of Ricci curvature $\rho=n\bar{c}/2$.
(ii) and (iii) are evident.
On the other hand, B. Smyth [4]., has proved the following theorem.

THEOREM 4.2. *If* $n \geqq 2$, *then*
(i) $P^n(C)$ *and the complex quadric* Q^n *are the only complex hypersurfaces of* $P^{n+1}(C)$ *which are complete and Einstein,*
(ii) D^n(*resp.* C^n) *is the only simply-connected complex hypersurface of* D^{n+1} (*resp.* C^{n+1}) *which is complete and Einstein.*

Thus, from Theorem 4.1. and Theorem 4.2., we have the following

THEOREM 4.3. *If* $n \geqq 2$, *then*
(i) *let* M *be a complete complex hypersurface of* $P^{n+1}(C)$ *which satisfies the condition* (*), *then* M *is* $P^n(C)$, *or* Q^n.
(ii) *let* M *be a simply-connected complete complex hypersurface of* D^{n+1} *which satisfies the condition* (*), *then* M *is* D^n.

Remark. If $c \neq 0$ *and* $n \geqq 2$, *then we can show that the condition* (*) *is equivalent to the condition,* $R(X, Y)\cdot S=0$.

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