Almost contact hypersurfaces in almost Hermitian manifolds

By

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1. Preliminaries

When, in a 2n-dimensional real differentiable manifold M^{2n} with local coordinates $\{x^\lambda\}$, there is given a mixed tensor field $F_\mu{}^\lambda$ satisfying $F_\mu{}^\nu F_\nu{}^\lambda = -\delta_\mu^\lambda$, we say that the manifold admits an almost complex structure $F_\mu{}^\lambda$ and we call such a manifold an almost complex manifold. Throughout the present paper the Greek indices take the values 1, 2, ……, 2n. If an almost complex manifold has a positive definite Riemannian metric tensor $G_{\mu\lambda}$ satisfying $F_\mu{}^\kappa F_\lambda{}^\nu G_{\kappa\nu} = G_{\mu\lambda}$, then the manifold is called an almost Hermitian manifold. In this case it is easily seen that $F_{\mu\lambda} = -F_{\lambda\mu}$, where $F_{\mu\lambda} = F_\mu{}^\nu G_{\nu\lambda}$.

Next, we shall give the definitions of various almost Hermitian manifolds [2]. If, in an almost Hermitian manifold, its structure tensor $F_\mu{}^\lambda$ satisfies

(1.1) $\nabla_\mu F_\lambda{}^\mu = 0,$

(1.2) $\nabla_\beta F_{\alpha\lambda} = -F_\beta{}^\nu F_\alpha{}^\mu \nabla_\nu F_{\mu\lambda}$ (i. e. $\nabla_\nu F_{\mu\lambda}$ is pure in ν and μ),

(1.3) $\nabla_\nu F_{\mu\lambda} + \nabla_\mu F_{\lambda\nu} + \nabla_\lambda F_{\nu\mu} = 0,$

(1.4) $\nabla_\nu F_{\mu\lambda} + \nabla_\mu F_{\nu\lambda} = 0,$

then the manifold is called an almost semi-Kählerian manifold, an *O-manifold, an almost Kählerian manifold (an H-manifold) or an almost Tachibana manifold (a K-manifold) respectively.

If the Nijenhuis tensor $N_{\nu\mu}{}^\lambda$ defined by

$$N_{\nu\mu}{}^\lambda = F_\nu{}^\sigma(\nabla_\sigma F_\mu{}^\lambda - \nabla_\mu F_\sigma{}^\lambda) - F_\mu{}^\sigma(\nabla_\sigma F_\nu{}^\lambda - \nabla_\nu F_\sigma{}^\lambda)$$

vanishes identically, the almost Hermitian manifold, the almost semi-Kählerian manifold and the *O-manifold are called an Hermitian manifold, a semi-Kählerian manifold and a Kählerian manifold respectively [6]. A necessary and sufficient condition for an almost Hermitian manifold to be a Kählerian manifold is given by

(1.5) $\nabla_\nu F_\mu{}^\lambda = 0.$

And it is well-known that in an *O-manifold the two conditions $\nabla_\mu F_\nu{}^\lambda = 0$ and

$N_{\mu\nu}{}^{\lambda}=0$ are equivalent to each other [2].

Next, in a (2n−1)-dimensional real differentiable manifold M^{2n-1} with local coordinates $\{x^i\}$, if there exist a mixed tensor field $\varphi_j{}^i$, a contravariant vector field ξ^i and a covariant vector field η_j satisfying the conditions $\xi^i\eta_i=1$, $\varphi_j{}^i\varphi_k{}^j=-\delta_k^i+\xi^i\eta_k$, then such a manifold is said to have an almost contact structure $(\varphi_j{}^i, \xi^i, \eta_j)$ and we call the manifold an almost contact manifold. Throughout the present paper the Latin indices run over the 1, 2,......, 2n−1. It is well-known that in a manifold with an almost contact structure $(\varphi_j{}^i, \xi^i, \eta_j)$, there exists a positive definite Riemannian metric g_{ji}, which is called a Riemannian metric associated with the almost contact structure, such that $\eta_i=g_{ij}\xi^j$, $g_{ji}\varphi_h{}^j\varphi_k{}^i=g_{hk}-\eta_h\eta_k$. We call the set $(\varphi_j{}^i, \xi^i, \eta_j, g_{ji})$ an almost contact metric structure and a manifold with an almost contact metric struture is called an almost contact metric (or Riemannian) manifold.

In a (2n−1)-dimensional differentiable manifold with an almost contact structure $(\varphi_j{}^i, \xi^i, \eta_j)$, the following relations hold true;

$$\begin{aligned} &\xi^i\eta_i=1, \quad \varphi_j{}^i\varphi_k{}^j=-\delta_k^i+\xi^i\eta_k, \\ &\varphi_j{}^i\xi^j=0, \quad \varphi_j{}^i\eta_i=0, \\ &\operatorname{rank}(\varphi_j{}^i)=2n-2. \end{aligned} \tag{1.6}$$

Furthermore, if this manifold has an associated metric and φ_{ji} is defined as $\varphi_j{}^h g_{hi}$, then in addition to the above relations the following are satisfied

$$\begin{aligned} &\varphi_{ji}=-\varphi_{ij}, \quad \operatorname{rank}(\varphi_{ji})=2n-2, \\ &\eta_i=g_{ij}\xi^j, \quad g_{ji}\varphi_h{}^j\varphi_k{}^i=g_{hk}-\eta_h\eta_k. \end{aligned} \tag{1.7}$$

If, in a (2n−1)-dimensional differentiable manifold M^{2n-1}, there exists a differentiable 1-form η such that $\eta\wedge(d\eta)^{n-1}\neq0$ everywhere, then such a manifold is called to have a contact structure η and we call the manifold a contact manifold [4]. It is well-known that in any contact manifold with a contact structure η there exists always an almost contact metric structure $(\varphi_j{}^i, \xi^i, \eta_j, g_{ji})$ such that $\varphi_j{}^h g_{hi}=\varphi_{ji}\equiv\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j)$ where, in terms of a local coordinate system x^i, η is expressed as $\eta=\eta_i dx^i$ and ∂_i denotes $\partial/\partial x^i$. Such an almost contact (metric) structure is simply called a contact (metric) structure.

If a (2n−1)-dimensional differentiable manifold has a contact metric structure $(\varphi_j{}^i, \xi^i, \eta_j, g_{ji})$ in the above sense, then the following relations hold true.

$$\begin{aligned} &\varphi_j{}^h g_{hi}=\varphi_{ji}\equiv\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j), \\ &\nabla_k\varphi_{ji}+\nabla_j\varphi_{ik}+\nabla_i\varphi_{kj}=0, \\ &\xi^j\nabla_j\eta_i=0, \quad \varphi^{ji}\nabla_j\varphi_{ih}=0, \quad \nabla_j\xi^j=0, \\ &\nabla_i\varphi_j{}^i=(2n-2)\eta_j, \end{aligned} \tag{1.8}$$

where ∇_j denotes the covariant differentiation with respect to the Riemannian connection.

Next, in a (an almost) contact manifold, the following four tensors are fundamental and called Nijenhuis tensors of the (almost) contact structure [4].

$$N_j = \eta^r(\nabla_r \eta_j - \nabla_j \eta_r),$$

$$N_{ji} = \varphi_j{}^r(\nabla_r \eta_i - \nabla_i \eta_r) - \varphi_i{}^r(\nabla_r \eta_j - \nabla_j \eta_r),$$

$$(1.9) \qquad N_j{}^i = \eta^r(\nabla_r \varphi_j{}^i - \nabla_j \varphi_r{}^i) - \varphi_j{}^r \nabla_r \eta^i,$$

$$N_{ji}{}^h = \varphi_j{}^r(\nabla_r \varphi_i{}^h - \nabla_i \varphi_r{}^h) - \varphi_i{}^r(\nabla_r \varphi_j{}^h - \nabla_j \varphi_r{}^h) + \eta_j \nabla_i \eta^h - \eta_i \nabla_j \eta^h,$$

where we use a notation η^i instead of ξ^i because of dealing with metric manifolds only in this paper.

The following theorem is well-known [4].

THEOREM. *If any one of N_{jk} and $N_j{}^i$ vanishes, then N_j vanishes. If $N_{jk}{}^i$ vanishes, then all the other tensors N_j, N_{jk} and $N_j{}^i$ vanish.*

The (almost) contact structure whose torsion tensor $N_{jk}{}^i$ vanishes identically is called a normal (almost) contact structure and the manifold with such a structure as a normal (almost) contact manifold. The contact metric structure whose tensor $N_j{}^i$ vanishes identically is called to be a K-contact metric structure. In this paper, we shall call the almost contact metric structure whose tensor $N_j{}^i$ vanishes identically as a K-almost contact metric structure, too.

2. Almost contact metric hypersurfaces in an almost Hermitian manifold

Let M^{2n} be a 2n-dimensional almost Hermitian manifold $(F_\mu{}^\nu, G_{\mu\nu})$ with local coordinate system X^λ. We now consider a (2n−1)-dimensional orientable submanifold M^{2n-1} differentiably immersed in M^{2n}. Let the submanifold M^{2n-1} be expressed by the equation $X^\lambda = X^\lambda(x^i)$, where x^i is a system of local coordinates in M^{2n-1}. If we put $B_j{}^\lambda = \partial_j X^\lambda (\partial_j = \partial/\partial x^j)$, then they are tangent to M^{2n-1} and linearly independent at each point of M^{2n-1}. And the induced Riemannian metric g_{ji} in M^{2n-1} is given by

$$(2.1) \qquad g_{ji} = B_j{}^\lambda G_{\lambda\mu} B_i{}^\mu.$$

Because of assuming the hypersurface to be orientable, we choose the unit normal vector C^κ to the hypersurface and put

$$(2.2) \qquad \varphi_j{}^i = B_j{}^\lambda F_\lambda{}^\mu B^i{}_\mu,$$

$$(2.3) \qquad \eta_j = B_j{}^\lambda F_\lambda{}^\mu C_\mu = B_j{}^\lambda F_{\lambda\mu} C^\mu,$$

where we have put $B^j{}_\lambda=G_{\lambda\mu}B_i{}^\mu g^{ij}$, $C_\lambda=C^\kappa G_{\kappa\lambda}$ and $F_{\lambda\mu}=F_\lambda{}^\kappa G_{\kappa\mu}$.
Then we have the relations

$$B_j{}^\lambda B^i{}_\lambda=\delta_j{}^i,\ B_j{}^\lambda C_\lambda=0,\ B^i{}_\lambda C^\lambda=0,$$

$$C_\lambda C^\lambda=1 \text{ and } B_j{}^\mu B^{j\lambda}+C^\mu C^\lambda=G^{\mu\lambda}. \tag{2.4}$$

And further it is easily verified that the following conditions hold true:

$\eta^i\eta_i=1$, $\varphi_j{}^i\eta^j=0$, $\varphi_j{}^i\eta_i=0$, $\varphi_j{}^i\varphi_k{}^j=-\delta_k{}^i+\eta_k\eta^i$, $g_{ji}\eta^i=\eta_j$ and rank $(\varphi_j{}^i)=2(n-1)$, where we have put $\eta^i=g^{ij}\eta_j$. Therefore the set $(\varphi_j{}^i, \eta^i, \eta_j, g_{ji})$ defines an almost contact metric structure in the hypersurface M^{2n-1} [5].

In the following we shall call an orientable hypersurface with the induced almost contact structure an almost contact hypersurface. We shall first recall the following formulas [1]

$$\nabla_j B_i{}^\lambda=H_{ji}C^\lambda \quad \text{(formula of Gauss)}, \tag{2.5}$$

$$\nabla_j C_\lambda=-H_{ji}B^i{}_\lambda \quad \text{(formula of Weingarten)}, \tag{2.6}$$

where H_{ji} is the second fundamental tensor of the hypersurface, and the left hand sides of these equations are so-called Bortolotti-van der Waerden covariant derivatives.

By making use of the formulas of Gauss and Weingarten, we have

$$\nabla_j\eta_i=-\varphi_i{}^r H_{rj}+B_j{}^\nu B_i{}^\mu C^\lambda\nabla_\nu F_{\mu\lambda}, \tag{2.7}$$

$$\nabla_j\varphi_{ih}=\eta_i H_{jh}-\eta_h H_{ji}+B_j{}^\nu B_i{}^\mu B_h{}^\lambda\nabla_\nu F_{\mu\lambda}. \tag{2.8}$$

Now, we introduce the following four tensor fields in an almost contact hypersurface:

$$\begin{aligned}\theta_{jih}&=B_j{}^\nu B_i{}^\mu B_h{}^\lambda\nabla_\nu F_{\mu\lambda},\\ \theta_{ji}&=C^\nu B_j{}^\mu B_i{}^\lambda\nabla_\nu F_{\mu\lambda},\\ \theta'_{ji}&=B_j{}^\nu B_i{}^\mu C^\lambda\nabla_\nu F_{\mu\lambda},\\ \theta_j&=C^\nu C^\mu B_j{}^\lambda\nabla_\nu F_{\mu\lambda},\end{aligned} \tag{2.9}$$

By virtue of the skew-symmetry of $F_{\mu\lambda}$ we have

$$\theta_{jih}+\theta_{jhi}=0, \tag{2.10}$$

$$\theta_{ji}+\theta_{ij}=0. \tag{2.11}$$

And the equations (2.7) and (2.8) are rewritten as

$$\nabla_j\eta_i=-\varphi_i{}^r H_{rj}+\theta'_{ji}, \tag{2.12}$$

$$\nabla_j\varphi_{ih}=\eta_i H_{jh}-\eta_h H_{ji}+\theta_{jih}. \tag{2.13}$$

Since η^i is a unit vector, multiplying η^i to (2.12) and contracting, we have $0=\eta^i\nabla_j\eta_i=\eta^i(-\varphi_i{}^rH_{rj}+\Theta'_{ji})$ $\quad\therefore\quad \eta^i\Theta'_{ji}=0.$

Again multiplying $\varphi_h{}^i$ to (2.12) and contracting we have

$$\varphi_h{}^i\nabla_j\eta_i=\varphi_h{}^i(-\varphi_i{}^rH_{rj}+\Theta'_{ji})=(\delta_h{}^r-\eta_h\eta^r)H_{rj}+\varphi_h{}^i\Theta'_{ji}$$

$$=H_{hj}-\eta_h\eta^rH_{rj}+\varphi_h{}^i\Theta'_{ji}.$$

On the other hand, by using (2.13) we have

$$\varphi_h{}^i\nabla_j\eta_i=-\eta_i\nabla_j\varphi_h{}^i=\eta^i\nabla_j\varphi_{ih}=\eta^i(\eta_iH_{jh}-\eta_hH_{ji}+\Theta_{jih})$$

$$=H_{jh}-\eta_h\eta^iH_{ji}+\eta^i\Theta_{jih}.$$

Therefore, since we get $\eta^r\Theta'_{jr}=0$ and $\varphi_h{}^r\Theta'_{jr}=\eta^r\Theta_{jrh}$, it follows that

(2.14) $$\Theta'_{ji}=\Theta_{jba}\varphi_i{}^b\eta^a.$$

Next, multiplying φ^{ih} to (2.13) and contracting we have

$$\varphi^{ih}\nabla_j\varphi_{ih}=\varphi^{ih}(\eta_iH_{jh}-\eta_hH_{ji}+\Theta_{jih})=\varphi^{ih}\Theta_{jih}.$$

On the other hand, since $\varphi^{ih}\varphi_{ih}=2(n-1)$, we have $\varphi^{ih}\nabla_j\varphi_{ih}=0$.

Therefore we have

(2.15) $$\Theta_{jba}\varphi^{ba}=0.$$

Lastly, with respect to the frame $(B_j{}^\lambda, C^\lambda)$, the components of the tensor $\nabla_\nu F_{\mu\lambda}$ are expressible as follows [1];

$$\nabla_\nu F_{\mu\lambda}=T_{jih}B^j{}_\nu B^i{}_\mu B^h{}_\lambda+T^{(1)}_{ji}B^j{}_\nu B^i{}_\mu C_\lambda+T^{(2)}_{jh}B^j{}_\nu C_\mu B^h{}_\lambda+T^{(3)}_{ih}C_\nu B^i{}_\mu B^h{}_\lambda$$

$$+T^{(1)}_hC_\nu C_\mu B^h{}_\lambda+T^{(2)}_iC_\nu B^i{}_\mu C_\lambda+T^{(3)}_jB^j{}_\nu C_\mu C_\lambda+TC_\nu C_\mu C_\lambda.$$

From this equation it is easily to be seen that

$$T_{jih}=\Theta_{jih},\ T^{(1)}_{ji}=-T^{(2)}_{ji}=\Theta'_{ji},\ T^{(3)}_{ji}=\Theta_{ji},$$

$$T^{(1)}_j=-T^{(2)}_j=\Theta_j\quad\text{and}\quad T^{(3)}_j=0=T.$$

Thus we have

(2.16) $$\nabla_\nu F_{\mu\lambda}=\Theta_{jih}B^j{}_\nu B^i{}_\mu B^h{}_\lambda+\Theta'_{ji}B^j{}_\nu(B^i{}_\mu C_\lambda-C_\mu B^i{}_\lambda)$$

$$+\Theta_{ji}C_\nu B^j{}_\mu B^i{}_\lambda+\Theta_jC_\nu(C_\mu B^j{}_\lambda-B^j{}_\mu C_\lambda).$$

Therefore we have the

THEOREM 2.1. *For an almost contact hypersurface in an almost Hermitian manifold, the relations*

(2.17) $$\Theta'_{ji}=\Theta_{jba}\varphi_i{}^b\eta^a,\quad \Theta_{jih}+\Theta_{jhi}=0,$$

$$\Theta_{ji}+\Theta_{ij}=0, \quad \Theta_{jba}\varphi^{ba}=0,$$

hold good.

THEOREM 2.2. *In order that an almost Hermitian manifold be a Kählerian manifold, it is necessary and sufficient that*

$$\Theta_{jih}=0, \quad \Theta_{ji}=0, \quad \Theta_j=0 \tag{2.18}$$

hold true in its every almost contact hypersurface.

Case I. Almost semi-Kählerian manifolds.

In an almost semi-Kählerian manifold, by definition (1.1) we have $\nabla_\nu F_\mu{}^\nu=0$. Transvecting (2.16) with $G^{\nu\lambda}$, we have

$$G^{\nu\lambda}\nabla_\nu F_{\mu\lambda}=(\Theta_{bja}g^{ba}-\Theta_j)B^j{}_\mu-\Theta'_{ba}g^{ba}C_\mu.$$

Therefore, in an almost semi-Kählerian manifold, by definition (1.1), we have $\Theta_j=-\Theta_{baj}g^{ba}$ and $\Theta'_{ba}g^{ba}=0$ which reduces to $\Theta_{cba}\varphi^{cb}\eta^a=0$.
Thus we have the

THEOREM 2.3. *For an almost contact hypersurface in an almost semi-Kählerian manifold, the relations*

$$\begin{aligned}&\Theta'_{ji}=\Theta_{jba}\varphi_i{}^b\eta^a, \quad \Theta_j=-\Theta_{baj}g^{ba}, \quad \Theta_{jih}+\Theta_{jhi}=0,\\ &\Theta_{ji}+\Theta_{ij}=0, \quad \Theta_{jba}\varphi^{ba}=0, \quad \Theta'_{ba}g^{ba}=0 \ (\text{or } \Theta_{cba}\varphi^{cb}\eta^a=0)\end{aligned} \tag{2.19}$$

hold good.

THEOREM 2.4. *In order that an almost semi-Kählerian manifold, it is necessary and sufficient that*

$$\Theta_{jih}=0, \quad \Theta_{ji}=0 \tag{2.20}$$

hold true in its every almost contact hypersurface.

THEOREM 2.5. *In order that an almost Hermitian manifold be almost semi-Kählerian, it is necessary and sufficient that*

$$\Theta_j=-\Theta_{baj}g^{ba}, \quad \Theta'_{ba}g^{ba}=0 \ (\text{or } \Theta_{cba}\varphi^{cb}\eta^a=0) \tag{2.21}$$

hold true in its every almost contact hypersuface.

Case II. *O-manifolds.

In an *O-manifold, by definition (1.2), we have $\nabla_\beta F_{\alpha\lambda}+F_\beta{}^\nu F_\alpha{}^\mu\nabla_\nu F_{\mu\lambda}=0$. Substituting (2.16) into this equation, we have

(1) $\Theta_{bah}+\varphi_b{}^j\varphi_a{}^i\Theta_{jih}-\varphi_b{}^j\eta_a\Theta'_{jh}+\eta_b\varphi_a{}^j\Theta_{jh}+\eta_b\eta_a\Theta_h=0,$

(2) $\Theta'_{ba}+\varphi_b{}^j\varphi_a{}^i\Theta'_{ji}-\eta_b\varphi_a{}^j\Theta_j=0,$

(3) $\theta'_{bh}+\varphi_b{}^j\eta^i\theta_{jih}+\eta^j\eta_b\theta'_{jh}=0$,

(4) $\theta_{ah}-\eta^j\varphi_a{}^i\theta_{jih}+\eta^j\eta_a\theta'_{jh}=0$,

(5) $\theta_h+\eta^j\eta^i\theta_{jih}=0$,

(6) $\theta_a+\eta^j\varphi_a{}^i\theta'_{ji}=0$,

(7) $\eta_b\eta^j\theta_j-\varphi_b{}^j\eta^i\theta'_{ji}=0$,

(8) $\eta^j\eta^i\theta'_{ji}=0$,

where the left hand sides of these equations are coefficients of $B^b{}_\beta B^a{}_\alpha B^h{}_\lambda$, $B^b{}_\beta B^a{}_\alpha C_\lambda$, $-B^b{}_\beta C_\alpha B^h{}_\lambda$, $C_\beta B^a{}_\alpha B^h{}_\lambda$, $C_\beta C_\alpha B^h{}_\lambda$, $-C_\beta B^a{}_\alpha C_\lambda$, $B^b{}_\beta B_\alpha C_\lambda$ and $C_\beta C_\alpha C_\lambda$ in $\nabla_\beta F_{\alpha\lambda}+F_\beta{}^\nu F_\alpha{}^\mu \nabla_\nu F_{\mu\lambda}$ respectively.

Transvecting (1) with g^{ba} and using (5), we have $\theta_j=-\theta_{baj}g^{ba}$.

Transvecting (2) with g^{ba} and using $\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a$ in (2.17), we have $\theta'_{ba}g^{ba}=0$. Thus an *O-manifold is necessarily almost semi-Kählerian. (Of course the fact is already known [2].)

The equation (8) is satisfied by $\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a$ in (2.17). The equation (7) is satisfied by (6) and $\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a$ in (2.17). The equation (6) is satisfied by $\theta_{jih}+\theta_{jhi}=0$ and $\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a$ in (2.17).

Multiplying (1) by $\eta^a\varphi_l{}^h$ and contracting, we get

$$\theta_{bah}\eta^a\varphi_l{}^h-\varphi_b{}^j\varphi_l{}^h\theta'_{jh}+\eta_b\varphi_l{}^h\theta_h=0.$$

And making use of $\theta_{jih}+\theta_{jhi}=0$ and $\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a$ in (2.17), we have the equation (2).

Substituting $\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a$ into the equation (2) and using $\theta_{jih}+\theta_{jhi}=0$, we have $\theta'_{ba}+\varphi_b{}^j\eta^h\theta_{jha}-\eta_b\varphi_a{}^j\theta_j=0$.

On the other hand transvecting (4) with η^a and using $\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a$ and the equation (5), we have $\eta^a\theta_{ah}=-\varphi_h{}^b\theta_b$.

Therefore we have $\theta'_{ba}+\varphi_b{}^j\eta^h\theta_{jha}+\eta_b\eta^j\theta_{ja}=0$, that is, the equation (3).

Accordingly the system of equations (1)~(8) are equivalent to

(1) $\theta_{bah}+\varphi_b{}^j\varphi_a{}^i\theta_{jih}-\varphi_b{}^j\eta_a\theta'_{jh}+\eta_b\varphi_a{}^j\theta_{jh}+\eta_b\eta_a\theta_h=0$,

(4) $\theta_{ah}-\eta^j\varphi_a{}^i\theta_{jih}+\eta^j\eta_a\theta'_{jh}=0$,

(5) $\theta_h+\eta^j\eta^i\theta_{jih}=0$.

Moreover, by using (4) we have $\eta_b\varphi_a{}^j\theta_{jh}+\eta_b\eta_a\theta_h=-\eta_b\eta^j\theta_{jah}$.

Thus we have

THEOREM 2.6. *For an almost contact hypersurface in an *O-manifold, the relations*

$$\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a,\quad \theta_j=-\theta_{baj}g^{ba},$$

$$\theta_{ji}=\varphi_j{}^a\eta^b\theta_{bai}-\eta_j\eta^a\theta'_{ai},$$

$$\theta_{jih}+\theta_{jhi}=0,\quad \theta_{ji}+\theta_{ij}=0,\quad \theta_{jba}\varphi^{ba}=0, \tag{2.22}$$

$$\Theta'_{ba} g^{ba} = 0 \text{ (or } \Theta_{cba} \varphi^{cb} \eta^a = 0),$$

$$\Theta_{baj} g^{ba} = \Theta_{baj} \eta^b \eta^a,$$

$$\Theta_{jih} + \varphi_j{}^b \varphi_i{}^a \Theta_{bah} = \varphi_j{}^a \eta_i \Theta'_{ah} + \eta_j \eta^a \Theta_{aih}$$

hold good.

THEOREM 2.7. *In order that an O^*-manifold be a Kählerian manifold, it is necessary and sufficient that*

$$\Theta_{jih} = 0 \tag{2.23}$$

hold true in its every almost contact hypersurface.

THEOREM 2.8. *In order that an almost Hermitian manifold be an $^*O$-manifold, it is necessary and sufficient that*

$$\begin{aligned} &\Theta_j = -\Theta_{baj} g^{ba}, \\ &\Theta_{ji} = \varphi_j{}^a \eta^b \Theta_{bai} - \eta_j \eta^a \Theta'_{ai}, \\ &\Theta'_{ba} g^{ba} = 0, \\ &\Theta_{baj} g^{ba} = \Theta_{baj} \eta^b \eta^a, \\ &\Theta_{jih} + \varphi_j{}^b \varphi_i{}^a \Theta_{bah} = \varphi_j{}^a \eta_i \Theta'_{ah} + \eta_j \eta^a \Theta_{aih} \end{aligned} \tag{2.24}$$

hold true in its every almost contact hypersurface.

THEOREM 2.9. *In order that an almost semi-Kählerian manifold be an $^*O$-manifold, it is necessary and sufficient that*

$$\begin{aligned} &\Theta_{ji} = \varphi_j{}^a \eta^b \Theta_{bai} - \eta_j \eta^a \Theta'_{ai}, \\ &\Theta_{baj} g^{ba} = \Theta_{baj} \eta^b \eta^a, \\ &\Theta_{jih} + \varphi_j{}^b \varphi_i{}^a \Theta_{bah} = \varphi_j{}^a \eta_i \Theta'_{ah} + \eta_j \eta^a \Theta_{aih} \end{aligned} \tag{2.25}$$

hold true in its every almost contact hypersurface.

Case III. Almost Tachibana manifolds.

By definition (1.4), in an almost Tachibana manifold we have $\nabla_\nu F_{\mu\lambda} + \nabla_\mu F_{\nu\lambda} = 0$. Substituting (2.16) into this equation, we have

$$\Theta_{jih} + \Theta_{ijh} = 0, \quad \Theta'_{ji} + \Theta'_{ij} = 0, \quad \Theta_{ji} = \Theta'_{ji}, \quad \Theta_j = 0.$$

Thus we have the

THEOREM 2.10. *In order that an almost Hermitian manifold be an almost Tachibana manifold, it is necessary and sufficient that*

(2.26) Θ_{jih} *is a skew-symmetric tensor,*

$$\theta_{ji}=\theta'_{ji}, \quad \theta_j=0$$

hold true in its every almost contact hypersurface.

THEOREM 2.11. *In order that an almost semi-Kählerian or an *O–manifold be an almost Tachibana manifold, it is necessary and sufficient that*

(2.27) θ_{jih} *is a skew-symmetric tensor,*

$$\theta_{ji}=\theta'_{ji}$$

hold true in its every almost contact hypersurface.

For an almost contact hypersurface, in an almost Tachibana manifold, we get

$$\begin{aligned}\varphi_j{}^a\eta_i\theta'_{ah}+\eta_j\eta^a\theta_{aih} &= \varphi_j{}^a\eta_i\theta_{ah}+\eta_j\theta_{iha}\eta^a\\ &=\eta_i\varphi_j{}^a\varphi_a{}^b\eta^l\theta_{lbh}+\eta_j\theta_{iha}\eta^a\\ &=(\eta_j\theta_{iha}-\eta_i\theta_{jha})\eta^a.\end{aligned}$$

Therefore we have the

THEOREM 2.12. *For an almost contact hypersuface in an almost Tachibana manifold, the relations*

$$\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a, \quad \theta_j=0, \quad \theta_{ji}=\varphi_j{}^a\eta^b\theta_{bai},$$

$$\varphi_j{}^a\theta_{aib}\eta^b=\varphi_i{}^a\theta_{jab}\eta^b \text{ (or } \theta_{ji}=\theta'_{ji}),$$

(2.28) θ_{jih} *is a skew-symmetric tensor,*

$$\theta_{ji}+\theta_{ij}=0, \quad \theta_{jba}\varphi^{ba}=0,$$

$$\theta_{jih}+\varphi_j{}^b\varphi_i{}^a\theta_{bah}=(\eta_j\theta_{iha}-\eta_i\theta_{jha})\eta^a$$

hold good.

Case IV. Almost Kählerian manifolds.

By definition (1.3), in an almost Kählerian manifold we have $\nabla_\nu F_{\mu\lambda}+\nabla_\mu F_{\lambda\nu}+\nabla_\lambda F_{\nu\mu}=0$. Substituting (2.16) into this equation, we have easily $\theta_{jih}+\theta_{ihj}+\theta_{hji}=0$, $\theta_{ji}=\theta'_{ij}-\theta'_{ji}$.

Thus we have

THEOREM 2.13. *In order that an almost Hermitian manifold be an almost Kählerian manifold, it is necessary and sufficient that*

(2.29) $$\theta_{jih}+\theta_{ihj}+\theta_{hji}=0, \quad \theta_{ji}=\theta'_{ij}-\theta'_{ji}$$

hold true in its every almost contact hypersurface.

THEOREM 2.14. *For an almost contact hypersurface in an almost Kählerian mani-*

fold, the relations

$$\theta'_{ji}=\theta_{jba}\varphi_i{}^b\eta^a,$$

$$\theta_j=-\theta_{baj}g^{ba}=-\theta_{baj}\eta^b\eta^a,$$

$$\theta_{ji}=\varphi_j{}^a\eta^b\theta_{bai}-\eta_j\eta^a\theta'_{ai},$$

(2.30)
$$\theta_{jih}+\theta_{jhi}=0,\quad \theta_{jih}+\theta_{ihj}+\theta_{hji}=0,$$

$$\theta_{ji}+\theta_{ij}=0,\quad \theta_{ji}=\theta'_{ij}-\theta'_{ji},$$

$$\theta_{jba}\varphi^{ba}=0,\quad \theta_{bah}\varphi^{ba}=0,$$

$$\theta_{jih}+\varphi_j{}^b\varphi_i{}^a\theta_{bah}=\varphi_j{}^a\eta_i\theta'_{ah}+\eta_j\eta^a\theta_{aih}$$

hold good.

3. Hypersurfaces in an *O-manifold

In order to express the Nijenhuis tensors of the (almost) contact structure in terms of θ_{jih}, θ_{ji}, θ'_{ji} and θ_j, substituting (2.12) and (2.13) into the right hand sides of the equations (1.9), we get the following expressions for an almost contact hypersurface in an almost Hermitian manifold.

$$N_j=-\varphi_j{}^b H_{ba}\eta^a+\eta^a\theta'_{aj},$$

$$N_{ji}=\eta_j H_{ia}\eta^a-\eta_i H_{ja}\eta^a+\varphi_j{}^a(\theta'_{ai}-\theta'_{ia})-\varphi_i{}^a(\theta'_{aj}-\theta'_{ja}),$$

(3.1)
$$N_j{}^l g_{li}=\eta_j H_{ia}\eta^a-H_{ji}+\varphi_j{}^b\varphi_i{}^a H_{ba}+\eta^a\theta_{aji}+\eta^a\theta_{jia}-\varphi_j{}^a\theta'_{ai},$$

$$N_{ji}{}^l g_{lh}=\eta_i(\varphi_j{}^a H_{ah}+\varphi_h{}^a H_{aj})-\eta_j(\varphi_i{}^a H_{ah}+\varphi_h{}^a H_{ai})$$
$$+\varphi_i{}^a(\theta_{jah}-\theta_{ajh})+\varphi_j{}^a(\theta_{aih}-\theta_{iah})-\eta_i\theta'_{jh}+\eta_j\theta'_{ih}.$$

Let us consider an almost contact hypersurface M^{2n-1} in an almost Tachibana manifold M^{2n}. In this case, by using (2.28) we can see easily that the equations (3.1) reduce to

$$N_j=-\varphi_j{}^b H_{ba}\eta^a,$$

$$N_{ji}=\eta_j H_{ia}\eta^a-\eta_i H_{ja}\eta^a-4\eta^a\theta_{aji},$$

(3.2)
$$N_j{}^l g_{li}=\eta_j H_{ia}\eta^a-H_{ji}+\varphi_j{}^b\varphi_i{}^a H_{ba}+3\eta^a\theta_{aji},$$

$$N_{ji}{}^l g_{lh}=\eta_i(\varphi_j{}^a H_{ah}+\varphi_h{}^a H_{aj})-\eta_j(\varphi_i{}^a H_{ah}+\varphi_h{}^a H_{ai})$$
$$+2(\varphi_i{}^a\theta_{jah}+\varphi_j{}^a\theta_{aih})+\eta_j\theta'_{ih}-\eta_i\theta'_{jh}.$$

At first we shall prove the following

LEMMA 3.1. *In an almost contact hypersurface of an almost Tachibana manifold, the following conditions are equivalent each other:*

(3.3) $N_j=0$,

(3.4) $H_{ja}\eta^a=\alpha\eta_j$ (*i. e.* η^i *defines a principal direction of* H_{ji}), *where* α *is a scalar.*

PROOF. It follows from $N_j=-\varphi_j{}^b H_{ba}\eta^a$ and $\varphi_j{}^a N_a=H_{ja}\eta^a-(H_{ba}\eta^b\eta^a)\eta_j$.

LEMMA 3.2. *In an almost contact hypersurface of an almost Tachibana manifold, the tensor* N_{ji} *vanishes if and only if*

(3.5) $\eta^a\theta_{aji}=0,\quad N_j=0.$

PROOF. It follows from the fact that $N_j=0$ if $N_{ji}=0$.

LEMMA 3.3. *In an almost contact hypersurface of an almost Tachibana manifold, the following conditions are equivalent one another:*

(3.6) $\nabla_j\eta_i+\nabla_i\eta_j=0$ (*i. e.* η^i *is a Killing vector*),

(3.7) $\varphi_j{}^a H_{ai}+\varphi_i{}^a H_{ja}=0$,

(3.8) $H_{ji}-\varphi_j{}^b\varphi_i{}^a H_{ba}=\alpha\eta_j\eta_i$,

where α *is a scalar.*

PROOF. By $\theta'_{ji}+\theta'_{ij}=0$ in (2.28) and (2.12), we have

$$\nabla_j\eta_i+\nabla_i\eta_j=-(\varphi_i{}^r H_{rj}+\varphi_j{}^r H_{ri}),$$

which shows that the conditions (3.6) and (3.7) are equivalent.

Multiplying (3.7) by η^i, we have $\varphi_j{}^a H_{ai}\eta^i=0$ and so $H_{ja}\eta^a=\alpha\eta_j$. Multiplying (3.8) by η^i, we have $H_{ja}\eta^a=\alpha\eta_j$ too.

The equivalence between (3.7) and (3.8) is deduced from

$$\varphi_j{}^b(\varphi_i{}^a H_{ba}+\varphi_b{}^a H_{ai})=\varphi_j{}^b\varphi_i{}^a H_{ba}-H_{ji}+\alpha\eta_j\eta_i$$

and

$$\varphi_j{}^a(H_{ai}-\varphi_a{}^b\varphi_i{}^c H_{bc}-\alpha\eta_a\eta_i)=\varphi_j{}^a H_{ai}+\varphi_i{}^a H_{ja}.$$

LEMMA 3.4. *In an almost contact hypersurface of an almost Tachibana manifold, it is necessary and sufficient for the hypersurface to be K-almost contact* (i. e. $N_j{}^i=0$) *that the conditions*

(3.9) $\varphi_j{}^a H_{ai}+\varphi_i{}^a H_{ja}=0,\quad \eta^a\theta_{aji}=0,$

are satisfied. And if the tensor $N_j{}^i$ *vanishes, then the tensor* N_{ji} *vanishes.*

PROOF. It is evident that if (3.9) hold true, then $N_j{}^i=0$.

Conversely if $N_j{}^i=0$, then we have

$$\eta_j H_{ia}\eta^a-H_{ji}+\varphi_j{}^b\varphi_i{}^a H_{ba}=-3\eta^a\theta_{aji}.$$

However, noting that if $N_j{}^i=0$, then $N_j=0$, the left hand side of the above equation is symmetric in j and i, but the right hand side is shew-symmetric in j and i. Therefore the both sides vanish separately.

The latter part of the lemma follows from the lemma 3.2.

Thus we have the

THEOREM 3.1. *For a K-almost contact hypersurface in an almost Tachibana manifold, we have*

$$\begin{aligned} &\theta'_{ji}=0, \quad \theta_{ji}=0, \quad \theta_j=0, \\ &\theta_{jih} \text{ is a skew-symmetric tensor,} \\ (3.10) \qquad &\eta^a\theta_{aji}=0, \\ &\theta_{jih}+\varphi_j{}^b\varphi_i{}^a\theta_{bah}=0 \quad (\text{or } \varphi_j{}^a\theta_{aih}=\varphi_i{}^a\theta_{jah}), \\ &N_{ji}{}^l g_{lh}=4\varphi_j{}^a\theta_{aih}. \end{aligned}$$

THEOREM 3.2. *If every hypersurface in an almost Tachibana manifold admits a normal almost contact structure, then the almost Tachibana manifold reduces to a Kählerian manifold.*

PROOF. If every hypersurface in an almost Tachibana manifold admits a normal almost contact structure, then from (3.10) we have $\theta_{jih}=0$.

Next we shall prove the following

THEOREM 3.3. *If every hypersurface in an *O-manifold admits a contact structure, then the *O-manifold reduces necessarily to an almost Kählerian manifold.*

PROOF. Suppose that every hypersurface in an *O-manifold admits a contact structure. Then from (1.8) we have $\nabla_j\varphi_{ih}+\nabla_i\varphi_{hj}+\nabla_h\varphi_{ji}=0$.

Substituting (2.13) into this equation, we have $\theta_{jih}+\theta_{ihj}+\theta_{hji}=0$.

By using the relations (2.22) in Theorem 2.6, we calculate as follows:

$$\eta^j(\theta_{ji}+\theta'_{ji}-\theta'_{ij})=-\eta^a\theta'_{ai}+\eta^j\theta'_{ji}=0$$

and

$$\begin{aligned} &\varphi_l{}^j(\theta_{ji}+\theta'_{ji}-\theta'_{ij}) \\ &=(-\delta_l^a+\eta_l\eta^a)\eta^b\theta_{bai}+\varphi_l{}^j\theta_{jba}\varphi_i{}^b\eta^a-(-\delta_l^b+\eta_l\eta^b)\theta_{iba}\eta^a \\ &=-\eta^b\theta_{bli}+\eta_l\eta^b\eta^a\theta_{bai}+\varphi_l{}^j\varphi_i{}^b\theta_{jba}\eta^a+\theta_{ila}\eta^a \\ &=\eta^b\theta_{bil}-\eta_l\eta^b\eta^a\theta_{bia}+\varphi_l{}^j\varphi_i{}^b\theta_{jba}\eta^a+\theta_{ila}\eta^a \\ &=\eta^a(\theta_{ail}+\theta_{ila}+\theta_{lai})+\theta_{lia}\eta^a+\varphi_l{}^b\varphi_i{}^a\theta_{bac}\eta^c-\eta_l\eta^b\eta^a\theta_{bia} \\ &=\eta^a(\theta_{ail}+\theta_{ila}+\theta_{lai}) \\ &=0. \end{aligned}$$

Therefore we have $\theta_{ji}+\theta'_{ji}-\theta'_{ij}=0$.

Thus our theorem is proved by Theorem 2.13.

From this theorem it follows immediately the following theorem [3].

THEOREM 3.4. *If every hypersurface in an almost Tachibana manifold admits a contact structure, then the almost Tachibana manifold reduces necessarily to a Kählerian manifold.*

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