A note on the submersions of bundle spaces

By

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1. Introduction

Let ξ be a $(k+1)$-plane bundle over a connected smooth manifold M^n and $B(\xi)$ the total space of the associated sphere bundle of ξ. $B(\xi)$ may be considered as a differentiable manifold. In this note, we shall prove the following

THEOREM. *Let $B(\xi)$ be as above and $B(\xi)_0$ denote $B(\xi)-\{x\}$, where x is a point of $B(\xi)$. Then $B(\xi)_0$ can be submersed in R^k.*

This is dual in the sence of [1] to the result, which is easily proved (see [6]); *Let $B(\xi)$ be the total space of the sphere bundle associated to a $(k+1)$-plane bundle over M^n. Then $B(\xi)$ can be immersed in R^{2n+k}.*

As application of the theorem, we consider submersion of $B(\xi)_0$, where ξ is a plane bundle over sphere, or real projective space.

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2. Notations and preliminary lemmas

In what follows, the word "differentiable" will mean "of class C^∞." A differentiable map of M^n in R^p is called a submersion if its differential has maximal rank at each point of M (we suppose $n \geqq p$). We will write $M^n \subseteq R^p$ when M is submersed in R^p. A. Phillips has proved the following result in [1].

THEOREM 2.1. *If M^n is open (M has no compact component), then the gradient map ∇: Sub $(M^n, R^p) \longrightarrow$ Sect T_pM is a weak homotopy equivalence, where Sub (M^n, R^p) is the space of submersions of M in R^p with C^1-topology, T_pM is the bundle of p-frames tangent to M and Sect T_pM is the space of sections of the bundle with the compact open topology.*

∇ is defined as follows. If $f_1, \ldots\ldots, f_p$ are the p coordinates of f, let $\nabla f(x)$ be p-frames $\nabla f_1(x), \ldots\ldots, \nabla f_p(x)$. By the theorem, the problem of submersion is reduced to the problem of the existence of cross-section of tangent bundle of M.

Further we shall need the following lemmas.

LEMMA 2.2 *Let M^n be an n-manifold with $H^n(M; Z)=0$ and ξ_i k-plane bundles over M^n $(k\geqq n)$, $i=1,2$. Then ξ_1 is stably equivalent to ξ_2 if and only if ξ_1 is equivalent to ξ_2.*

PROOF. The if-part is trivial. We may assyme that $\xi_1\oplus\varepsilon^1$ is equivalent to $\xi_2\oplus\varepsilon^1$. We identify them and denote it by ξ^{k+1}. Let (ξ) be the associated sphere bundle of ξ and S_i $(i=1,2)$ its non-zero cross - sections. Define two bundle monoorphisms $u, v: \varepsilon^1\longrightarrow\xi$ by the formulas;

$$u(b, a)=aS_1(b) \text{ and } v(b, a)=aS_2(b) \text{ for } (b, a)\in E(\varepsilon^1).$$

A homotopy of monomorphisms is determined by a cross-section of $(\xi)\times I=(\xi\times I)$ over $M\times[0, 1]$, where $S|M\times 0$ corresponds to u and $S|M\times 1$ to v. Since $H^{n+1}(M\times I, M\times\{0,1\}; \pi_n(S^n))$ vanishes by assumption, we have a prolongation of S to $M\times I$ as a cross-section of $\xi\times I$. This cross-section S^* determines a monomorphism $w: \varepsilon^1\longrightarrow\xi\times I$. Since coker $w|M\times 0$ is isomorphic to coker u and coker $w|M\times 1$ is isomorphic to coker v, there is an isomorphism between coker u and coker v. Thus we have proved that ξ_1 is isomorphic to ξ_2.

LEMMA 2.3 *Let ξ be $(k+1)$-plane bundle over an n-manifold M^n and $B(\xi)$ the associated sphere bundle. Then we have $\tau(B(\xi))\oplus\varepsilon^1=\pi^*(\tau(M)\oplus\xi)$, where $\tau(M)$ denotes the tangent bundle of M and $\pi: B(\xi)\rightarrow M$ is the projection map.*

PROOF. Let $\xi=(E, P, M)$ be a plane bundle and $\hat{\xi}$ the bundle along the fibres. As is well known, we have $\tau(E)=P^*(\tau(M))\oplus\hat{\xi}$.
We can prove that the sequence;

$$0\longrightarrow P^*(\xi)\longrightarrow\tau(E)\longrightarrow P^*\tau(M)\longrightarrow 0 \tag{2.1}$$

is exact and hence we have $\hat{\xi}=P^*(\xi)$. For each point $x\in M$, we have an inclusion E_x(= the fibre of ξ at x)$\longrightarrow E$ and hence a natural inclusion $\tau(E_x)\longrightarrow\tau(E)$. It follows from definition that the total space of $P^*\xi$ consists of pairs of vectors (v, w) lying over the same base point, in other words, the fibre of x is $E_x\times E_x$. Since E_x is a euclidean space, $E_x\times E_x$ is naturally identified with $\tau(E)_x$. Hence we have a bijection $(P^*\xi)_x\longrightarrow\tau(E)_x$ for each x. It follows from this that $P^*\xi$ and $\hat{\xi}$ are equivalent. The exactness of the sequence (2.1) implies that $\tau(E)=P^*(\tau(M)\oplus\xi)$. Thus we have $\tau(B)\oplus\varepsilon^1=\pi^*(\tau(M)\oplus\xi)$.

3. The proof of Theorem

We shall prove the theorem in Introduction.

THEOREM 3.1 *Let ξ be a $(k+1)$-plane bundle over M^n and $B(\xi)$ the total space of* *the associated k-sphere bundle of* ξ. *Then we have* $B(\xi)_0 \subseteq R^k$, *where* $B(\xi)_0$ *denotes* $B(\xi)-\{x\}$ *for some point of* $B(\xi)$.

PROOF. Let $\pi: B(\xi)\to M^n$ be the projection. By lemma 2.3, we have

$$\tau(B)\oplus\varepsilon^1=\pi^*(\tau(M)\oplus\xi).$$

We denote $\tau(M)\oplus\xi$ by ζ. Obstructions to the existence of $(k+1)$ linearly independent cross-sections of ζ lie in $H^{i+1}(M^n;\ \{\pi_i(V_{n+k+1,\ k+1})\})$, where $\{\pi_i(V_{n+k+1,\ k+1})\}$ denotes the bundle of coeffiecients. Since $H^{i+1}(M^n;\ \{\pi_i(V_{n+k+1,\ k+1})\})$ vanishes for $i<n$, we have $\zeta=\varepsilon^{k+1}\oplus\eta''$ and $\tau(B(\xi))\oplus\varepsilon^1=\varepsilon^{k+1}\oplus\eta'$, where $\eta'=\pi^*\eta''$. Using lemma 2.2, we have $\tau(B(\xi)_0)=\varepsilon^k+\eta$, where $\eta=\eta'|B(\xi)_0$.
We have completed the proof of Theorem 3.1.

4. Sphere bundles over spheres

In this section, we shall consider submersion of the total spaces of sphere bundles over spheres. Let ξ be a $(k+1)$-plane bundle over S^n and $B(\xi)$ the total space of the associated sphere bundle with projection π. We obtain the following

THEOREM 4.1 (i) *If n is congruent to* 3, 5, 6 *or* 7 *mod* 8, *then* $B(\xi)_0\subseteq R^{n+k}$.
(ii) *If n is congruent to* 1 *mod* 8 *and greater then* 8, *then* $B(\xi)_0\subseteq R^{k+3}$.
(iii) *If n is congruent to* 2 *mod* 8 *and greater then* 17, *then* $B(\xi)_0\subseteq R^{k+6}$.
(iv) *If n is divisible by* 8 *and not equal to* 4 *or* 8, *then* $B(\xi)_0\subseteq R^{k+1}$.

PROOF. We denote $\tau(S^n)\oplus\xi$ by ζ. Since $\pi_{n-1}(SO)=0$ for $n=3, 5, 6$ or 7 mod 8, the result of (i) holds. The obstruction to the existence of $(k+4)$ linearly independent cross-sections of ζ is an element of $H^n(S^n;\ \pi_{n-1}(V_{n+k+1,k+4}))$. Since $\pi_{8s}(V_{8s+k+2,k+4})=0$ for $s\geqq 1$ (see [2]), we obtain the result of (ii). Similarly we obtain result of (iii) using the fact $\pi_{8s+1}(V_{8s+k+3,k+7})=0$ for $s\geqq 2$. In order to prove (iv), we use the result in [3]; the n-th Stiefel-Whitney class $w_n(\zeta)$ of ζ vanishes for $n\neq 4.8$. Thus we have proved Theorem 4.1.

We next consider k-sphere bundles over S^n for $n\leqq 4$. We use the following notation. By the bundle classification theorem, the equivalence classes of k-sphere bundle over S^n are in one to one correspondence with elements of $\pi_{n-1}(SO(k+1))$. $B_m^{(2,k)}$ denotes the total space of the k-sphere bundle over S^n which corresponds to the element m of $\pi_{n-1}(SO(k+1))$.

THEOREM 4.2 (i) $(B_m^{(2,k)})_0\subseteq R^k (k\geqq 2)$ *and* $(B_0^{(2,k)})_0\subseteq R_{k+1}$. *This is best possible.*
(ii) $(B_m^{(4,k)})_0\subseteq R^{k+1}$ *if m is even and* $k\geqq 4$.
(iii) $(B_m^{(4,k)})_0\subseteq R^k$ *if m is odd and* $k\geqq 4$. *This is best possible.*

PROOF. (i) In this cace, we can choose the associated bundle of $\theta\oplus\varepsilon^{k-1}$ as (ξ),

where θ is the canonical 2 plane bundle over $S^2=CP_1$. Since θ has the total Chern class $c(\theta)=1+a$, where a is a generator of $H^2(S^2)$ and $w_2(\xi)=a$ mod 2. Submersibility follows from Thoerm 3.1. Since $w_2(B_{m^{(2,k)}})_0\neq 0$, this is best possible.
(iii) This is a direct consequence of Theorm 3.1. We shall prove (ii). Let $\xi_{m^{(4,k)}}$ be the bundle with characteristic map $i(m\sigma)$, where i is the inclusion: $SO(4)\longrightarrow SO(k+1)(k\geqq 5)$ and σ is the map $S^3\longrightarrow SO(4)$ given by $\sigma(u)v=uv$, where u and v denote quaternions with norm 1. By a result of [5], we have $O(\hat{\xi}_{m^{(4,k)}})=\pm m\alpha$, where α is a generator of $H^4(S^4)$ and $O(\hat{\xi}_{m^{(4,k)}})$ is defined as follows; Let $\hat{\xi}_{m^{(4,k)}}$ be the associated principal bundle of $\xi_{m^{(4,k)}}$. The restriction of it to the 3-skelton of S^4 has a cross section. Then $O(\hat{\xi}_{m^{(4,k)}})$ is the obstruction to extending the cross section over S^4. Moreover we have $w_4(\xi_{m^{(4,k)}})=P^*O(\hat{\xi}_{m^{(4,k)}})$. Hence we have $w_4(\xi_{m^{(4,k)}})=0$ if and only if m is even. This proves (ii). Finally we prove the best possibility of (iii). This is a direct consequence of the fact that $w_4(B_{m^{(4,k)}})\neq 0$, which follows from that $w_4(B_{m^{(4,k)}})=\pi^*(w_4(\xi_{m^{(4,k)}})$ and that π^* is an isomophism.

5. Sphere bundles over real brojective spaces

In this section, we shall consider submersion of total spaces of sphere bundles over real projective space $P_n(n\leqq 4)$. Let $B(\ )$ and $B(\ )_0$ be similar as above and L the canonical line bundle over P_n. We quote from [4] the results of the classification of vector bundles over P_n.

(5.1) k-sphere bundles over $P_2(k\geqq 1)$.
We obtain the following results.

(i) $B(\varepsilon^{k+1})_0\subseteq R^k$.

Since $w_2(B(\varepsilon^{k+1})_0)\neq 0$, this is best possible.

(ii) $B(L\oplus\varepsilon^k)_0\subseteq R^{k+1}$.

By lemma 2.3, we have $\tau(B(L\oplus\varepsilon^k))\oplus\varepsilon'=\pi^*(\tau(P_2)\oplus L\oplus\varepsilon^k)$. We denote $\tau(P_2)\oplus L\oplus\varepsilon^k$ by ζ. The obstruction to the existence of $(k+2)$ linearly independent cross-sections of ζ is $w_2(\zeta)\in H^2(P_2;\ \pi_1(V_{k+3,k+2}))$. Since $w(\zeta)=(1+\alpha)^4=1$, we have $w_2(\zeta)=0$, where α is a generator of $H^1(P_2;\ Z_2)$. This proves (ii).

(iii) $B(2L\oplus\varepsilon^{k-1})_0\subseteq R^{k+1}(k\geqq 2)$.

This follows from Theorm 3.1. This result is best possible. In fact, we have $w(B(2L\oplus\varepsilon^{k-1}))=\pi^*(1+\alpha)^5$. Since π^* is an isomorphism by the exactness of the Gysin sequence. Thus we have $w_1(B(2L\oplus\varepsilon^{k-1}))\neq 0$. The inclusion map i: $B(2L\oplus\varepsilon^{k-1})_0\longrightarrow B(2L\oplus\varepsilon^{k-1})$ induces an isomorphism i^*: $H^1(B;\ Z_2)\longrightarrow H^1(B_0;\ Z_2)$. Thus we have $w_1(B(2L\oplus\varepsilon^{k-1})_0)\neq 0$. This implies the best possibility of (iii).

(iv) $B(3L\oplus\varepsilon^{k-2})_0\subseteq R^k$ $(k\geqq 3)$.

This is also best possible.

(5.2) k-sphere bundles over P_3.

Since P_3 is paralelizable, we obtain the following results.

(i) $B(\varepsilon^{k+1})_0\subseteq R^{k+3}$. $(k\geqq 1)$
(ii) $B(L\oplus\varepsilon^{k})_0\subseteq R^{k+2}$ $(k\geqq 1)$.
(iii) $B(2L\oplus\varepsilon^{k+1})_0\subseteq R^{k+1}$ $(k\geqq 2)$.
(iv) $B(3L\oplus\varepsilon^{k-2})_0\subseteq R^k$ $(k\geqq 3)$.

These are all best possible.

(5.3) k-sphere bundles over P_4.

We obtain the following results.

(i) $B(\varepsilon^{k+1})_0\subseteq R^k$ $(k\geqq 1)$
(ii) $B(L\oplus\varepsilon^{k})_0\subseteq R^k$ $(k\geqq 1)$
(iii) $B(2L\oplus\varepsilon^{k-1})_0\subseteq R^k$ $(k\geqq 2)$

These results follow from Theorem 3.1 and are best possible.

(iv) $B(3L\oplus\varepsilon^{k-2})_0\subseteq R^{k+1}$ $(k\geqq 3)$

This is proved as follows. We have $(B(3L\oplus\varepsilon^{k-2}))\oplus\varepsilon^1=\pi^*(\tau(P_4)\oplus 3L\oplus\varepsilon^{k-2})$. We denote $\tau(P_4)\oplus 3L\oplus\varepsilon^{k-2}$ by ζ. The obstruction to the existence of $k+2$ linearly independent cross sections of ζ is $w_4(\zeta)\in H^4(P_4;\pi_3(V_{k+5,k+2}))$. Since we have $w(\zeta)=(1+\alpha)^8$, $w_4(\zeta)=0$. This proves (iv).

Similarly we can prove the following results.

(v) $B(4L\oplus\varepsilon^{k-3})_0\subseteq R^{k+1}$ and $\nsubseteq R^{k+1}$ $(k\geqq 4)$
(vi) $B(5L\oplus\varepsilon^{k-4})_0\subseteq R^{k+1}$ and $\nsubseteq R^{k+3}$ $(k\geqq 5)$
(vii) $B(6L\oplus\varepsilon^{k-5})_0\subseteq R^{k+1}$ $(k\geqq 6)$
(viii) $B(7L\oplus\varepsilon^{k-6})_0\subseteq R^k$ $(k\geqq 7)$

The results of (vii) and (viii) are best possible.

6. Dold's manifolds

In this section, we shall consider the submersion of Dold's manifolds of type $(n, 1)$. We denote it by $P(n, 1)$. $P(n, 1)$ is defined as follows. Let S^n be the unit sphere and CP_1 the complex 1-dimensional projective space. Now $P(n, 1)$ is the manifold obtained from $S^n\times CP_1$ by identifying (x, z) with $(-x, \bar{z})$, where $-x$ denotes the antipodal point of x and $\bar{z}$ the conjugate of z. It is obvious that ρ: $P(n,$

$1) \longrightarrow P_n$ defined by $\rho(x, z)=x$ is a fibre map. We denote this bundle by δ; $\delta=\{P(n, 1), \delta, P_n, CP_1, O(1)\}$. We can prove that $P(n, 1)$ is the total space of the associated sphere bundle of a vector bundle ξ^3, with cross section. According to a result of [4], it is known that the stable class of ξ is the stable class of L if $n>2$. The Stiefel-Whitney class of $P(n, 1)$ is given by $w(P(n, 1))=\pi^*(w(P_n)w(\xi))=\pi^*(1+\alpha)^{n+2}$, where π is the projection of ξ. Since ξ has a non-zero cross section, the homomorphism $\pi^*: H^*(P_n; Z_2) \longrightarrow H^*(P(n, 1); Z_2)$ is a monomorphism. The inclusion map $i: P(n, 1)_0 \longrightarrow P(n, 1)$ induces an isomorphism $i^*: H^r(P(n, 1); Z_2) \longrightarrow H^r(P(n, 1)_0; Z_2)$ for $r \leqq 2n$. Thus we have $w_i(P(n, 1)_0)=\binom{n+2}{i}\alpha^i$. We define σ as follows;

$$\sigma=max\{i \leqq n; \binom{n+2}{i} \neq 0 \text{ mod } 2\}.$$

We can prove the following

THEOREM 6.1. $P(n, 1)_0 \subseteq R^2$ and $\nsubseteq R^{n+3-\sigma}$ $(n>2)$.

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