On complex hypersurfaces of C^{n+1} satisfying a certain condition on the curvature tensor

By

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1. Introduction

If a Riemannian manifold is locally symmetric, then its curvature tensor R satisfies

$$(*) \qquad R(X, Y)\cdot R=0$$

for all tangent vectors X and Y, where the endomorphism $R(X, Y)$ operates on R as a derivation of tensor algebra at a point of M.

Conversely, does this algebraic condition (*) on the curvature tensor field R imply that M is locally symmetric (i. e. $\nabla R=0$)? In fact, if M is a compact Einstein space, then the statement above is affirmative[1].

K. Nomizu has conjectured that the answer is affirmative in the case where M is irreducible and complete and dim. $M \geqq 3$. And recently he [2] gived an affirmative answer in the case where M is a complete hypersurfac in a Enclidean space.

In this paper, we shall consider a complex hypersurface of C^{n+1} such that its curvature tensor R satisfies (*) and we shall see that the type number at any point of this manifold is 0 or 2. This result will lead directly to the main theorem by virtue of the result by B. Smith [3]. In § 2, we shall state some properties of a complex hypersurface of a Kähler manifold and then we shall confine our attention to a complex hypersurface of complex n+1-dimensional Euclidean space C^{n+1} endowed with the usual flat Kähler structure.

§3 will be devoted to the proof of our main theorem.

1) see for example [1]

2. Couplex hypersurface

Let $\tilde{M}$ be a Kähler manifold of complex dimension $n+1$. The Kähler structure and Kähler metric of $\tilde{M}$ is denoted by $\tilde{J}$ and $\tilde{g}$ respectively. And M will be a complex manifold of complex dimension n which is a complex hypersurface of $\tilde{M}$, i.e. there exists a complex analytic mapping $\varphi: M \rightarrow \tilde{M}$ whose differential φ_* is 1-1 at each point of M.

It is well known that the complex structure J of M is a Kähler structure with $g=\varphi^*\tilde{g}$.

In order simplify the presentation, we identify for each point $x \in M$, the tangent space $T_x(M)$ with $\varphi_*(T_x(M)) \subset T_{\varphi(x)}(\tilde{M})$ by means of φ_*. A vector in $T_{\varphi(x)}(\tilde{M})$ which is orthogonal, with respect to $\tilde{g}$, to the subspace $\varphi_*(T_x(M))$ is said to be normal to M at x.

If we denote the Riemannian covariant differentiation on $\tilde{M}$ by $\tilde{\nabla}$ and by X, Y and Z, vector fields on a coordinate neighborhood $U(x)$ of M, or vector fields tangent to M, we may write

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N + k(X, Y)\tilde{J}N$$

where $\nabla_X Y$ denotes the component of $\tilde{\nabla}_X Y$ tangent to M, and N a unit vector field normal to M at each point of $U(x)$ Then, we can see that ∇ is the Riemannian covariant differentiation with respect to g, and h and k are symmetric covariant tensor fields of degree 2 on $U(x)$ satisfying

$$h(X, JY) = -k(X, Y) \tag{2.1}$$
$$k(X, JY) = h(X, Y)$$

for any pair of vectors X and Y tangent to M at a point of $U(x)$.

Moreover, the identity $\tilde{g}(N, N)=1$ implies $\tilde{g}(\tilde{\nabla}_X N, N)=0$ for any vector field X on $U(x)$. We may therefore write

$$\tilde{\nabla}_X N = -A(X) + s(X)\tilde{J}N \tag{2.2}$$

where $A(X)$ is tangent to M. In this case, A and s are tensor fields on $U(x)$ of type (1.1) and (0.1) respectively. Furthermore they satisfy

$$AJ = -JA$$
$$h(X, Y) = g(AX, Y) \tag{2.3}$$
$$k(X, Y) = g(JAX, Y)$$

for any pair of vectors X, Y tangent to M at a point of $U(x)$.

The following lemma will be useful in this paper.

LEMMA 2.1.[2] *Let V be a $2n$-dimensional real vector space with a complex structure and a positive definite inner product g which satisfiee $g(JX, JY)=g(X, Y)$ for all X, $Y \in V$. If A is symmetric (i. e. $g(AX, Y)=g(X, AY)$ for all $X, Y \in V$) and $AJ=-JA$, there exists an orthogonal basis $\{e_1, \ldots\ldots, e_n, Je_1, \ldots\ldots, Je_n\}$ of V with respect to which the matrix of A is diagonal of the from*

$$\begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ 0 & & & & & -\lambda_n \end{pmatrix}$$

In particular Trace $A=$Trace $JA=0$.

Now, the following proposition is well known.

PROPOSITION 2.2. *If $\tilde{R}$ and R denote the Riemannian curvature tensors of $\tilde{M}$ and M respectively, then for any vector fields X, Y, Z and W on $U(x)$, we have*

$$\begin{aligned} \tilde{R}(X, Y)W = R(X, Y)W &- \{g(AY, W)AX - g(AX, W)AY\} \\ &- \{g(JAY, W)JAX - g(JAX, W)JAY\} \\ &+ g((\nabla_X A)Y - (\nabla_Y A)X - s(X)JAY + s(Y)JAX, W)N \\ &+ g((\nabla_X(JA))Y - (\nabla_Y(JA))X + s(X)AY - s(Y)AX, W)\tilde{J}N, \end{aligned}$$

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= \tilde{g}(\tilde{R}(X, Y)W, Z) \\ &= R(X, Y, Z, W) \\ &\quad - \{g(AX, Z)g(AY, W) - g(AX, W)g(AY, Z) \\ &\quad - \{g(JAX, Z)g(JAY, W) - g(JAX, W)g(JAY, Z)\} \end{aligned} \tag{2.4}$$

If $\tilde{M}$ is flat i. e. $\tilde{R}=0$, then from (2.4), we have

$$\begin{aligned} R(X, Y, Z, W) &= g(AX, Z)g(AY, W) - g(AX, W)g(AY, Z) \\ &\quad + g(JAX, Z)g(JAY, W) - g(JAX, W)g(JAY, Z) \end{aligned} \tag{2.5}$$

and hence

$$\begin{aligned} R(X, Y)W &= g(AY, W)AY - g(AX, W)AY \\ &\quad + g(JAY, W)JAX - g(JAX, W)JAY \end{aligned} \tag{2.6}$$

for any vectors X, Y, Z and W tangent to M at a point of $U(x)$.

Moreover, by S we denote the Ricci tensor of M. Then $S(Y, W)$ at a point

2) see for example [3]

$y \in U(x)$ is the trace of the linear endomorphism of $T_y(M)$ determined by $X \to R(X, Y)W$. Hence by using $JA=-AJ$, Trace A=Trace $JA=0$ and the general fact that, for any 1-from ω the trace of the linear mapping $X \to \omega(X)Y$ is equal to $\omega(Y)$, we have

(2.7) $$S(Y, W)=-2g(A^2Y, W)$$

on $U(x)$.

For further computation, we derive the complexification $T_y^c(M)$ of the tangent space $T_y(M)$ where y is any point of $U(x)$, and we denote the basis by $\left(\frac{\partial}{\partial z^1}\right)_y, \ldots, \left(\frac{\partial}{\partial z^n}\right)_y, \left(\frac{\partial}{\partial \bar{z}^1}\right)_y, \ldots, \left(\frac{\partial}{\partial \bar{z}^n}\right)_y$ where $(z^1, \ldots, z^n)$ are local complex coordinates of M.

Now, we shall express the components of the foregoing tensors with respect to this basis. From now on, the indices $i, j, k, \ldots$ take the value $1, 2, 3, \ldots, n$.

We put

$$g_{\bar{j}i}=g\left(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^i}\right), \quad R_{\bar{k}ji\bar{h}}=R\left(\frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^h}\right),$$

$$R_{\bar{k}ji}{}^h=g^{h\bar{a}}R_{\bar{k}ji\bar{a}}, \quad R_{\bar{j}i}=S\left(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial x^i}\right),$$

$$h_{ji}=h\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^i}\right), \quad h_{\bar{j}\bar{i}}=h\left(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^i}\right),$$

$$h_{\bar{j}}{}^i=g^{i\bar{a}}h_{\bar{j}\bar{a}}, \quad h^{ji}=g^{j\bar{b}}g^{i\bar{a}}h_{\bar{b}\bar{a}} \quad \text{and etc.}$$

Then, we have

$$\overline{h_{\bar{j}\bar{i}}}=h_{ji}, \quad \overline{h_{\bar{j}}{}^i}=h_j{}^{\bar{i}} \quad \text{and etc.}$$

And from (2.3), we get

$$A\left(\frac{\partial}{\partial z^i}\right)=h_i{}^{\bar{j}}\frac{\partial}{\partial \bar{z}^j} \quad \text{and} \quad A\left(\frac{\partial}{\partial \bar{z}^i}\right)=h_{\bar{i}}{}^j\frac{\partial}{\partial z^j}.$$

Moreover, the well known fact that $J\left(\frac{\partial}{\partial z^i}\right)=\sqrt{-1}\frac{\partial}{\partial z^i}$, $J\left(\frac{\partial}{\partial \bar{z}^i}\right)=-\sqrt{-1}\frac{\partial}{\partial \bar{z}^i}$ and (2.1) imply that

$$\sqrt{-1}\,k_{ji}=h_{ji} \quad \text{and} \quad -\sqrt{-1}\,k_{\bar{j}\bar{i}}=h_{\bar{j}\bar{i}}.$$

Thus, from (2.6) and (2.7), we get

(2.3) $$R_{\bar{k}ji\bar{h}}=-2h_{\bar{k}\bar{h}}h_{ji}$$

and

(2.9) $$R_{\bar{j}i}=-2h_{\bar{j}}{}^a h_{ia}$$

respectively.

3. Complex hypersurfaces of C^{n+1} satisfying the condition (*)

In this section, we consider a complex hypersurface M of complex n+1-dimensional Euclidean space C^{n+1} endowed with the usual flat Kähler structure such that its curvature tensor R satisfies the condition (*).

In this case, we have, of course,

$$R(X, Y)\cdot S=0 \tag{3.1}$$

for all tangent vectors X and Y of M.

The condition (*) is equivalent to

$$-R_{\bar m l \bar h}{}^{\bar s} R_{\bar k j i \bar s}-R_{\bar m l i}{}^{s} R_{\bar k j s \bar h}$$

$$-R_{\bar m l j}{}^{s} R_{\bar k s i \bar h}-R_{\bar m l \bar k}{}^{\bar s} R_{\bar s j i \bar h}=0$$

where the components of R are the same things at the end of the last section. Substituting (2.8) in the last equation, we have

$$\begin{aligned} P_{\bar m l \bar k j i \bar h} \overset{def}{=} & h_{\bar m \bar h} h_l{}^{\bar s} h_{\bar k \bar s} h_{ji}-h_{\bar m}{}^{s} h_{li} h_{\bar k \bar h} h_{js} \\ & -h_{\bar m}{}^{s} h_{lj} h_{\bar k \bar h} h_{si}-h_{\bar m \bar k} h_l{}^{\bar s} h_{\bar s \bar h} h_{ji} \\ & =0. \end{aligned} \tag{3.2}$$

Transvecting the last equation with $g^{l\bar k}$, we have

$$\begin{aligned} Q_{\bar m j i \bar h} \overset{def}{=} & h_{\bar m \bar h} h^{\bar k \bar s} h_{\bar k \bar s} h_{ji}-h_{\bar m}{}^{s} h^{\bar k}{}_{i} h_{\bar k \bar h} h_{js} \\ & -h_{\bar m}{}^{s} h^{\bar k}{}_{j} h_{\bar k \bar h} h_{si}+h_{\bar m \bar k} h^{\bar k \bar s} h_{\bar s \bar h} h_{ji} \\ & =0. \end{aligned} \tag{3.3}$$

On the other hand, (3.1) is equivalent to

$$-R_{\bar m l i}{}^{s} R_{\bar j s}-R_{\bar m l \bar j}{}^{\bar s} R_{\bar s i}=0,$$

i. e.

$$T_{\bar m l \bar j i} \overset{def}{=} h_{\bar m}{}^{s} h_{li} h_{\bar j}{}^{a} h_{sa}-h_{\bar m \bar j} h_l{}^{\bar s} h_{\bar s}{}^{a} h_{ia}=0. \tag{3.4}$$

By long but straighforward computation of

$$P \overset{def}{=} g^{\bar m f} g^{l \bar e} g^{\bar k d} g^{j \bar c} g^{i \bar b} g^{\bar h a} P_{\bar m l \bar k j i \bar h} P_{f \bar e d \bar c \bar b a}$$

$$Q \overset{def}{=} g^{\bar m f} g^{j \bar c} g^{i \bar b} g^{\bar h a} Q_{\bar m j i \bar h} Q_{f \bar c \bar b a}$$

$$T \overset{def}{=} g^{\bar m f} g^{l \bar e} g^{j \bar c} g^{i \bar b} T_{\bar m l \bar j i} T_{f \bar e \bar c \bar b},$$

we get

$$P=4(\alpha^2 \beta+\alpha \delta-2\gamma)$$

$$Q=\alpha^4+2\beta\alpha^2-3\delta\alpha+2\beta^2-2\gamma$$

$$T=2(\alpha\delta-\beta^2)$$

where we have put

$$\alpha=h^{ji}\, h_{ji}$$

$$\beta=h^{kj}\, h_{ji}\, h^{ih}\, h_{hk}$$

$$\delta=h^{mh}\, h_{hi}\, h^{il}\, h_{ls}\, h^{sk}\, h_{km}$$

$$\gamma=h^{mh}\, h_{hk}\, h^{ks}\, h_{sl}\, h^{li}\, h_{ij}\, h^{jt}\, h_{tm}.$$

Hence, by virtue of (3.2), (3.3) and (3.4), we have three equations

$$\alpha^2+\alpha\delta-2\gamma=0$$

$$\alpha^4+2\beta\alpha^2-3\delta\alpha+2\beta^2-2\gamma=0$$

$$\alpha\delta-\beta^2=0.$$

From these equations, we get

$$\alpha^2=\beta$$

i. e.

$$h^{ji}\, h_{ji}\, h^{kh}\, h_{kh}=h^{kj}\, h_{ji}\, h^{ih}\, h_{hk}. \tag{3.5}$$

On the other hand, taking account of the property of the tensor h, we see that

$$\text{Trace } A^2=2h^{ji}\, h_{ji}$$

$$\text{Trace } A^4=2h^{kj}\, h_{ji}\, h^{ih}\, h_{hk}.$$

Hence, from (3.5), we can deduce

$$(\text{Trace } A^2)^2=2 \text{ Trace } A^4.$$

LEMMA 3.1. *If M is a complex hypersurface of C^{n+1} such that its curvature tensor R satisfies*

$$R(X, Y)\cdot R=0,$$

then the equality

$$(\text{Trace } A^2)^2=2(\text{Trace } A^4) \tag{3.6}$$

is valid, where A is given by (2. 2).

Now, we shall express the equality (3.6) with respect to the basis of lemma 2.1.
At any fixed point x_0 if we take the basis of lemma 2.1, then A^2 and A^4 are repre-

sented by the matrices

$$\begin{pmatrix} \lambda_1^2 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n^2 & & & \\ & & & \lambda_1^2 & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_n^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_1^4 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n^4 & & & \\ & & & \lambda_1^4 & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_n^4 \end{pmatrix}$$

respectively and hence we have

$$\text{Trace } A^2 = 2(\lambda_1^2 + \ldots\ldots + \lambda_n^2)$$

$$\text{Trace } A^4 = 2(\lambda_1^4 + \ldots\ldots + \lambda_n^4)\,.$$

Therefore (3. 6) is given by

$$\{2(\lambda_1^2 + \ldots\ldots + \lambda_n^2)\}^2 = 2\{2(\lambda_1^4 + \ldots\ldots + \lambda_n^4)\}$$

i. e.

$$\lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \ldots\ldots + \lambda_{n-1}^2 \lambda_n^2 = 0\,.$$

This means that the set $\{\lambda_1, \ldots\ldots, \lambda_n\}$ contains at most one non-zero element. Consepuently, we have the following lemma.

LEMMA 3. 2. *If M is complex hypersurface of C^{n+1} such that its curvature tensor R satisfies*

$$R(X, Y)\cdot R = 0,$$

then at each point of M, the type number i. e. the rank of A is 0 or 2, that is, at any point $x \in M$, A_x is represented by the matrix

$$\begin{pmatrix} 0 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & \lambda(x) & & & & \\ & & & & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & \\ 0 & & & & & & & -\lambda(x) \end{pmatrix}$$

with respect to the basis of lemma 2. 1, where $\lambda(x)$ is zero or not.

If $\lambda(x) = 0$ at any point of M, then the curvature tensor $R = 0$ on M by virtue of (2. 6). As a result, M is, of course, symmetric.

Next, using (2. 7) and the property of the basis of lemma (2. 1), the Ricci tensor S_x and the metric tensor g_x at $x \in M$ are represented by the matrices

$$\begin{pmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & -2\lambda^2(x) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & 0 & \\ 0 & & & & & & -2\lambda^2(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$

respectively. Therefore, if $n \geqq 2$ (the complex dimension of M) and if there exists a point $x \in M$ satisfying $\lambda(x) \neq 0$, then M is not Einstein (A Riemannian manifold M with the Riemannian metric g is called an Einstein space if the Ricci tensor S satisfies $S=\rho g$ where ρ is a certain scalar field.). As a result, by [3], M is not symmetric.

If $n=1$, then

$$S_x = -2\lambda^2(x) g_x$$

for any point $x \in M$, that is, M is a Einstein space with non-positive scalar curveture.

Thus, we have the following main theorem.

THEOREM 3.3. *If M is a complex hypersurface of C^{n+1} such that its curvature tensor R satisfies $R(X, Y) \cdot R=0$, then the rank of A is 0 or 2 at each point of M, and then we can conclude that*

(i) *if the rank of $A=0$ over M, then M is locally flat,*

(ii) *if there exists a point where the rank of $A=2$, then M is not symmetric ($n \geqq 2$)*

(iii) *if $n=1$, then M is an Einstenin space with non-positive scalar curvature.*

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