On the higher order non-singular immersion

By

Chikô YOSHIOKA

(Received October 31, 1967)

1. Introduction

In [1] and [2], W. F. Pohl and E. A. Feldman have considered higher order tangent bundles of a smooth manifold M and the higher order non-singular immersion of M into euclidean spaces. Subsequently, H. Suzuki obtained some higher order non-immersion theorems for real projective spaces in [3] (Theorem 7. 1) etc.) and for complex projective spaces in [4] (the case of $p=2$ in Theorem 7. 2) etc.) by the calculations of Stiefel-Whitney classes of the higher order tangent bundles of these spaces and the other reasons. The purpose of this paper is to show a higher order non-immersion theorem for complex projective spaces and Dold manifolds, using the methods which are similar to [3], [4].

In §2, we will prepare brief definitions, general results etc. of higher order tangent bundle, higher order non-singular immersion and the others. In §3, we will prove the main theorem, Theorem 7, of this paper which is stated together with the case of real projective spaces.

The author wishes to express his thanks to Prof. K. Aoki, Prof. T. Kaneko and T. Watabe for their kind encouragements, and to Prof. H. Suzuki for his valuable suggestions.

2. Preliminaries

Let M be an n-dimensional smooth manifold and $(x_1,\cdots,x_n)$ be the local coordinate of M. Let $\tau_p(M)_x$ $(x\in M)$ be the real vector space spanned by $\left\{\left(\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}}\right)_x ; 1\leqq k \leqq p,\ 1\leqq i_1\leqq\cdots\leqq i_k\leqq n\right\}$ and $\tau_p(M)=\bigcup_{x\in M}\tau_p(M)_x$. Then we have that $\tau_p(M)$ is a smooth $\nu(n, p)$-vector bundle over M where $\nu(n, p)=n+\binom{n+1}{2}+\cdots+\binom{n+p-1}{p}=\binom{n+p}{p}-1$. $\tau_p(M)$ is called the p-th order tangent bundle of M (See [1] p189).

Let f be a smooth map of M into the m-dimensional euclidean space R^m. Let f_p: $\tau_p(M)\longrightarrow\tau_p(R^m)$ be the p-th order differential of f (See [1] p190) and D_j: $\tau_j(R^m)\longrightarrow\tau_{j-1}(R^m)$ $(j=2, 3,\cdots,p)$ be the vector bundle homomorphisms which are defined by

$$D_j(v_{j-1}+\sum a_{i_1}\cdots_{i_j}\frac{\partial^j}{\partial x_{i_1}\cdots\partial x_{i_j}})=v_{j-1},$$

where $v_{j-1}\in\tau_{j-1}(R^m)$ (See [2] p176). Then $D_2\cdots D_p f_p$: $\tau_p(M)\longrightarrow\tau_1(R^m)=\tau(R^m)$ is a vector bundle homomorphism covering f. If $D_2\cdots D_p f_p$ is of maximal rank ($=\min\{\nu(n, p), m\}$) on each fibre, we say that f is a p-th order non-singular immersion and denote that f: $M\subseteq_p R^m$ (See [3] p270). We have (See [1] p217):

THEOREM 1. *Let $n>1$. Then $M\subseteq_p R^{\nu(n,p)+n}$, and if $p>1$, $M\subseteq_p R^{\nu(n,p)-n}$.*

Throughout this paper, we identify a vector bundle over M with its class in $KO(M)$. Let O^i: $KO(M)\longrightarrow KO(M)$ ($i=0,1,\cdots$) be the symmetric i-th power operations which have the following properties (See [3] p272):

P1. $O^0x=1$,

P2. $O^1x=x$,

P3. $O^i(x+y)=\sum_{j=0}^{i} O^jx\cdot O^{i-j}y$ for x, $y\in KO(M)$.

We have (See [2] Theorem 2.1, [3] p273):

THEOREM 2. *The sequence*

$$0\rightarrow\tau_{p-1}(M)\rightarrow\tau_p(M)\rightarrow O^p\tau(M)\rightarrow 0$$

is a split exact sequence.

In general, if x is an n-vector bundle, $O^i(x)$ is an $\binom{n+i-1}{i}$-vector bundle (Cf. [2] p173).

Let M be an n-dimensional compact connected smooth manifold and $w^p(M)$, $\bar{w}^p(M)$, $w^p{}_i(M)$ and $\bar{w}^p{}_i(M)$ be total, dual total, i-dimensional and dual i-dimensional Stiefel-Whitney class of $\tau_p(M)$, respectively. We have the following theorems which are the sources of our results.

THEOREM 3. ([3] Lemma 2.2) $\tau_p(M)=O^p(\tau(M)+1)-1$.

THEOREM 4. ([3] Theorem 1.1) *If $\bar{w}^p{}_i(M)\neq 0$, then M cannot be immersed into $R^{\nu(n,p)+k}$ without singularities of order p; $M\not\subseteq_p R^{\nu(n,p)+k}$ ($0\leqq k<i$). If $w^p{}_i(M)\neq 0$, then $M\not\subseteq_p R^{\nu(n,p)+k}$ ($-i<k\leqq 0$).*

Finally, we prepare some lemmas which serve to compute $w^p(M)$. Let x and y be n- and n'-vector bundle, respectively, and

$$w(x)=1+w_1(x)+\cdots+w_n(x)=\Pi_{i=1}^{n}(1+t_i)$$

$$w(y)=1+w_1(y)+\cdots+w_{n'}(y)=\Pi_{i=1}^{n'}(1+t_i')$$

be formal factorizations of total Stiefel-Whitney classes of x and y. We have:

LEMMA 1.

1) $w(x\cdot y)=\Pi_{1\leq i\leq n,\ 1\leq j\leq n'}(1+(t_i+t_j'))$.

2) ([2] Lemma 3.24) $w(O^p(x))=\Pi_{1\leq i_1\leq\cdots\leq i_p\leq n}(1+(t_{i_1}+\cdots+t_{i_p}))$.

LEMMA 2. *Let p and r be positive integers. Then*

$$O^p(rx)=\sum_p \binom{r}{s} O^{i_1}x\cdots O^{i_s}x,$$

where the sum $\sum_p$ is taken for the positive integers $i_1,\cdots,i_s$ ($s\geqq 1$) such that $i_1+\cdots+i_s=p$.

We can prove Lemma 2 by the induction on r.

LEMMA 3. *In Lemma 2, we have*

$$\sum_p\left\{\binom{r}{s}\Pi_{k=1}^{s}\binom{n+i_k-1}{i_k}\right\}=\binom{nr+p-1}{p}.$$

In particular, if $n=1$, then $\sum_p\binom{s}{r}=\binom{r+p-1}{p}$ and if $n=2$, then $\sum_p\left\{\binom{r}{s}\Pi_{k=1}^{s}(i_k+1)\right\}$ $=\binom{2r+p-1}{p.}$.

3. Non-immersion theorem and its proof

Let RP_m, CP_n and $P(m,n)$ be m-dimensional real, n-dimensional complex projective space and Dold manifold of type (m,n), respectively. In this section we will prove the higher order non-immersion theorem for RP_m, CP_n and $P(m,n)$.

From Theorem 3, it follows:

THEOREM 5.

1) ([3] p274) $\tau_p(RP_m)=O^p((m+1)\xi)-1$,

2) ([4] Theorem 1.1) $\tau_p(CP_n)=O^p((n+1)\eta)-O^{p-1}((n+1)\eta)-1$,

3) $\tau_p(P(m,n))=\sum_{i=0}^{p}O^i(m\xi)\cdot O^{p-i}((n+1)\eta)-\sum_{i=0}^{p-1}O^i(m\xi)\cdot O^{p-1-i}((n+1)\eta)-1$,

where ξ in 1) is the canonical line bundle over RP_m, η in 2) is the realization of the canonical line bundle over CP_n, and ξ,η in 3) are the bundles which are defined in [5] *Proposition* 1.4: $i^*\xi=(\xi$ *in* 1)), $j^*\eta=(\eta$ *in* 2)) *where* i: $RP_m\to P(m,n)$, j: $CP_n\to P(m,n)$ *are inclusions.*

PROOF of 3). By Theorem 1.5 in [5], $\tau(P(m,n))\oplus\xi\oplus 2=(m+1)\xi\oplus(n+1)\eta$. Hence, by Theorem 3 and P3, we have

$$\begin{aligned}\tau_p\ (P(m,n))&=O^p(m\xi+(n+1)\eta-1)-1\\&=O^p(m\xi+(n+1)\eta)-O^{p-1}(m\xi+(n+1)\eta)-1,\end{aligned}$$

Thus, Theorem 5.3) follows from P3. (q. e. d.)

Using Theorem 5 and Lemmas, we obtain:

THEOREM 6.

1) $w_p(RP_m)=\begin{cases}1 & \text{(for even p)}\\ (1+\alpha)^{\binom{m+p}{p}} & \text{(for odd p).}\end{cases}$

2) $w^p(CP_n)=\begin{cases}(1+\beta)^{-\frac{1}{2}\binom{2n+p}{p-1}} & \text{(for even p)}\\ (1+\beta)^{\frac{1}{2}\binom{2n+p+1}{p}} & \text{(for odd p).}\end{cases}$

3) $w^p(P(m,n))=(1+c)^A(1+c+d)^B$,

where α, β *are the generators of* $H^1(RP_m\,;\,Z_2)$, $H^2(CP_n\,;\,Z_2)$, *respectively*; c, d *are the classes which are defined in* [5] $p284$: $c\in H^1(P(m,n)\,;Z_2)$, $d\in H^2(P(m,n)\,;\,Z_2)$, $c^{m+1}=d^{n+1}=0$, $Sq^1(d)=cd$, $w(\xi)=1+c$, $w(\eta)=1+c+d$; *and*

$$A=\frac{1}{2}\sum_{0\leq even\ i\leq p}\left\{\binom{2n+i+1}{i}+(-1)^{p-1}\frac{m+2(p-i)-1}{m-1}\binom{n+\frac{i}{2}}{\frac{i}{2}}\right\}\binom{m+p-i-2}{m-2},$$

$$B=\frac{1}{2}\sum_{0<odd\ i\leq p}\binom{2n+i+1}{i}\binom{m+p-i-2}{m-2}.$$

PROOF.

1) (Cf. [3] p274) Since $w(\xi)=1+w_1(\xi)=1+\alpha$, by Lemma 1, we have $w(O^i\xi)=1+i\alpha$. Hence,

$$w(O^{i_1}\xi\cdots\cdots O^{i_s}\xi)=1+(\sum^{s}_{k=1}i_k)\alpha=\begin{cases}1 & (\text{if } \sum i_k \text{ is even})\\ 1+\alpha & (\text{if } \sum i_k \text{ is odd}).\end{cases}$$

On the other hand, by Lemma 2, we have $O^p((m+1)\xi)=\sum_p\binom{m+1}{s}O^{i_1}\xi\cdots O^{i_s}\xi$. Hence, from Theorem 5 and Lemma 3, it follows

$$w^p(RP_m)=w(O^p((m+1)\xi))=\Pi_p w(O^{i_1}\xi\cdots O^{i_s}\xi)^{\binom{m+1}{s}}$$

$$=\begin{cases}1^{\binom{m+1}{s}}=1 & (\text{for even } p)\\ \Pi_p(1+\alpha)^{\binom{m+1}{s}}=(1+\alpha)^{\sum_p\binom{m+1}{s}}=(1+\alpha)^{\binom{m+p}{p}} & (\text{for odd } p).\end{cases}$$

2) (Cf. [4] p389) Let $w(\eta)=1+w_2(\eta)=1+\beta=(1+\sigma_1)(1+\sigma_2)$ be a formal factorization of total Stiefel-Whitney class of η : $\sigma_1+\sigma_2=0$, $\sigma_1\sigma_2=\beta$. By Lemma 1, we have

$$w(O^i\eta)=\Pi^i_{j=0}(1+j\sigma_1+(i-j)\sigma_2)=\begin{cases}1^{i+1}=1 & (\text{for even } i)\\ \{(1+\sigma_1)(1+\sigma_2)\}^{\frac{i+1}{2}}=(1+\beta)^{\frac{i+1}{2}} & (\text{for odd } i).\end{cases}$$

By induction on s, we obtain

$$w(O^{j_1}\eta\cdots O^{j_s}\eta)=\begin{cases}1 & (\text{if } \sum j_k \text{ is even})\\ (1+\beta)^{\frac{1}{2}\Pi^s_{k=1}(j_k+1)} & (\text{if } \sum j_k \text{ is odd}).\end{cases}$$

Here, note that $\Pi(j_k+1)$ is even, when $\sum j_k$ is odd, since it exists at least one k such that j_k is odd. Hence, from Lemmas 2 and 3, it follows

$$w((O^q(n+1)\eta))=w(\sum_q\binom{n+1}{s}O^{j_1}\eta\cdots O^{j_s}\eta)=\Pi_q w(O^{j_1}\eta\cdots O^{j_s}\eta)^{\binom{n+1}{s}}$$

$$=\begin{cases}1 & (\text{for even } q)\\ (1+\beta)^{\frac{1}{2}\sum_q\{\binom{n+1}{s}\Pi(j_k+1)\}}=(1+\beta)^{\frac{1}{2}\binom{2n+q+1}{q}} & (\text{for odd } q).\end{cases}$$

Thus, by Theorem 5, we have

$$w^p(CP_n)=w(O^p((n+1)\eta))w(O^{p-1}((n+1)\eta))^{-1}$$

$$=\begin{cases}w(O^{p-1}((n+1)\eta))^{-1}=(1+\beta)^{-\frac{1}{2}\binom{2n+p}{p-1}} & (\text{for even } p)\\ w(O^p((n+1)\eta))=(1+\beta)^{\frac{1}{2}\binom{2n+p+1}{p}} & (\text{for odd } p).\end{cases}$$

3) Since $w(\xi)=1+c$, similarly to 1), we have

$$(1)\quad w(O^q(m\xi))=\begin{cases}1^{\binom{m+q-1}{q}}=1 & \text{(for even } q)\\ (1+c)^{\binom{m+q-1}{q}} & \text{(for odd } q).\end{cases}$$

Let $w(\eta)=1+w_1(\eta)+w_2(\eta)=1+c+d=(1+\sigma_1)(1+\sigma_2)$ be a formal factorization of $w(\eta)$: $\sigma_1+\sigma_2=c$, $\sigma_1\sigma_2=d$. By Lemma 1, we have

$$w(O^i\eta)=\Pi^i{}_{j=0}(1+j\sigma_1+(i-j)\sigma_2)=\begin{cases}1^{\frac{i}{2}+1}(1+\sigma_1+\sigma_2)^{\frac{i}{2}}=(1+c)^{\frac{i}{2}} & \text{(for even } i)\\ \{(1+\sigma_1)(1+\sigma_2)\}^{\frac{i+1}{2}}=(1+c+d)^{\frac{i+1}{2}} & \text{(for odd } i).\end{cases}$$

By induction on s, we obtain

(2) $w(O^{j_1}\eta\cdots O^{j_s}\eta)$

$$=\begin{cases}1^{\frac{1}{2}(\Pi(j_k+1)+1)}\cdot(1+\sigma_1+\sigma_2)^{\frac{1}{2}(\Pi(j_k+1)-1)}=(1+c)^{\frac{1}{2}(\Pi(j_k+1)-1)} & \text{(if any } j_k \text{ is even)}\\ \{1\cdot(1+\sigma_1+\sigma_2)\}^{\frac{1}{2}\Pi(j_k+1)}=(1+c)^{\frac{1}{2}\Pi(j_k+1)} & \text{(if } \sum j_k \text{ is even and it exists at least one } k \text{ such that } j_k \text{ is odd)}\\ \{(1+\sigma_1)(1+\sigma_2)\}^{\frac{1}{2}\Pi(j_k+1)}=(1+c+d)^{\frac{1}{2}\Pi(j_k+1)} & \text{(if } \sum j_k \text{ is odd).}\end{cases}$$

Note that $\Pi(j_k+1)-1$ and $\Pi(j_k+1)$ in (2) are even. Hence, from Lemmas 2 and 3, it follows

$$w(O^q((n+1)\eta))=w\Big(\sum_q\binom{n+1}{s}O^{j_1}\eta\cdots O^{j_s}\eta\Big)=\Pi_q w(O^{j_1}\eta\cdots O^{j_s}\eta)^{\binom{n+1}{s}}$$
$$=\begin{cases}(1+c)^h & \text{(for even } q)\\ (1+c+d)^{\frac{1}{2}\sum_q\{\binom{n+1}{s}\Pi(j_k+1)\}}=(1+c+d)^{\frac{1}{2}\binom{2n+q+1}{q}} & \text{(for odd } q).\end{cases}$$

Now, we must determine the value of above h. Let

$$w(O^q((n+1)\eta))=\Pi_q w(O^{j_1}\eta\cdots O^{j_s}\eta)^{\binom{n+1}{s}}=1^g\cdot(1+\alpha_1+\alpha_2)^h \quad \text{(for even } q).$$

Then

$$g+h=\binom{2n+q+1}{q} \quad (=\text{fibre dimension of } O^q((n+1)\eta)).$$

Next, we consider $\sum'_q\binom{n+1}{s}O^{j_1}\eta\cdots O^{j_s}\eta$, where the sum $\sum'_q$ is taken for the positive even numbers $j_1,\cdots,j_s$ such tnat $j_1+\cdots+j_s=q$. Then, we have $\sum'_q\binom{n+1}{s}=\binom{n+\frac{q}{2}}{\frac{q}{2}}$ and from the first expression in (2), it follows

$$g-h=\binom{n+\frac{q}{2}}{\frac{q}{2}}.$$

Hence,

$$g=\frac{1}{2}\Big\{\binom{2n+q+1}{q}+\binom{n+\frac{q}{2}}{\frac{q}{2}}\Big\},\quad h=\frac{1}{2}\Big\{\binom{2n+q+1}{q}-\binom{n+\frac{q}{2}}{\frac{q}{2}}\Big\}.$$

Here, note that $\binom{2n+q+1}{q}\pm\binom{n+\frac{q}{2}}{\frac{q}{2}}$ is even, because $\binom{2n+q+1}{q}\equiv\binom{n+\frac{q}{2}}{\frac{q}{2}}$ (mod 2).

Thus,

(3) $w(O^q(n+1)\eta))$

$$=\begin{cases} 1^{\frac{1}{2}\{\binom{2n+q+1}{q}+\binom{n+\frac{q}{2}}{\frac{q}{2}}\}}\cdot(1+\sigma_1+\sigma_2)^{\frac{1}{2}\{\binom{2n+q+1}{q}-\binom{n+\frac{q}{2}}{\frac{q}{2}}\}}=(1+c)^{\frac{1}{2}\{\binom{2n+q+1}{q}-\binom{n+\frac{q}{2}}{\frac{q}{2}}\}} & \text{(for even } q) \\ \{(1+\sigma_1)(1+\sigma_2)\}^{\frac{1}{2}\binom{2n+q+1}{q}}=(1+c+d)^{\frac{1}{2}\binom{2n+q+1}{q}} & \text{(for odd } q). \end{cases}$$

From (1) and (3), it follows

$w(O^j(m\xi)\cdot O^i((n+1)\eta))$

$$=\begin{cases} (1+\sigma_1+\sigma_2)^{\frac{1}{2}\{\binom{2n+i+1}{i}-\binom{n+\frac{i}{2}}{\frac{i}{2}}\}\binom{m+j-1}{j}}=(1+c)^{\frac{1}{2}\{\binom{2n+i+1}{i}-\binom{n+\frac{i}{2}}{\frac{i}{2}}\}\binom{m+j-1}{j}} & \text{(for even } j,\ i) \\ \{(1+\sigma_1)(1+\sigma_2)\}^{\frac{1}{2}\binom{2n+i+1}{i}\binom{m+j-1}{j}}=(1+c+d)^{\frac{1}{2}\binom{2n+i+1}{i}\binom{m+j-1}{j}} & \text{(for even } j, \text{ odd } i) \\ (1+c)^{\frac{1}{2}\{\binom{2n+i+1}{i}+\binom{n+\frac{i}{2}}{\frac{i}{2}}\}\binom{m+j-1}{j}} & \text{(for odd } j, \text{ even } i) \\ \{(1+c+\sigma_1)(1+c+\sigma_2)\}^{\frac{1}{2}\binom{2n+i+1}{i}\binom{m+j-1}{j}}=(1+c+d)^{\frac{1}{2}\binom{2n+i+1}{i}\binom{m+j-1}{j}} & \text{(for odd } j,\ i) \end{cases}$$

$$(4)\quad =\begin{cases} (1+c)^{\frac{1}{2}\{\binom{2n+i+1}{i}+(-1)^{j-1}\binom{n+\frac{i}{2}}{\frac{i}{2}}\}\binom{m+j-1}{j}} & \text{(for even } i) \\ (1+c+d)^{\frac{1}{2}\binom{2n+i+1}{i}\binom{m+j-1}{j}} & \text{(for odd } i). \end{cases}$$

Hence, by Theorem 5 and (4), we have

$w^p(P(m,\ n))$

$=\Pi^p_{i=0}w(O^{p-i}(m\xi)\cdot O^i((n+1)\eta))\Pi^{p-1}_{i=0}w(O^{p-1-i}(m\xi)\cdot O^i((n+1)\eta))$

$=(1+c)^{\frac{1}{2}\sum_{\text{even } i}\{\binom{2n+i+1}{i}+(-1)^{p-i-1}\binom{n+\frac{i}{2}}{\frac{i}{2}}\}\binom{m+p-i-1}{m-1}}(1+c+d)^{\frac{1}{2}\sum_{\text{odd } i}\binom{2n+i+1}{i}\binom{m+p-i-1}{m-1}}$

$\cdot(1+c)^{-\frac{1}{2}\sum_{\text{even } i}\{\binom{2n+i+1}{i}+(-1)^{p-i}\binom{n+\frac{i}{2}}{\frac{i}{2}}\}\binom{m+p-i-2}{m-1}}(1+c+d)^{-\frac{1}{2}\sum_{\text{odd } i}\binom{2n+i+1}{i}\binom{m+p-i-2}{m-1}}.$

Computing the power numbers of $(1+c)^{\frac{1}{2}}$ and $(1+c+d)^{\frac{1}{2}}$ in above, we obtain the required results:

$$2A=\sum_{0\leq \text{even } i\leq q}\left\{\binom{2n+i+1}{i}+(-1)^{p-1}\frac{m+2(p-i)-1}{m-1}\binom{n+\frac{i}{2}}{\frac{i}{2}}\right\}\binom{m+p-i-2}{m-2}$$

$$2B=\sum_{0<\text{odd } i\leq p}\binom{2n+i+1}{i}\binom{m+p-i-2}{m-2}. \qquad \text{(q. e. d.)}$$

From Theorem 6, we can get the maximal integers i, j such that $\bar{w}^p{}_i\neq 0$, $w^p{}_j\neq 0$ Thus, by Theorem 4, we will obtain our non-immersion theorem.

THEOREM 7.

1) ([3] Theorem 1.2) *When p is odd, let*

$$d(m,\ p)=max\{i\,|\,0<i\leq m,\ \binom{\binom{m+p}{p}}{i}\not\equiv 0\ \ (mod\ 2)\},$$

$$s(m,\ p)=max\{i\,|\,0<i\leq m,\ \binom{\binom{m+p}{p}+i-1}{i}\not\equiv 0\ \ (mod\ 2)\}.$$

If k is an integer such that $-d(m, p)<k<s(m, p)$, *then* $RP_m \nsubseteq_p R^{\nu(m,p)+k}$.

2) (Cf. [4] Theorem 1.5) *Let*

$$d_{1,p}=max\{i\,|\,0<i\leqq n,\ \binom{\frac{1}{2}\binom{2n+p}{p-1}+i-1}{i}\not\equiv 0 \pmod 2\},$$

$$d_{2,p}=max\{i\,|\,0<i\leqq n,\ \binom{\frac{1}{2}\binom{2n+p+1}{p}}{i}\not\equiv 0 \pmod 2\}.$$

When p is even, if k is an integer such that $-2d_{1,p}<k<2d_{2,p-1}$, *and when p is odd, if k is an integer such that* $-2d_{2,p}<k<2d_{1,p+1}$, *then* $CP_n \nsubseteq_p R^{\nu(2n,p)+k}$.

3) *Let*

$$b_1=max\{i\,|\,0<i=\alpha+2\beta\leqq m+2n,\ \sum_{0\leqq\gamma\leqq min(\alpha,B-\beta)}\binom{A}{\alpha-\gamma}\frac{B!}{(B-\beta-\gamma)!\gamma!\beta!}\not\equiv 0 \pmod 2\},$$

$$b_2=max\{i\,|\,0<i=\alpha+2\beta\leqq m+2n,\ \sum_{0\leqq\gamma\leqq min(\alpha,2^t-B-\beta)}\binom{A-1+\alpha-\gamma}{\alpha-\gamma}\frac{(2^t-B)!}{(2^t-B-\beta-\gamma)!\gamma!\beta!}\not\equiv 0 \pmod 2\},$$

where A and B are ones in Theorem 6, t is an integer such that $2^t>max\{m, n, B-1\}$. *If k is an integer such that* $-b_1<k<b_2$, *then* $P(m, n)\nsubseteq_p R^{\nu(m+2n,p)+k}$.

PROOF.

2) When p is even, by Theorem 6.2), we have

$$\bar{w}^p(CP_n)=(1+\beta)^{\frac{1}{2}\binom{2n+p}{p-1}},\quad w^p(CP_n)=(1+\beta)^{-\frac{1}{2}\binom{2n+p}{p-1}}.$$

Hence,

$$\max\{i\,|\,0<i\leqq n,\ \bar{w}^p_{2i}(CP_n)\neq 0\}=\max\{i\,|\,0<i\leqq n,\ \binom{\frac{1}{2}\binom{2n+p}{p-1}}{i}\not\equiv 0 \pmod 2\}=d_{2,p-1},$$

$$\max\{i\,|\,0<i\leqq n,\ \bar{w}^p_{2i}(CP_n)\neq 0\}=\max\{i\,|\,0<i\leqq n,\ \binom{\frac{1}{2}\binom{2n+p}{p-1}+i-1}{i}\not\equiv 0 \pmod 2\}=d_{1,p}.$$

Thus, the first half of Theorem 7.2) follows directly from Theorem 4. The latter half of Theorem 7.2) follows also by similar reasons.

3) By Theorem 6.3),

$$w^p(P(m, n))=(1+c)^A(1+c+d)^B,$$

$$\bar{w}^p(P(m, n))=(1+c)^{-A}(1+c+d)^{-B}\equiv(1+c)^{-A}(1+c+d)^{2^t-B} \pmod 2,$$

where $2^t>\max\{m, n, B-1\}$: $(1+c+d)^{2^t}\equiv 1+c^{2^t}+d^{2^t}=1 \pmod 2$, $2^t-B\geqq 0$. Hence, the coefficient of $c^\alpha d^\beta$ in the expansion of $w^p(P(m,n))$ is equal to $\sum_{0\leqq\gamma\leqq min(\alpha,B-\beta)}\binom{A}{\alpha-\gamma}\frac{B!}{(B-\beta-\gamma)!\gamma!\beta!}$, and the coefficient of $c^\alpha d^\beta$ in the expansion of $\bar{w}^p(P(m, n))$ is equal to $\sum_{0\leqq\gamma\leqq min(\alpha,2^t-B-\beta)}\binom{A-1+\alpha-\gamma}{\alpha-\gamma}\frac{(2^t-B)!}{(2^t-B-\beta-\gamma)!\gamma!\beta!}$. Thus, Theorem 7.3) follows similarly to the proof of 2) from Theorem 4. (q. e. d.)

NIIGATA UNIVERSITY

References

1. E. A. FELDMAN, *The geometry of immersions. I*, Trans. Amer. Math. Soc., 120 (1965), 185–224.
2. W. F. POHL, *Differential geometry of higher order*, Topology, 1 (1962), 169–211.
3. H. SUZUKI, *Bounds for dimensions of odd order nonsingular immersions of* RP_n, Trans. Amer. Math. Soc., 121 (1966), 269–275.
4. ———, *Characteristic classes of some order tangent bundles of complex projective spaces*, J. Math. Soc. Japan, 18 (1966), 386–393.
5. J. J. UCCI, *Immersions and embeddings of Dold manifolds*, Topology, 4 (1965), 283–293.