A remark on the embeddability of n-manifolds in (2n-2)-space

By

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1. Introduction

In this note, we shall prove the following

THEOREM 1. *Let M be a closed, smooth simply-connected n-manifold whose homology groups are torsion free. Then the immersibility of M in R^{2n-3} ((2n−3)-dimensional Euclidean space) implies the embeddability of M in R^{2n-2} if $n \geq 7$.*

This is a corollary to the following

THEOREM 2. *Let M be a closed, smooth $(n-k-1)$-connected n-manifold whose homology groups are torsion free. If M is immersible in R^{n+k} with vanishing Euler class, $2k \geq n+3$ and $k \leq n-2$, then M is embeddable in R^{n+k}.*

We shall sketch an outline of our proof. Let M be immersed in R^{n+k} with normal disk bundle ν. Then the total space E of ν is parallelizable manifold and contains M as a submanifold (the image of the zero cross section). Let E_0 be the total space of the restriction of ν to $M_0 = M - \text{int } D^n$. Then E_0 is also parallelizable manifofd and contains M_0 as a submanifold. Note that bM_0 is embedded in bE_0. We kill homotopy groups of E_0 and obtain a contractible manifold C in which M_0 is embedded and bM_0 in bC. We can show that bC is simply connected and hence C is an $(n+k)$-disk. Thus we have an embedding of M_0 in D^{n+k}. By attaching a cone on bM_0 in the complemetary disk to D^{n+k} in S^{n+k}, we have a piecewise linear embedding of M in S^{n+k}. By a result of Haefliger, this is approximated by a differentiable embedding under a suitable assumption on n and k.

This note is motivated by the method of the proof of THEOREM 14 in Wall's paper "Classification Problems in Differential Topology. V On certain 6-manifolds." Invent. Math., 335–374('66).

2. Statements of results

Throughout this section, M denotes a closed, smooth simply-connected n-manifold whose homology groups are torsion free. Let ξ be a k-dimensional vector bundle over M and E the total space of the associated disk bundle and bE the total space of the associated sphere bundle. Let M_0 denote the complement of an embedded open n-disk D^n in M. We define E_0 as the total space of restriction of E to M_0. We shall assume that E is parallelizable and the Euler class of ξ vanishes. By performing surgery on E, we can prove the following proposition·

PROPOSITION 1. *We can construct two sequences of parallelizable manifolds;*

$$C_1=E_0,\ C_2,\ \cdots\cdots,\ C_{k-2},$$

and

$$D_1=E,\ D_2,\ \cdots\cdots,\ D_{k-2},$$

with the properties;

(1) $C_h=C_{h-1}\cup(\bigcup_{i=1}^{r_{h-1}} D^{h+1}\times D_i^{n+k-h-1})$
$D_h=D_{h-1}\cup C_h$

(2) C_h *and* bC_{h-1} *are* h*-connected and* $H_i\ (C_h)=H_i(C_h)=H_i(C_{h-1})$ *for* $i\geq h+1$,

(3) D_h *and* bD_h *are* h*-connected and* $H_i\ (D_h)=H_i\ (D_{h-1})$ *for* $i\geq h+2$,

(4) $H_{h+1}\ (C_h)=H_{h+1}\ (D_h)$

and

(5) bD_h *is* h*-connected and* $H_{h+1}\ (bD_h)$ *has no torsion,*

where $2\leq h\leq k-2$. *Moreover* M *embeds in* D_h *and* $(M_0,\ bM_0)$ *in* $(C_h,\ bC_h)$.

We construct C_h and D_h inductively. From (5), we can choose maps $f_i\ :\ S^{h+1}\to bD_h$ $(i=1,2,\cdots\cdots,r_h)$ representing basis of $H_{h+1}\ (bD_h)$. If $n+k-1\geq 2\ (h+1)+1$, we may assume that f_i's are embeddings with trivial normal bundles and have disjoint images. It is easy to show that $f_i\ (S^{h+1})$ can be pushed into $bD_h\cap C_h$. By attaching handles $D^{h+2}\times D_i^{n+k-h-2}$ to C_h with attaching maps f_i, we construct $C_{h+1}=C_h\cup(\bigcup_{i=1}^{r_h} D^{h+2}\times D_i^{n+k-h-2}.)$

It is known that C_{h+1} is a smooth manifold Moreover C_{h+1} is parallelizable. Define D_{k+1} by $D_h\cup C_{h+1}$, then D_{h+1} is also parallelizable manifold. Using the Mayer-Vietoris exact sequence, we can show that C_{h+1} and D_{h+1} satisfy the properties (1)~(5)

We can show the following

PROPOSITION 2. *The* $(k-1)$*-th homology group of* bD_{k-2} *contains the infinite cyclic group generated by the fiber over the center of the disk which is to be removed from* M *to construct* M_0 *as a direct summand;* $H_{k-1}\ (bD_{k-2})=Z\oplus G$.

By the same arguments as above, we can kill the group G and obtain the following proposition;

PROPOSITION 3. *We can construct smooth manifolds C_{k-1} and D_{k-1} with the properties;*

(1) *C_{k-1} and bC_{k-1} are $(k-1)$-connected and $H_i(C_{k-1}) = H_i(C_{k-2})$ for $i \geq k$,*

(2) *$H_i(D_{k-1}) = H_i(D_{k-2})$ for $i \geq k+1$,*

and

(3) *bD_{k-1} is $(k-2)$-connected and $H_{k-1}(bD_{k-1}) = Z$.*

Since bD_{k-1} is $(k-2)$-connected, the Hurwicz homomorphism $H: \pi_k(bD_{k-1}) \rightarrow H_k(bD_{k-1})$ is an epimorphism. Note that $H_k(bD_{k-1})$ is free. By performing surgery on elements of $H_k(bD_{k-1})$, we can construct a smooth manifold C with the properties;

PROPOSITION 4. *C is k-connected and $H_i(C)$ is isomorphic with $H_i(M_0)$ for $i \geq k+1$. Moreover (M_0, bM_0) is embeddable in (C, bC).*

We shall apply the arguments above to the normal bundle ν of an immersion of M in R^{n+k}, which is assumed to have vanishing Euler class. We can construct a smooth manifold C with the properties of proposition 4. The $(n-k-1)$-connectedness of M implies that all homotopy groups of C vanish. We can show that bC is simply-connected and hence C is an $(n+k)$-disk Thus M_0 is embeddable in D^{n+k} and bM_0 in $bD^{n+k} = S^{n+k-1}$. The desired embedding of M in S^{n+k} is obtained by attaching a cone to $bM_0 = S^{n-1}$ in the complementary disk of C in S^{n+k}. This proves Theorem 2.

3. Surgery on a disk bundle

We use same notations in the preceding section. Moreover we assume that $k \leqq n-2$. By the assumptions, E, E_0, bE and bE_0 are simply-connected and s-parallelizable, and their homology groups have no torsion. It is easy to prove the following lemma;

LEMMA. *Let f be an embedding of S^r in bE such that $f(S^r)$ does not meet the fibre over the center of D^n (the disk which is to removed from M to construct M_0). Then $f(S^r)$ can be pushed into $bE \cap E_0$. Hence if $r < n-1$, then we may assame that $f(S^r) \subset bE \cap E_0$.*

We kill $H_2(bE)$. Since all elements are spherical, we can find maps $f_i: S^2 \longrightarrow bE$ $(i = 1 \cdots\cdots, r_2)$ which represent a base of $H_2(bE)$. If $n+k-1 \geq 5$, we may assume that f_i's are embeddings with trivial normal bundles and their images are disjoint. Let C_2 be the manifold obtained from E_0 by attaching handles $D^3 \times D_i^{n+k-3}$

$(i=1, \cdots\cdots, r_2)$ to $E_0 \cap bE$ by the maps f_i. Note that C_2 is parallelizable manifold. We put $D_2=E \cup C_2$. Using the Mayer-Vietoris exact sequence, we can immeadiately prove the properties (2)~(5) of Proposition 1 for C_2 and D_2.

Suppose that we have constructed manifolds $C_1=E_0$, $C_2, \cdots\cdots$, C_h and $D_1=E$, $D_2, \cdots\cdots$, D_h, with the properties (1)~(5) in Proposition 1 for $2 \leqq h \leqq k-3$. Since all elements of H_{h+1} (bD_h) are spherical, we can find maps f_i: $S^{h+1} \longrightarrow bD_h$ $(i=1, \cdots\cdots, r_h)$ which represent a base of $H_{h+1}(bD_h)$, where $r_h=$rank of H_{h+1} (bD_h). Since $n+k-1>h+k+1$, we may assume that $f_i(S^{h+1}) \subset bD_{h-1} \cap C_h$ by the same argument of Lemma. Thus we can construct $C_{h+1}=C_h \cup (\cup_{i=1}^{r_h} D^{h+2} \times D^{n+k-h-2})$ and $D_{h+1}=D_h \cup C_{h+1}$. It is easy to show (1), (2), (3) and (5). We prove (4). Consider the following commutative diagram;

$$\begin{array}{ccccccccc} H_{h+3}(D_{h+1}, D_h) & \longrightarrow & H_{h+2}(D_h) & \longrightarrow & H_{h+2}(D_{h+1}) & \longrightarrow & H_{h+2}(D_{h+1}, D_h) & \longrightarrow & H_{h+1}(D_h) \\ i_{-2} \uparrow \simeq & & i_{-1} \uparrow & & i_0 \uparrow & & i_1 \uparrow \simeq & & i_2 \uparrow \simeq \\ H_{h+3}(C_{h+1}, C_h) & \longrightarrow & H_{h+2}(C_h) & \longrightarrow & H_{h+2}(C_{h+1}) & \longrightarrow & H_{h+2}(C_{h+1}, C_h) & \longrightarrow & H_{h+1}(C_h) \end{array}$$

By excision, $H_i(D_{h+1}, D_h)=H_i(C_{h+1}, C_h)$ for all i. From the following commutative pagarm, in which all homomorphisms are induced by inclusion and all homomorphisms except i_{-1} are isomorphisms, it follows that i_{-1} is an isomorphism.

$$\begin{array}{ccccccc} H_{h+2}(D_h) & \longleftarrow & H_{h+2}(D_{h-1}) & \longleftarrow \cdots\cdots \longleftarrow & H_{h+2}(E) \\ i_{-1} \uparrow & & & & \uparrow \simeq \\ H_{h+2}(C_h) & \longleftarrow & H_{h+2}(C_{h-1}) & \longleftarrow \cdots\cdots \longleftarrow & H_{h+2}(E_0). \end{array}$$

By 5-lemma, we have $H_{h+2}(D_{h+1})=H_{h+2}(C_{h+1})$ (note that i_2 is an isomorphism). This completes the proof of Proposition 1.

We shall prove Proposition 2. Write $Y=bD_{k-2} \cap C_{k-2}$, and $Y=bC_{k-2}-S^{n-1} \times D^k =bD_{k-2}-D^n \times S^{k-1}$. Consider the Mayer-Vietoris exact sequence;

$$0 \longrightarrow H_k(Y) \longrightarrow H_k(bC_{k-2}) \longrightarrow H_{k-1}(S^{n-1} \times S^{k-1}) \longrightarrow H_{k-1}(Y) \longrightarrow H_{k-1}(bC_{k-2}) \longrightarrow 0.$$

Since $H_k(bC_{k-2})$ is free, $H_k(Y)$ is also free and has the same rank as $H_k(bC_{k-2})$. Hence we have an exact sequence;

$$0 \longrightarrow H_{k-1}(S^{n-1} \times S^{k-1}) \longrightarrow H_{k-1}(Y) \longrightarrow H_{k-1}(bC_{k-2}) \longrightarrow 0.$$

Since $H_{k-1}(bC_{k-2})$ is free, we have $H_{k-1}(Y)=H_{k-1}(S^{n-1} \times S^{k-1}) \oplus H^{k-1}(bC_{k-2})$. From the Mayer-Vietoris exact sequence;

$$0 \longrightarrow H_{k-1}(S^{n-1} \times S^{k-1}) \longrightarrow H_{k-1}(Y) \oplus H_{k-1}(D^n \times S^{k-1}) \longrightarrow H_{k-1}(bD_{k-2}) \longrightarrow 0,$$

and the fact that $H_{k-1}(bD_{k-2})$ is free, we have $H_{k-1}(bD_{k-2})=Z \oplus G$, where Z is generated by the fibre over the center of the disk and G is isomorphic with H_{k-1} (bC_{k-2}). This completes the proof of Proposition 2. By selecting embeddings f_i: $S^{k-1} \longrightarrow bD_{k-2} \cap C_{k-2}$ with trivial normal bundles, which represent a base of G, we construct smooth manifolds C_{k-1} and D_{k-1} as before. We shall prove Proposition 3. It is not difficult to show (1) and (2). Write $Y=bD_{k-1}\cap C_{k-1}$. Applying the Mayer-Vietoris exact sequence to $bC_{k-1}=Y\cup S^{n-1}\times D^k$, we have $H_{k-1}(Y)=Z$ (note tht $H_k(Y)\longrightarrow H_k(bC_{k-1})$ is an epimorphism). Since $H_i(bD_{k-1}, Y)=H_i(D^n\times S^{k-1}, S^{n-1}\times S^{k-1})$, we have $H_{k-1}(Y)=H_{k-1}(bD_{k-1})$ and hence $H_{k-1}(bD_{k-1})=Z$. This implies Proposition 3.

Finally we shall prove Proposition 4. Since $C=C_{k-1}\cup(\cup_{i=1}^{rk}D^{h+1}\times D_i^{n-1})$ by the Mayer-Vietoris exact sequence, we have $H_i(C)=0$ for $i\leqq k$-1 and $H_i(C)=H_i(C_{k-1})$ for $i\geqq k+2$. Consider the exact sequence;

$$0\longrightarrow H_{k+1}(C_{k-1})\longrightarrow H_{k+1}(C)\longrightarrow H_k(\cup_i S^k\times D_i^{n-1})\xrightarrow{j_*} H_k(C_{k-1})\longrightarrow H_k(C)\longrightarrow 0.$$

We show that j_* is an isomorphism. In fact, we consider the following commutative diagram;

$$\begin{array}{ccccc} & \swarrow & H_k(\cup_i S^k\times D_i^{n-1}) & \xrightarrow{j_*} & H_k(C_{k-1}) \\ H_k(bD_{k-1}) & \cong & \downarrow & & \uparrow\cong \\ & \nwarrow i_* & H_k(bD_{k-1}\cap C_{k-1}) & \xrightarrow{j'_*} & H_k(bC_{k-1}). \end{array}$$

Since $H_i(bD_{k-1}, bD_{k-1}\cap C_{k-1})=H_i(D^n\times S^{k-1}, S^{n-1}\times S^{k-1})$, if $k\leqq n$-2, i_* is an isomorphism. From the exact sequence;

$$0\longrightarrow H_k(bD_{k-1}\cap C_{k-1})\xrightarrow{j_*} H_k(bC_{k-1})\longrightarrow H_k(S^{n-1}\times D^k, S^{n-1}\times S^{k-1})$$
$$\longrightarrow H_{k-1}(bD_{k-1}\cap C_{k-1})\longrightarrow 0,$$

it follows that j'_* is an isomorphism and hence j_* is an isomophism. Therefore $H_k(C)=0$, and hence $H_i(C)=H_i(C_{k-1})$ for $i\geqq k+1$.

4. Application

In this section, we shall prove Theorem 2 and hence Theorem 1. Let M be a closed, smooth simply-connected manifold whose homology groups have no torsion. Suppose that M be immersed in R^{n+k} with normal bundle ν whose Euler class vanishes. Applying the arguments in Section 3 to the associated disk bundle, we have a smooth manifold C with boundary with following properties;

(1) (M_0, bM_0) is embeddable in (C, bC)

(2) C is k-connected and $H_i(C)=H_i(M_0)$ for $i\geqq k+1$,

and

(3) bC is simply-connected.

If M is $(n-k-1)$-connected, then $H_i(C)=0$ for all i and hence C is an $(n+k)$-disk $(n+k\geqq 6)$. We obtain a piecewise linear embedding of M in S^{n+k} by the method mentioned in Introduction. By a result of Haefliger, if $2k\geqq n+3$, this embedding can be approximated by a differentiable one. Thus we have Theorem 2. It is easy to deduce Theorem 1 from Theorem 2.

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