# A characterization of almost analytic tensor fields in almost complex manifolds

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The purpose of the present paper is to give a characterization of almost analytic tensor fields in an almost complex manifold. For a contravariant vector field, it is defined as a vector field $X$ such that it makes the structure tensor field $F$ invariant i. e., $\pounds_X F=0$ where $\pounds_X$ denotes the operator of Lie differentiation with respect to $X$.

Our characterization of a contravariant almost analytic vector field is different from the above one.

In §1 we shall give the definition of pure tensor fields. In §2 we shall state some properties about the complete lift of tensor fields introduced by K. Yano and S. Kobayashi [9] and Lie derivative. In §3, in terms of components, we shall deal with almost analytic tensor fields and prove an important lemma by which almost analytic tensor fields can be characterized. §4 will be devoted to a characterization of almost analytic tensor fields. Some applications will be given in the last section.

## 1. Pure tensor fields

Throughout this paper by $M$ we shall always mean an n-dimensional almost complex manifold with a local coordinate $(x^1,\cdots,x^n)$ and by $F$ a structure tensor field of type $(1,1)$. Following notations and terminologies of [9] let $T(M)=\bigcup_{x\in M} T_x(M)$ be the tangent bundle over $M$ and let $T_q{}^p(M)$ be the space of tensor fields of type $(p, q)$, i. e., contravariant degree $p$ and covariant degree $q$, on $M$. We put

$$\mathcal{T}(M)\equiv\sum_{p,q}\mathcal{T}_q{}^p(M)=\mathcal{T}^*(M)\otimes\mathcal{T}_*(M)$$

where $\quad \mathcal{T}^*(M)\equiv\sum_p \mathcal{T}_0{}^p(M)$ and $\mathcal{T}^*(M)\equiv\sum_q \mathcal{T}_q{}^0(M)$.

An element of $\mathcal{T}_0{}^1(M)$ is a vector field and is denoted by $X$, $Y$ or $Z$.
An element of $\mathcal{T}_1{}^0(M)$ is a 1-form and is denoted by $X^*$, $Y^*$ or $Z^*$.
An element of $\mathcal{T}_0{}^0(M)$ is a function (of $C^\infty$) and is denoted by $f$.

Of course, our structure tensor field $F$ is an element of $\mathcal{T}_1^1(M)$ and since it can be considered as an endomorphism of $\mathcal{T}_0^1(M)$, we shall denote its image of $X\in\mathcal{T}_0^1(M)$ by $FX$. Similarly we shall define $FX^*$ for $X^*\in\mathcal{T}_1^0(M)$ (cf. § 4). An element $T$ of $\mathcal{T}_q^p(M)$ is a tensor field of type $(p, q)$ and can be considered as a multilinear mapping of

$$\mathcal{T}_0^1(M)\times\cdots\times\mathcal{T}_0^1(M)\times\mathcal{T}_1^0(M)\times\cdots\times\mathcal{T}_1^0(M) \text{ into } \mathcal{T}_0^0(M)$$
$$(\mathcal{T}_1^0(M)\ p \text{ times},\ \mathcal{T}_0^1(M)\ q \text{ times}).$$

Then we shall define a pure tensor field as follows. If a tensor field $T$, say, of type(1,2) satisfies

$$T(X, Y, Z^*)=-T(FX, FY, Z^*)=-T(X, FY, FZ^*) \quad (1.1)$$
$$\text{for any } X, Y\in\mathcal{T}_0^1(M), \text{ any } Z^*\in\mathcal{T}_1^0(M),$$

then we shall call $T$ a pure tensor [8].

In terms of components, (1.1) is equivalent to

$$T_{jk}{}^i=-F_j{}^aF_k{}^bT_{ab}{}^i=-F_k{}^aF_b{}^iT_{ja}{}^b \quad (1.2)$$

or

$$F_j{}^aT_{ak}{}^i=F_k{}^aT_{ja}{}^i=F_a{}^iT_{jk}{}^a. \quad (1.3)$$

## 2. Lie derivations and complete lifts

Let $X$ and $Y$ be two elements in $\mathcal{T}_0^1(M)$. The Lie derivative with respect to $X$ has the following properties:

$$\pounds_X(S\otimes T)=(\pounds_X S)\otimes T+S\otimes(\pounds_X T) \quad \text{for } S, T\in T(M), \quad (2.1)$$

$$\pounds_X f=Xf \quad \text{for } f\in\mathcal{T}_0^0(M), \quad (2.2)$$

$$\pounds_X df=d\pounds_X f \quad \text{for } f\in\mathcal{T}_0^0(M), \quad (2.3)$$

$$\pounds_X Y=[X, Y], \quad (2.4)$$

$$[\pounds_X, \pounds_Y]=\pounds_{[X,Y]}, \quad (2.5)$$

$$\pounds_X\omega=d\circ\iota_X\omega+\iota_X\circ d\omega \quad \text{for any differential form } \omega \quad (2.6)$$

where $d$ denotes exterior differentiation and $\iota X$ skew-derivation.

Owing to K. Yano and S. Kobayashi [9], the complete lift is a linear mapping of $\mathcal{T}(M)$ into $\mathcal{T}(T(M))$ and if we denote the complete lift of $T\in\mathcal{T}(M)$ by $T^c$, then we have the following properties:

$$\pounds_{X^c}T^c=(\pounds_X T)^c \quad \text{for } T\in\mathcal{T}_q^p(M), \quad (2.7)$$

$$(FT)^c=F^cT^c \text{ for a pure tensor field } T\in\mathcal{T}_q^p(M) \text{ (for } FT \text{ see § 4)}, \quad (2.8)$$

$$T^c(X_1{}^c,\cdots,X_q{}^c,X_1{}^{*c},\cdots,X^*{}_p{}^c)=(T(X_1,\cdots,X_q,X_1{}^*,\cdots,X_p{}^*))^c \quad (2.9)$$

where $X_i \in \mathcal{T}_0^1(M)$, $X_i^* \in \mathcal{T}_1^0(M)$, $T \in T_q^p(M)$.

By (1.1) and (2.9), we have the following

Proposition. *If $T$ in an almost complex manifold with a structure tensor field $F$ is a pure tensor field, then $T^c$ is also a pure tensor field in $T(M)$ with the almost complex structure tensor field $F^c$.*

## 3. Almost analytic tensor fields

In this section, we shall prove a lemma which is essential for our characterization of almost analytic tensor fields.

Let $T$ be a pure tensor field of type $(p, q)$ and in terms of components, we shall denote $T$ and $F$ by

$$T = T_{j_q \cdots j_1}{}^{i_p \cdots i_1} \frac{\partial}{\partial x^{i_p}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_1}} \otimes dx^{j_q} \otimes \cdots \otimes dx^{j_1} \text{ and } F = F_j{}^i \frac{\partial}{\partial x^i} \otimes dx^j$$

respectively.

Now Tachibana's operator $\Phi_h$ is defined for $T$ as follows

$$\Phi_h T_{j_q \ldots j_1}{}^{i_p \cdots i_1} = F_h{}^r \partial_r T_{j_q \ldots j_1}{}^{i_p \cdots i_1} - \partial_h \tilde{T}_{j_q \ldots j_1}{}^{i_p \cdots i_1} + \sum_{r=1}^{q} (\partial_{j_r} F_h{}^l) T_{j_q \ldots l \ldots j_1}{}^{i_p \cdots i_1} + \sum_{r=1}^{p} (\partial_h F_l{}^{i_r} - \partial_l F_h{}^{i_r}) T_{j_q \ldots j_1}{}^{i_p \cdots l \cdots i_1} \tag{3.1}$$

where $\tilde{T}_{j_q \ldots j_1}{}^{i_p \cdots i_1} = F_l{}^{i_r} T_{j_q \ldots j_1}{}^{i_p \cdots l \cdots i_1} = F_{j_s}{}^l T_{j_q \ldots l \ldots j_1}{}^{i_p \cdots i_1} (r = 1, \cdots, p,\ s = 1, \cdots, q)$.

If $\Phi_h T_{j_q \ldots j_1}{}^{i_p \cdots i_1} = 0$, then $T$ is called an almost analytic tensor field or $T$ is almost analytic. This is a generalization of analytic tensor fields in a Kählerian manifold to an almost complex manifold [1], [4], [7].

First of all, we consider two vector fields $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$. By the definition of Lie derivation, we have

$$\begin{aligned} &\mathcal{L}_X \tilde{Y}^i - \mathcal{L}_{\tilde{X}} Y^i \\ &= X^r \partial_r (F_t{}^i Y^t) - F_t{}^s Y^t \partial_s X^i - [(F_t{}^r X^t) \partial_r Y^i - Y^t \partial_t (F_s{}^i X^s)] \\ &= Y^t [X^r \partial_r F_t{}^i - F_t{}^s \partial_s X^i + F_s{}^i \partial_t X^s] \\ &\qquad - X^t [F_t{}^r \partial_r Y^i - \partial_t (F_r{}^i Y^r) + (\partial_t F_l{}^i - \partial_l F_t{}^i) Y^l] \\ &= (\mathcal{L}_X F_t{}^i) Y^t - X^t \Phi_t Y^i \end{aligned} \tag{3.2}$$

from which we have

$$\mathcal{L}_X (F_t{}^i Y^t) - (\mathcal{L}_X F_t{}^i) Y^t = \mathcal{L}_{\tilde{X}} Y^i - X^t \Phi_t Y^i \tag{3.3}$$

which is equivalent to

$$F_t{}^i \mathcal{L}_{\tilde{X}} Y^t = \mathcal{L}_{\tilde{X}} \tilde{Y}^i - X^t \Phi_t Y^i. \tag{3.4}$$

Consequently, if $Y$ is a contravariant almost analytic vector field, we have

$$F_t{}^i \mathcal{L}_X Y^t = \mathcal{L}_{\tilde{X}} Y^i. \tag{3.5}$$

Conversely, if (3.5) holds good for any $X \in \mathcal{T}_0^1(M)$, $Y$ is contravaяiant almost analytic. In fact, from (3.4), we have $X^t \Phi_t Y^i = 0$ i.e., $\Phi_t Y^i = 0$.

Secondly, we consider a covariant pure tensor field of type (o, $q$) $T = T_{j_q \ldots j_1} dx^{j_q} \otimes \cdots \otimes dx^{j_1}$. Just as for (3.1), we have

$$\begin{aligned}
&\mathcal{L}_X \tilde{T}_{j_q \ldots j_1} - \mathcal{L}_{\tilde{X}} T_{j_q \ldots j_1} \\
&= X^s \partial_s \tilde{T}_{j_q \ldots j_1} + \sum_{r=1}^{q} \tilde{T}_{j_q \ldots s \ldots j_1} \partial_{j_r} X^s - (\tilde{X}^s \partial_t T_{j_q \ldots j_1} + \sum_{r=1}^{q} T_{j_q \ldots t \ldots j_1} \partial_{j_r} \tilde{X}^t) \\
&= X^s \partial_s \tilde{T}_{j_q \ldots j_1} + \sum_{r=1}^{q} \tilde{T}_{j_q \ldots s \ldots j_1} \partial_{j_r} X^s - [F_s{}^t X^s \partial_t T_{j_q \ldots j_1} + \sum_{r=1}^{q} T_{j_q \ldots t \ldots j_1} (\partial_{j_r} F_s{}^t) X^s \\
&\qquad + \sum_{r=1}^{q} T_{j_q \ldots t \ldots j_1} F_s{}^t \partial_{j_r} X^s] \\
&= -X^s (F_s{}^t \partial_t T_{j_q \ldots j_1} - \partial_s \tilde{T}_{j_q \ldots j_1} + \sum_{r=1}^{q} T_{j_q \ldots t \ldots j_1} \partial_{j_r} F_s{}^t) = -X^s \Phi_s T_{j_q \ldots j_1}.
\end{aligned} \tag{3.6}$$

Accordingly if $T$ is covariant almost analytic, then from (3.6), it follows that

$$\mathcal{L}_X \tilde{T}_{j_q \ldots j_1} = \mathcal{L}_{\tilde{X}} T_{j_q \ldots j_1}. \tag{3.7}$$

Conversely, if (3.7) holds good for any $X \in \mathcal{T}_0^1(M)$, then the pure tensor field $T$ becomes an almost anlytic tensor field.

Finally, we consider a pure tensor field of type ($p$, $q$)

$$T = T_{j_q \ldots j_1}{}^{i_p \cdots i_1} \frac{\partial}{\partial x^{i_p}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_1}} \otimes dx^{j_q} \otimes \cdots \otimes dx^{j_1}.$$

For this tensor field, from the forms of (3.2) and (3.6), it will be expected that the following equation holds good for any vector field $X \in \mathcal{T}_0^1(M)$, i.e.,

$$\mathcal{L}_X \tilde{T}_{j_q \ldots j_1}{}^{i_p \cdots i_1} - \mathcal{L}_{\tilde{X}} T_{j_q \ldots j_1}{}^{i_p \cdots i_1} = \sum_{r=1}^{p} (\mathcal{L}_X F_s{}^{i_r}) T_{j_q \ldots j_1}{}^{i_p \cdots s \cdots i_1} - X^s \Phi_s T_{j_q \ldots j_1}{}^{i_p \cdots i_1}. \tag{3.8}$$

Indeed, it is verified by straightforward calculation but since it is complicated, we shall not go into it.

Thus we have the following

LEMMA. *In an almost complex manifold, a pure tensor field $T$ of type $(p, q)$ is almost analytic if and only if it satisfies*

$$\mathcal{L}_X \tilde{T}_{j_q \ldots j_1}{}^{i_p \cdots i_1} = \mathcal{L}_{\tilde{X}} T_{j_q \ldots j_1}{}^{i_p \cdots i_1} + \sum_{r=1}^{p} (\mathcal{L}_X F_s{}^{i_r}) T_{j_q \ldots j_1}{}^{i_p \cdots s \cdots i_1}$$

*for any vector field $X \in \mathcal{T}_0^1(M)$.*

## 4. A characterization of almost analytic tensor fields

We now define the notation of contraction $C_{(1)}{}^{(r)}$ for two tensor fields $S = S_j{}^i \frac{\partial}{\partial x^i} \otimes dx^j$ and $T = T_{j_q \ldots j_1}{}^{i_p \cdots i_1} \frac{\partial}{\partial x^{i_p}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_1}} \otimes dx^{j_q} \otimes \cdots \otimes dx^{j_1}$ as follows

$$C_{(1)}^{(r)}S\otimes T$$
$$\equiv S_j{}^i T_{j_q\ldots j_1}{}^{i_p\cdots i_1}\left\langle\frac{\partial}{\partial x^{i_r}},\ dx^j\right\rangle\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^{i_p}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_{r+1}}}\otimes\frac{\partial}{\partial x^{i_{r-1}}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_1}}$$
$$\otimes dx^{j_q}\otimes\cdots\otimes dx^{j_1}$$

where (1) of $C_{(1)}^{(r)}$ denotes the 1-st covariant index of $S_j{}^i$ and $(r)$ denotes the $r$-th contravariant index of $T_{j_q\ldots j_1}{}^{i_p\cdots i_1}$. If $S=F$ and $T$ is a pure tensor field, then by (1. 3), $C_{(1)}^{(r)}F\otimes T$ $(r=1, 2, \cdots p)$ are all the same tensor field, so we denote it briefly by $FT$. Similarly for a covariant tensor field $T=T_{j_q\ldots j_1}dx^{j_q}\otimes\cdots\otimes dx^{j_1}$, we define as follows

$$C^{(1)}{}_{(r)}S\otimes T\equiv S_j{}^iT_{j_q\ldots j_1}\left\langle\frac{\partial}{\partial x^i},\ dx^{j_r}\right\rangle dx^j\otimes dx^{j_q}\otimes\cdots\otimes dx^{j_{r+1}}\otimes dx^{j_{r-1}}\otimes\cdots\otimes dx^{j_1}$$

but when $S=F$ and $T$ is a pure tensor field, for the same reason, we denote it briefly by $FT$.

Then, by virtue of Lemma in § 3, we can characterize almost analytic tensor fields in an almost complex manifold $M$ with a structure tensor field $F$ as in the following.

(1) A contravariant almost analytic vector field $Y$ is a vector field satisfying

$$\mathcal{L}_XFY=\mathcal{L}_{FX}Y+(\mathcal{L}_XF)Y \qquad \text{for any } X\in\mathcal{T}_0{}^1(M)$$

which is equivalent to

$$F[X,Y]=[FX,Y] \qquad \text{for any } X\in\mathcal{T}_0{}^1(M).$$

(2) A covariant almost analytic tensor field $T$ is a pure tensor field of type (o, $q$) satisfying

$$\mathcal{L}_XFT=\mathcal{L}_{FX}T \qquad \text{for any } X\in\mathcal{T}_0{}^1(M).$$

(3) An almost analytic tensor field of mixed type $T$ is a pure tensor field of type $(p, q)$ satisfying

$$\mathcal{L}_XFT=\mathcal{L}_{FX}T+\sum_{r=1}^{p}C_{(1)}^{(r)}(\mathcal{L}_XF)\otimes T \qquad \text{for any } X\in\mathcal{T}_0{}^1(M).$$

Remark. For an almost analytic tensor field $T$ of type $(p, q)$ and a contravariant almost analytic vector field $X$, from (3), it follows that

$$\mathcal{L}_XFT=\mathcal{L}_{FX}T.$$

Hence if a pure tensor field $T$ of type $(p,q)$ satisfies this equation for any contravariant almost analytic vector field $X$, then $T$ is an almost analytic tensor field.

## 5. Applications

(1) It is well known that the set of all contravariant almost analytic vector fields is involutive. Indeed, let $Y$ and $Z$ be two contravariant almost analytic

vector fields, then by Jacobi's identity, the following equation:

$$F[X,[YZ]]=[FX,[YZ]] \qquad \text{for any } X\in\mathcal{T}_0^1(M)$$

can be easily verified and hence $[Y, Z]$ is almost analytic. This also follows from (2.5).

(2) If a covariant tensor field $T$ of type $(o, q)$ is almost analytic, by (2) in § 4, we have

$$\pounds_X FT=\pounds_{FX}T \qquad \text{for any } X\in\mathcal{T}_0^1(M).$$

But since $F(FT)=-T$, if we put $Y=FX$, then we have

$$\pounds_X F(FT)-\pounds_{FX}FT=\pounds_{FY}T-\pounds_Y FT=0.$$

Hence $FT$ is also almost analytic [7].

(3) Since $\pounds_X$ commutes with every contraction of a tensor field, if a covariant tensor field $T$ of type $(o, q)$ and a contravariant vector field $Y$ are both almost analytic, then we have

$$\pounds_X F\iota_Y T-\pounds_{FX}\iota_Y T=\iota_Y(\pounds_X FT-\pounds_{FX}T)=0 \quad \text{for any } X\in\mathcal{T}_0^1(M).$$

Hence $\iota_Y T$ is also almost analytic

In particular, suppose $M$ is an *$O$-space, a kind of an almost Hermitian manifold (cf. [1]).

It is known that in a compact *$O$-space a skew-symmetric covariant almost analytic tensor field is closed [5]. Consequently in a compact *$O$-space, for a skew-symmetric covariant almost analytic tensor field $T$ and a contravariant almost analytic vector field $X$, we have immediately
$\pounds_X T=0$ by virtue of (2.6) for $T$ [5].

In this case, although $FX$ is not necessarily analytic, if we notice that $FT$ is skew-symmetric, from (2) in § 4, it follows that $\pounds_{FX}T=0$.

(4) Let $T$ be an almost analytic tensor field of type $(p, q)$ in an almost complex manifold $M$, then from (3) in § 4, we have

$$\pounds_X FT=\pounds_{FX}T+\sum_{r=1}^{q}C_{(1)}{}^{(r)}(\pounds_X F)\otimes T \qquad \text{for any } X\in\mathcal{T}_0^1(M)$$

from which, by (2.7) and (2.8), we have

$$\pounds_{X^c}F^cT^c=\pounds_{F^cX^c}T^c+\sum_{r=1}^{q}C_{(1)}{}^{(r)}(\pounds_{X^c}F^c)\otimes T^c \quad \text{for any } X\in\mathcal{T}_0^1(M).$$

On the other hand, by virtue of Proposition in § 2, $T^c$ is also a pure tensor field of the same type in the tangent bundle $T(M)$.

Accordingly, it follows that the complete lift of an almost analytic tensor field of type $(p, q)$ in $M$ is also almost analytic in $T(M)$. For a contravariant vector field see [3].

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