On the non-embeddability of Dold's manifold II

By

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1. Introduction

Let $P(m,n)$ be Dold's manifold of type (m,n) (See [5]). The purpose of this paper is to prove the non-embeddability theorem of $P(m,n)$ in the euclidean space R^k.

Let $w, \overline{w}$ and $\overline{w_i}$ be the total, dual total and dual i-dimensional Stiefel-Whitney class of $P(m,n)$, respectively. By Dold [2] we know that:
(1) $H^*(P(m,n))$, the cohomology ring of the coefficient Z_2 of $P(m,n)$, is generated by $c\in H^1(P(m,n))$ and $d\in H^2(P(m,n))$ such that $c^{m+1}=0$, $d^{n+1}=0$ and $Sq^1d=cd$.
(2) $w=(1+c)^m(1+c+d)^{n+1}$.

We consider all pairs of two non-negative integers (except for $(m,n)=(0,0)$). From (1), (2) and $w\overline{w}=1$, we obtain
(3) $\overline{w}=(1+c)^{-(m+1)}$ for $(m,n)=(m,0)$ $(m>0)$
and
(4) $\overline{w}=(1+c)^{-m}(1+c+d)^{-(2^t+r+1)}$ for $(m,n)=(m,2^t+r)$ $(m\geqq 0, 0\leqq r<2^t, t\geqq 0)$.
By the expansion of (3), (4) we will obtain Theorem A in §2. The easy calculations of $\overline{w}$ in the proof of Theorem A was suggested to the author by the method which has used in Ucci [4].

In §3, we will prove Theorem B which improve slightly the results of Theorem A with a few exceptions. In the proof of Theorem B, we will use the same method with it in Massey [3], Andô [1], and Yoshioka [5]. This theorem is a generalization of Theorem in [5] p4.

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2. Theorem A and its proof

We will denote by M the maximal number of integers i such that $\overline{w_i}\neq 0$. We have the well-known theorem:

"If an n-dimensional differentiable manifold M^n is embeddable in R^{n+k}, then

$\overline{w_i}(M^n)=0$ for $i\geqq k$."

Hence, it follows that to know the value of M means to get the best result which is obtainable by this theorem. We will know the value of M by the expansion of (3), (4).

In the calculation of binomial coefficients, we use frequently the following fact: "If $a=\sum a_i 2^i$, $b=\sum b_i 2^i$ $(0\leqq a_i, b_i<2)$, then

$$\binom{a}{b}\equiv \prod_i \binom{a_i}{b_i} \pmod 2."$$

THEOREM A.

(1) *If* $(m, n)=(m, 0)$, *then* $M=m-2q-1$; *where* $m=2^s+q$ $(0\leqq q<2^s, s\geqq 0)$.

(2) *If* $(m, n)=(m, 2^t+r)$ $(m\geqq 0, 0\leqq r<2^t, t\geqq 0)$, *then* $M=m+2n-2q-4r-2$; *where* q *is the integer which is defined uniquely as follows:*

i) *when* $m=0$, $q=0$,

ii) *when* $0<m<2^{t+1}$, $m=2^s+q$ $(0\leqq q<2^s, 0\leqq s\leqq t)$,

iii) *when* $m\geqq 2^{t+1}$, *let* $m=\sum_{i=1}^{k} 2^{t+i}+q'$ $(k\geqq 1, 0\leqq q'<2^{t+k+1})$,

a) *if* $q'=0$, $q=\sum_{i=1}^{k} 2^{t+i-1}(=m/2)$,

b) *if* $0<q'<2^{t+1}$, $q'=2^s+q$ $(0\leqq q<2^s, 0\leqq s\leqq t)$,

c) *if* $q'\geqq 2^{t+1}$, $q'=2^t+q$.

Hence,

(3) $P(m, 0)\not\subset$ (*cannot be embedded*) $R^{2m-2q-1}$ *for* m *in* (1).

(4) $P(m, n)\not\subset R^{2(m+2n)-2q-4r-2}$ *for* (m, n) *in* (2).

PROOF.

(1) From §1. (3),

$$\overline{w}=(1+c)^{-(2^s+q+1)}=(1+c)^{2^s-(q+1)} \quad (\text{Because } (1+c)^{2^{s+1}}=1)$$

$$=(1+c)^{m-2q-1}=1+\cdots\cdots+c^{m-2q-1}.$$

(2) i) From §1. (4), similarly to (1),

$$\overline{w}=(1+d)^{2^t-(r+1)}=1\cdots\cdots+d^{n-2r-1}.$$

ii) Since $(1+c)^{-2^{t+1}}=1$, from §1. (4),

$$\overline{w}=(1+c)^{-m}(1+c+d)^{2^t-(r+1)}=\sum \varepsilon_{\alpha,\beta}c^{m-\alpha}d^{2^t-(r+1)-\beta}$$

where

$$\varepsilon_{\alpha,\beta}=\sum_{0\leqq\gamma\leqq\min(m-\alpha,\beta)}\binom{(m-1)+(m-\alpha-\gamma)}{m-\alpha-\gamma}\frac{(2^t-(r+1))!}{(\beta-\gamma)!\gamma!(2^t-(r+1)-\beta)!}=0 \text{ or } 1$$

and the sum $\sum$ is taken for the integers α, β such that $0\leqq\alpha\leqq m$, $0\leqq\beta\leqq 2^t-(r+1)$.

If $\varepsilon_{\alpha,\beta}=1$ $(\beta>0)$, then there exists at least one γ such that

$$\binom{(m-1)+(m-\alpha-\gamma)}{m-\alpha-\gamma}=1 \quad (0\leqq\gamma\leqq\min(m-\alpha,\ \beta)).$$

But, for this γ,

$$\varepsilon_{\alpha+\gamma,0}=\binom{(m-1)+m-(\alpha+\gamma)}{m-(\alpha+\gamma)}=1$$

and dim $(c^{m-\alpha}d^{2^t-(r+1)-\beta})<\dim(c^{m-\alpha-\gamma}d^{2^t-(r+1)})$ (Because $0<2\beta-\gamma$). Thus, to obtain the value of M, it suffices to consider $\varepsilon_{\alpha,0}$ only. We set $\varepsilon_{\alpha,0}=\varepsilon_\alpha$, and look for the minimal number of integers α such that $\varepsilon_\alpha=1$.

Now,

$$\varepsilon_{2q}=\binom{(m-1)+(m-2q)}{m-2q}=\binom{2^{s+1}-1}{2^s-q}=1$$

and

$$\varepsilon_\alpha=\binom{(m-1)+(m-\alpha)}{m-\alpha}=\binom{2^{s+1}+2q-\alpha-1}{2^s+q-\alpha} \quad (0\leqq\alpha<2q).$$

Since $0\leqq 2q-\alpha-1<2^{s+1}$ and $2^s+q-\alpha<2^{s+1}$, if $\varepsilon_\alpha=1$, then $2q-\alpha-1\geqq 2^s+q-\alpha$. But, $q<2^s+1$. Hence, $\varepsilon_\alpha=0$. Therefore, we have $M=m-2q+2(n-2r-1)$.

iii) From §1. (4),

$$\overline{w}=(1+c)^{-(2^{t+1}+m)}(1+c+d)^{2^t-(r+1)}=\sum\varepsilon_{\alpha,\beta}c^{m-\alpha}d^{2^t-(r+1)-\beta}$$

where

$$\varepsilon_{\alpha,\beta}=\sum_{0\leqq\gamma\leqq\min(m-\alpha,\ \beta)}\binom{(2^{t+1}+m-1)+(m-\alpha-\gamma)}{m-\alpha-\gamma}\frac{(2^t-(r+1))!}{(\beta-\gamma)!\gamma!(2^t-(r+1)-\beta)!}$$

$=0$ or 1

and the sum $\sum$ is taken for the integers α, β such that $0\leqq\alpha\leqq m$, $0\leqq\beta\leqq 2^t-(r+1)$. Similarly to ii), to obtain the value of M, it suffices to consider

$\varepsilon_\alpha=\varepsilon_{\alpha,0}=\binom{2^{t+1}+2m-\alpha-1}{m-\alpha}$ only.

a) Since $2q=m$,

$$\varepsilon_{2q}=\binom{2^{t+1}+m-1}{0}=1.$$

And,

$$\varepsilon_\alpha=\binom{\sum_{i=1}^{k+1}2^{t+i}-\alpha-1}{\sum_{i=1}^{k}2^{t+i}-\alpha}=\binom{2^{t+k+1}+\sum_{i=1}^{k}2^{t+i}-\alpha-1}{\sum_{t=i}^{k}2^{t+i}-\alpha} \quad (0\leqq\alpha\leqq 2q).$$

Since $0\leqq\sum_{i=1}^{k}2^{t+i}-\alpha-1<2^{t+k+1}$ and $\sum_{i=1}^{k}2^{t+i}-\alpha<2^{t+k+1}$, if $\varepsilon_\alpha=1$, then $\sum_{i=1}^{k}2^{t+i}-\alpha-1\geqq\sum_{i=1}^{k}2^{t+i}-\alpha$. But, it is impossible. Hence, $\varepsilon_\alpha=0$. Therefore, we have $M=2(n-2r-1)$.

b) Since

$$\varepsilon_\alpha=\binom{\sum_{i=1}^{k+1}2^{t+i}+2^{s+1}+2q-\alpha-1}{\sum_{i=1}^{k}2^{t+i}+2^s+q-\alpha},$$

similarly to ii), we have $\varepsilon_{2q}=1$, $\varepsilon_\alpha=0$ for $0\leqq\alpha<2q$. Thus, $M=m-2q+2(n-2r-1)$.

c) Let $q=\sum_{i=1}^{h}2^{t+i-1}+q''$ $(0\leqq q''<2^{t+h},\ 1\leqq h\leqq k)$. Then $m=\sum_{i=1}^{k}2^{t+i}+2^{t+h}+q''$. Now,

$$\varepsilon_{2q}=\binom{2^{t+1}+\sum_{i=1}^{k}2^{t+i+1}+2^{t+h+1}-\sum_{i=1}^{h}2^{t+i}-1}{\sum_{i=1}^{k}2^{t+i}+2^{t+h}-\sum_{i=1}^{h}2^{t+i}-q''}=\binom{2^{t+k+2}-1}{\sum_{i=h}^{k}2^{t+i}-q''}=1$$

and

$$\varepsilon_\alpha=\binom{2^{t+1}+\sum_{i=1}^{k}2^{t+i+1}+2^{t+h+1}+2q''-\alpha-1}{\sum_{i=1}^{k}2^{t+i}+2^{t+h}+q''-\alpha}$$

$$=\binom{2^{t+k+2}+\sum_{i=1}^{h}2^{t+i}+2q''-\alpha-1}{\sum_{i=1}^{k}2^{t+i}+2^{t+h}+q''-\alpha}\quad(0\leqq\alpha<2q).$$

Since $0\leqq\sum_{i=1}^{h}2^{t+i}+2q''-\alpha-1<2^{t+k+2}$ and $\sum_{i=1}^{k}2^{t+i}+2^{t+h}+q''-\alpha<2^{t+k+2}$, if $\varepsilon_\alpha=1$, then $\sum_{i=1}^{h}2^{t+i}+2q''-\alpha-1\geqq\sum_{i=1}^{k}2^{t+i}+2^{t+h}+q''-\alpha$. But, $q''<\sum_{i=h}^{k}2^{t+i}+1$. Hence, $\varepsilon_\alpha=0$. Therefore, we have $M=m-2q+2(n-2r-1)$. (q.e.d.)

3. Theorem B and its proof

In this section, we will show that we can improve only "1" the results of Theorem A for (m,n) as follows:

(1) $(2^s+q,0)$ $(0<q<2^s-1,\ s\geqq2)$,

(2) $(0,2^t+r)$ $(0<r<2^t-1, t\geqq2)$,

(3) $(2^s+q,2^t+r)$ or $(\sum_{i=1}^{k}2^{t+i}+2^s+q,2^t+r)$ $(0\leqq q<2^s, 0<r<2^t, k\geqq1)$,

i) $s=0, t=1$, ii) $s=0, t\geqq2, r\equiv0$ or 3 (mod 4); $m=1$, $r\equiv1$ or 2 (mod 4),

iii) $s=t=1$, iv) $s=1, t\geqq2$, v) $2\leqq s\leqq t$,

(4) $(\sum_{i=1}^{k}2^{t+i},2^t+r)$ $(0<r<2^t-1,\ t\geqq2, k\geqq1)$,

(5) $(\sum_{i=0}^{k}2^{t+i}+q,2^t+r)$ $(2^t\leqq q<2^{t+k}+\cdots\cdots+2^{t+1}+2^t, 0<r<2^t)$

i) $t=1$, ii) $t\geqq2$.

Expanding §1. (3), (4) for those cases, we obtain the following table.

Table of the coefficients in $\overline{w_i}$ $(i=1,2,3)$ and $\overline{w_M}$.

		(1)		(2)		(3) i)		(3) ii) (r : mod 4)						(3) iii)		(3) iv) (r : mod 2)							
q		even	odd	$m=q=0$		0		0						0	1	0				0			
r		$n=r=0$		even	odd	1, $m=1$	1, $m>1$	$m=1$: 0	1	2	3	$m>1$: 0	3	1		0	1	2	3	0	1	2	3
$\overline{w_1}$	c	1	0	0	0	1	1	0	1	0	1	0	1	0	1	1	0	1	0	0	1	0	1
$\overline{w_2}$	c^2			0	0	0	1	0	0	0	0	1	1	1	0	0	0	1	1	0	1	1	0
	d			1	0	0	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0
$\overline{w_3}$	c^3					0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0
	cd					0	0	1	0	1	0	1	0	0	0	0	0	0	0	1	0	1	0
$\overline{w_M}$		c^{m-2q-1}		d^{n-2r-1}		c^m		$c^m d^{n-2r-1}$						$c^m c^{m-2}$		$c^m d^{n-2r-1}$				$c^{m-2}d^{n-2r-1}+\varepsilon' c^m d^{n-2r-2}$			

		(3) v), (4), (5) ii) (q, r : mod 4)																(5) i) (q : mod 4)			
q		0				1				2				3				0	1	2	3
r		0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	1			
$\overline{w_1}$	c	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	0	1	0	1
$\overline{w_2}$	c^2	1	1	0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	0	0	1
	d	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0
$\overline{w_3}$	c^3	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	1
	cd	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
$\overline{w_M}$		$c^{m-2q}d^{n-2r-1}+\sum_{\alpha>0}\varepsilon'_\alpha c^{m-2q+2\alpha}d^{n-2r-1-\alpha}$																c^{m-2q}			

Using this table, we will prove by the method of Massey [3] the following theorem. See [5] for the calculaions relating to Sq-operation.

THEOREM B.
(1) $P(m, n)\not\subset R^{2m-2q}$ *for* (m, n) *of case* (1).
(2) $P(m, n)\not\subset R^{2(m+2n)-2q-4r-1}$ *for* (m, n) *of cases* (2), (3), (4) *and* (5).

PROOF.

In this proof, the coefficients of $c^i d^j$ f, g, u, v etc. are equal to 0 or 1.

(1) Suppose $P(m, 0)\subset R^{2m-2q}$. Let $(E, p, P(m, 0), S^{m-2q-1})$ be the normal sphere bundle associated with this embedding. Then, by Massey [3], we have the following results.

There exist a subring $A^*=\sum A^i$ of the cohomology ring $H^*(E)$ and an element a of A^{m-2q-1} such that

1) A^* is closed under any cohomology operation,
2) $H^i(E)=p^*(H^i(P(m, 0)))+A^i$ (direct sum, $0<i<2m-2q-1$),

3) $A^{2m-2q-1}=0$,
4) any element $y\in H^*(E)$ can be expressed uniquely in the form $y=p^*(y_1)+ap^*(y_2)$ where $y_i\in H^*(P(m,0))$,
5) if $y=Sq^i(a)$ in 4), then $y_2=\overline{w_i}$.

From (5) and Table,

$$a^2=ap^*(c^{m-2q-1})+\begin{cases}0 & (m>4q+2)\\ p^*(uc^{2m-4q-2}) & (m\leqq 4q+2).\end{cases}$$

$$Sq^1(a)=\begin{cases}p^*(fc^{m-2q})+ap^*(c) & (q:\text{even})\\ 0 & (q:\text{odd}).\end{cases}$$

Now, we consider an element x of A^{m-q-1} such that $x=p^*(vc^{m-q-1})+ap^*(c^q)$. Then

$$Sq^1(x)=p^*(vc^{m-q})+\begin{cases}p^*(fc^{m-q})+ap^*(c^{q+1}) & (q:\text{even})\\ ap^*(c^{q+1}) & (q:\text{odd}).\end{cases}$$

Hence,

$$0=A^{2m-2q-1}\ni xSq^1(x)=\left.\begin{cases}fap^*(c^m)\\ 0\end{cases}\right\}+a^2p^*(c^{2q+1})=\begin{cases}(1+f)ap^*(c^m) & (q:\text{even})\\ ap^*(c^m) & (q:\text{odd}).\end{cases}$$

Here, note that $m<2m-2q-1$. On the other hand, when q is even, from

$$0=Sq^1Sq^1(a)=p^*(fc^{m-2q+1}),$$

we have $f=0$. Thus, $xSq^1(x)=ap^*(c^m)\neq 0$. But, it is a contradiction. Therefore, we obtain $P(m,0)\not\subset R^{2m-2q}$.

(2) Suppose $P(m,n)\subset R^{2(m+2n)-2q-4r-1}$. Let $(E,p,P(m,n),S^{m+2n-2q-4r-2})$ be the normal sphere bundle assosciated with this embedding. Similarly to (1), we have the following results.

There exist a subring $A^*=\sum A^i$ of $H^*(E)$ and an element a of $A^{m+2n-2q-4r-2}$ such that 1), 4), 5) are the same as (1),
2) $H^i(E)=p^*(H^i(P(m,n)))+A^i$ (direct sum, $0<i<2(m+2n)-2q-4r-2$),
3) $A^{2(m+2n)-2q-4r-2}=0$.

From (5),

$$Sq^1(a)=p^*(\cdots\cdots+fc^{m-2q-1}d^{n-2r}+f'c^{m-2q-3}d^{n-2r+1}+\cdots\cdots)+ap^*(\overline{w_1})$$

$$Sq^2(a)=p^*(\cdots\cdots+gc^{m-2q}d^{n-2r}+g'c^{m-2q-2}d^{n-2r+1}+\cdots\cdots)ap^*(\overline{w_2})$$

$$Sq^3(a)=p^*(\cdots\cdots+uc^{m-2q-1}d^{n-2r+1}+\cdots\cdots)+ap^*(\overline{w_3})$$

$$a^2=aq^*(c^{m-2q}d^{n-2r-1}+\textstyle\sum_{\alpha>0}\varepsilon_\alpha' c^{m-2q+2\alpha}d^{n-2r-1-\alpha})+p^*(\sum_{i,j}y_{i,j}c^id^j),$$

where if the power number of c or d is negative, then the coefficient of its term is equal to zero (Henceforth, the same notice is omitted.), and $i+2j=2(m+2n)-4q-8r-4$.

We consider an element x of $A^{m+2n-q-2r-2}$ such that

$$x=p^*(\cdots\cdots+vc^{m-q}d^{n-r-1}+v'c^{m-q-2}d^{n-r}+\cdots\cdots)+ap^*(c^qd^r).$$

Then,

$$Sq^2(x)=p^*(\cdots\cdots+(v+v')c^{m-q}d^{n-r}+\cdots\cdots)$$
$$+Sq^2(a)p^*(c^qd^r)+Sq^1(a)p^*(Sq^1(c^qd^r))+ap^*(Sq^2(c^qd^r))$$

where $v'=0$ for (m, n) of cases (3) i) $m>1$ and (3) iv).
Noticing that

$$p^*(\cdots\cdots+vc^{m-q}d^{n-r-1}+v'c^{m-q-2}d^{n-r}+\cdots\cdots)\cdot p^*(\cdots\cdots+(v+v')c^{m-q}d^{n-r}+\cdots)\text{ etc.}=0,$$
$$a^2p^*(c^{2q+2}d^{2r})=0,$$
$$a^2p^*(c^{2q}d^{2r+1})=ap^*(c^md^n)\quad(\text{Because } m+2n<2(m+2n)-2q-4r-2.)$$

we obtain

$$\begin{aligned}xSq^2(x)=ap^*(&(v+v'+g)c^md^n+\overline{w}_2(vc^md^{n-1}+v'c^{m-2}d^n)\\&+(fc^{m-q-1}+v'\overline{w}_1c^{m-q-2})d^{n-r}Sq^1(c^qd^r)\\&+(vc^{m-q}d^{n-r-1}+v'c^{m-q-2}d^{n-r})Sq^2(c^qd^r))\\&+a^2p^*(\overline{w}_2c^{2q}d^{2r}+c^qd^qSq^2(c^qd^r)).\end{aligned}$$

Since

$$Sq^1(c^qd^r)=\begin{cases}c^{q+1}d^r & (q+r:\text{odd})\\ 0 & (q+r:\text{even}),\end{cases}$$

$$Sq^2(c^qd^r)=\begin{cases}0 & (q\equiv0\text{ or }1,\ r\equiv0;\ q\equiv2\text{ or }3,\ r\equiv2\ (\text{mod }4))\\ c^qd^{r+1} & (q\equiv0\text{ or }3,\ r\equiv1;\ q\equiv1\text{ or }2,\ r\equiv3\ (\text{mod }4))\\ c^{q+2}d^r & (q\equiv0\text{ or }1,\ r\equiv2;\ q\equiv2\text{ or }3,\ r\equiv0\ (\text{mod }4))\\ c^{q+2}d^r+c^qd^{r+1} & (q\equiv0\text{ or }3,\ r\equiv3;\ q\equiv1\text{ or }2,\ r\equiv1\ (\text{mod }4)),\end{cases}$$

it follows

$$xSq^2(x)=\left\{\begin{matrix}(v+v'+g)\\(1+v'+f+g)\\(v+g)\\(1+f+g)\\(v+v'+f+g)\end{matrix}\right\}ap^*(c^md^n)+\left\{\begin{matrix}0\\v'\\0\\v'\\v'\end{matrix}\right\}ap^*(\overline{w}_1c^{m-1}d^n)\quad\begin{matrix}(q\equiv0,\ r\equiv0;\ q\equiv2,\ r\equiv2)\\(q\equiv0,\ r\equiv1;\ q\equiv2,\ r\equiv3)\\(q\equiv0,\ r\equiv2;\ q\equiv2,\ r\equiv0)\\(q\equiv0,\ r\equiv3;\ q\equiv2,\ r\equiv1)\\(q\equiv1,\ r\equiv0;\ q\equiv3,\ r\equiv2)\end{matrix}\begin{pmatrix}\text{mod}\\4\end{pmatrix}$$

$$\begin{vmatrix}(1 & & +g)\\ (& v+ & f+g)\\ (1 & +v' & +g)\end{vmatrix} \qquad \begin{vmatrix}0\\ v'\\ 0\end{vmatrix} \qquad \begin{matrix}(q\equiv 1,\ r\equiv 1;\ q\equiv 3,\ r\equiv 3)\\ (q\equiv 1,\ r\equiv 2;\ q\equiv 3,\ r\equiv 0)\\ (q\equiv 1,\ r\equiv 3;\ q\equiv 3,\ r\equiv 1)\end{matrix}$$

$$+ap^*(\overline{w}_2(vc^md^{n-1}+v'c^{m-2}d^n))+a^2p^*(\overline{w}_2c^{2q}d^{2r}).$$

Next, using Table, again it follows

$$xSq^2(x)=\left\{\begin{matrix}(1+g)\\ (1+f+g)\end{matrix}\right\}ap^*(c^md^n) \qquad \begin{matrix}(q+r:\text{even})\\ (q+r:\text{odd}).\end{matrix}$$

Since $xSq^2(x)\in A^{2(m+2n)-2q-4r-2}=0$, if $g=0$ or $f+g=0$, then we have a contradiction. Therefore, $P(m,n)\not\subset R^{2(m+2n)-2q-4r-1}$.

Henceforth, we set $C=1+g$ or $1+f+g$, and will show that $C=1$ for each (m,n) of cases (2), (3), (4) and (5).

Case (2). When r is even, $C=1+g$. Then,

$$0=Sq^3Sq^1(a)=Sq^2Sq^2(a)=p^*(gd^{n-2r+1})$$

implies $g=0$.

when r is odd, $C=1$ since $\overline{w}_1=\overline{w}_2=0$.

Case (3) i). When $m=1$, $C=1+f$ since $\overline{w}_2=0$. Then,

$$Sq^2Sq^1(a)=p^*(fd^2)=0.$$

When $m>1$, $C=1+f+g$. Then from

$$Sq^3(a)=Sq^1Sq^2(a)=p^*((f'+g')c^{m-1}d^2+\cdots)+ap^*(c^3)$$

$$Sq^2Sq^1(a)=p^*((f+f'+g')c^{m-1}d^2+\cdots)=0$$

$f+u=0$. And, from

$$Sq^2Sq^2(a)=p^*((g+g')c^md^2+\cdots\cdots)$$

$$Sq^3Sq^1(a)=p^*((g'+u)c^md^2+\cdots\cdots)$$

and $Sq^2Sq^2=Sq^3Sq^1$, $g=u$. Hence, $f+g=0$.

Case (3) ii). When $r\equiv 0\pmod 4$; $m=1$, $r\equiv 2\pmod 4$, $C=1+g$. Then,

$$0=Sq^3Sq^1(a)=Sq^2Sq^2(a)=p^*((g+g')c^md^{n-2r+1}+\cdots\cdots)$$

$$Sq^3(a)Sq^1Sq^2(a)=p^*(\cdots\cdots)+ap^*(cd)$$

$$Sq^2Sq^3(a)=p^*((g'+u)c^{m-1}d^{n-2r+2}+\cdots\cdots)=0$$

imply $g+g'=u=g'+u=0$ (where if $m=1$, then $g'=0$). Hence, $g=0$.

When $r\equiv 3\pmod 4$; $m=1$, $r\equiv 1\pmod 4$, $C=1+f+g$. Then,

$$Sq^3(a)=Sq^1Sq^2(a)=p^*((f'+g')c^{m-1}d^{n-2r+1}+\cdots)+ap^*(c^3)$$

$$Sq^2Sq^1(a)=p^*((f+f'+g')c^{m-1}d^{n-2r+1}+\cdots\cdots)=0$$

$$Sq^2Sq^2(a)=p^*((g+g')c^m d^{n-2r+1}+\cdots\cdots)$$

$$Sq^3Sq^1(a)=p^*((g'+u)c^m d^{n-2r+1}+\cdots\cdots)$$

imply $f'+g'=u$, $f+f'+g'=0$, $g+g'=g'+u$ (where if $m=1$, then $f'=g=g'=u=0$). Hence, $f+g=0$.

Case (3) iii). When $q=0$, $C=1+g$ since $\overline{w_1}=0$. Then,

$$0=Sq^3Sq^1(a)=Sq^2Sq^2(a)=p^*(gc^m d^2+\cdots\cdots).$$

When $q=1$, $C=1$ since $\overline{w_2}=0$.

Case (3) iv). When $q=0$, $r\equiv 0$ or 2 (mod 4), $C=1+g$. Then,

$$Sq^2Sq^2(a)=p^*((f+g)c^m d^{n-2r+1}+\cdots\cdots)+ap^*(c^2d)$$

$$Sq^3Sq^1(a)=p^*((g'+u)c^m d^{n-2r+1}+\cdots\cdots)+ap^*(c^2d)$$

$$Sq^3(a)=Sq^1Sq^2(a)=p^*((f+g')c^{m-1}d^{n-2r+1}+\cdots\cdots)+ap^*(c^3)$$

imply $f+g=g'+u$, $f+g'=u$ (where if $r\equiv 0$, then $u=0$). Hence, $g=0$.

When $q=0$, $r\equiv 1$ (mod 4); $q=1$, $r\equiv 3$ (mod 4), $C=1$ since $\overline{w_1}=0$ or $\overline{w_2}=0$.

When $q=0$, $r\equiv 3$ (mod 4); $q=1$, $r\equiv 0$ or 1 or 2 (mod 4), $C=1+g$ since $\overline{w_1}=0$ (except for $r\equiv 1$).

Then,

$$Sq^2Sq^2(a)=p^*(gc^{m-2q}d^{n-2r+1}+\cdots\cdots)=0.$$

Case (3) v), (4), (5). When $q\equiv r\equiv 0$ or 2 (mod 4), $C=1+g$. Then,

$$Sq^2Sq^2(a)=p^*(\cdots\cdots+(f+g)c^{m-2q}d^{n-2r+1}+\cdots\cdots)+ap^*(c^2d)$$

$$Sq^3Sq^1(a)=p^*(\cdots\cdots+(g'+u)c^{m-2q}d^{n-2r+1}+\cdots\cdots)+ap^*(c^2d)$$

$$Sq^3(a)=Sq^1Sq^2(a)=p^*(\cdots\cdots+(f+f'+g')c^{m-2q-1}d^{n-2r+1}+\cdots)+ap^*(c^3)$$

$$0=Sq^1Sq^1(a)=p^*(\cdots\cdots+f'c^{m-2q-2}d^{n-2r+1}+\cdots\cdots)$$

imply $f+g=g'+u$, $f+f'+g'=u$, $f'=0$. Hence, $g=0$.

When $q\equiv 0$, $r\equiv 1$ (mod 4); $q\equiv 2$, $r\equiv 3$ (mod 4), $C=1+g$ since $\overline{w_1}=0$. Then,

$$0=Sq^3(a)=Sq^1Sq^2(a)=p^*(\cdots\cdots+gc^{m-2q+1}d^{n-2r}+\cdots\cdots).$$

When $q\equiv 0$, $r\equiv 2$ (mod 4) or vice versa, $C=1+g$. Then,

$$Sq^2Sq^2(a)=p^*(\cdots\cdots+(f+g)c^{m-2q}d^{n-2r+1}+\cdots)+ap^*(c^2d)$$

$$Sq^3Sq^1(a)=p^*(\cdots\cdots+g'c^{m-2q}d^{n-2r+1}+\cdots\cdots)+ap^*(c^2d)$$

$$0=Sq^3(a)=Sq^1Sq^2(a)=p^*(\cdots\cdots+(f+g')c^{m-2q-1}d^{n-2r+1}+\cdots\cdots)$$

imply $f+g=g'$, $f+g'=0$. Hence, $g=0$.

When $q\equiv 0$, $r\equiv 3 \pmod 4$; $q\equiv r\equiv 1$ or $3 \pmod 4$; $q\equiv 2$, $r\equiv 1 \pmod 4$, $C=1$ since $\overline{w}_1=0$ or $\overline{w}_2=0$.

When $q\equiv 1$ or 3, $r\equiv 0$ or $2 \pmod 4$, $C=1+g$ since $\overline{w}_1=0$. Then,

$$0=Sq^3Sq^1(a)=Sq^2Sq^2(a)=p^*(\cdots\cdots+gc^{m-2q}d^{n-2r+1}+\cdots\cdots).$$

When $q\equiv 1$, $r\equiv 3 \pmod 4$ or vice versa, $C=1+g$. Then,

$$Sq^2Sq^2(a)=p^*(\cdots gc^{m-2q}d^{n-2r+1}+\cdots)=0. \qquad \text{(q. e. d.)}$$

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