Remarks on certain 14-manifolds

By

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1. Introduction

The object of this note is to give the classification up to diffeomorphism of closed, 5-connected 14-manifolds. All of our results are valid only for manifolds with torsion free homology which are boundaries of certain 15-manifolds. The proofs of our rsults are straightforward applications of the results of [6] and [8].

Throughout this note, we are only concerned with 14-manifolds M which satisfy the hypothesis;

(H) M is closed, 5-connected and the homology of M is torsion free.

By an (H)-manifold, we shall mean a 14-manifold satifying the hypothesis (H).

2. Splitting theorem

THEOREM 1. *Let M be an (H)-manifold. Then we can write M as a connected sum*

$$M=M_1\#(S^7\times S^7)\#\cdots\cdots\#(S^7\times S^7),$$

where M_1 is an (H)-manifold with $H_7(M_1)=0$.

Since the proof of this is analogous to that of theorem 1 in [8], we shall give an outline of the proof.

It is known that $H_7(M)$ admits a symplectic basis $\{e_i, e_i'\}$ $(1\leqq i\leqq k)$ so that

$$e_i\cap e_j = e_i'\cap e_j' = 0$$

and

$$e_i\cap e_j' = \delta_{ij}.$$

Since M is 5-connected, the Hurwicz homomorphism $H:\pi_7(M)\longrightarrow H_7(M)$ is an epimorphism. Thus we have mappings $\overline{f_i}$ and $\overline{f_i'}$ of S^7 in M which represent e_i and e_i, respectively. By a theorem of Haefliger [2], a general position argument and the method of Whithey, we may assume that for each i, $\overline{f_i}$ and $\overline{f_i'}$ are embeddings, the image spheres meet each other transversely in a finite set of points, none of which lies on more than two of the spheres, and for each i, $\overline{f_i}$ and $\overline{f_i'}$ intersect in a single point and others do not intersect. Let p_0 be the base point of S^7. We may suppose that our intersections are $\overline{f_i}(p_0)=\overline{f_i'}(p_0)$. We have embeddings

$$h=\overline{f_i}\times p_0\cup p_0\times\overline{f_i'}:S^7\times p_0\cup p_0\times S^7\longrightarrow M$$

and

$$\overline{h}:D^7\times D^7\longrightarrow M,$$

where $D^7\times D^7$ is a neighborhood of $p_0\times p_0$ in $S^7\times S^7$. Since $\pi_6(SO(7))=0$, $\overline{f_i}$ and $\overline{f_i'}$ can be extended to embeddings

$$f_i\ :S^7\times D^7\longrightarrow M$$

$$f_i':S^7\times D^7\longrightarrow M.$$

Combing these embeddings, we have an embedding F_i of a neighborhood N of $S^7\times p_0\cup p_0\times S^7$ in $S^7\times S^7$ in M;

$$F_i:N\longrightarrow M.$$

By a suitable choise of N, we can construct a closed 14-manifold M_1 by

$$M_1=(M-\cup intF_i(N))\cup\bigcup_{i=1}^{k}D_i^{14},$$

where a cell D_i^{14} is attached by the map F_i. Obviousely M is diffeomorphic to a connected sum $M_1\#(S^7\times S^7)\#\cdots\cdots\#(S^7\times S^7)$. It is not difficult to see that M_1 is an (H)-manifold with additional property $H_7(M_1)=0$.

3. A normal form

We shall first prove the following lemma.

Lemma 2. *Let M be an (H)-manifold with $H_7(M)=0$. Then M can be obtained from a homotopy 14-sphere Σ by surgery on a disjoint set of embeddings $g_i:S^7\times D^7\longrightarrow\Sigma$ $(1\leqq i\leqq k)$.*

Although the proof of lemma 2 is similar to that of theorem 2 in [8], we shall give a complete proof, since some notations used in the proof are needed later.

Proof of lemma. Let $\{e_i\}$ be a free basis for $H_6(M)$. By the Hurwicz theorem, for each i, e_i can be represented by a map $\overline{f_i}:S^6\longrightarrow M$; by a general position argument, these maps may be supposed disjoint embeddings. Since $\pi_5(SO(8))=0$, for each i, $\overline{f_i}(S^6)$ has a trivial normal bundle, and hence $\overline{f_i}$ extends to an embedding $f_i:S^6\times D^8\longrightarrow M$. Form W from $M\times I$ by using the map f_i to attach handle $D^7\times D_i^8$ to $M\times 1$. Evindently W has the same homotopy type as $M\cup\bigcup_{i=1}^{k}D_i^7$. Hence we have

$$H_7(W,M;Z)=\begin{cases}Z+\cdots\cdots+Z & \text{if } i=7\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that W is 7-connected. Let Σ be the component other than $M\times 0$ of ∂W. We shall show that Σ is a homotopy sphere, i. e. Σ is 7-connected. In fact, from the homology exact sequence of the pair (W, Σ), we have $H_i(\Sigma)=0$ for $i\leqq 6$. Consider the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H_8(W) & \longrightarrow & H_8(W, \Sigma) & \longrightarrow & H_7(\Sigma) & \longrightarrow 0 \\ & \downarrow i_* \approx & & \downarrow \approx & & & \\ H_8(M)=H^6(M) & & \xrightarrow{\delta} & H^7(W, M), & & & \end{array}$$

where the top horizontal sequence is a part of the homology exact sequence of that pair (W, Σ), i_* the homomorphism induced by the inclusion $M\longrightarrow W$ and δ the coboundary homomophism. Since i_* and δ are isomorphisms, $H_7(\Sigma)=0$. Clearly Σ is simply connected, and hence Σ is 7-connected.

Reversing the construction above, we see that M can be obtained from Σ by surgery on a disjoint set of embeddings $g_i : S^7\times D^7 \longrightarrow \Sigma (1\leqq i\leqq k)$. This completes the proof of lemma 2.

Combining theorem 1 and lemma 2, we have shown that an (H)-manifold M can be written as a connected sum $M_1\#(S^7\times S^7)\#\cdots\cdots\#(S^7\times S^7)$, where M_1 is obtained from a homotopy 14-sphere Σ by surgery on a disjoint set of embeddings $g_i : S^7\times D^7 \longrightarrow \Sigma$. If Σ is the standard sphere S^{14}, then M_1 is boundary of a handlebody $W\in\mathcal{H}(15, k, 8)$. Since Wall has given a classification up to diffemorphism of elements of $\mathcal{H}(15, k, 8)$ [6], we can classify (H)-manifolds such that M_1 bounds a handlebody.

In what follows, M_s denotes the sum $(S^7\times S^7)\#\cdots\cdots\#(S^7\times S^7)$.

Assume that M bounds a manifold W with $w_2(W)$ (the second Stiefel-Whitney class)$=0$ and $p_1(W)$ (the first Pontrjagin class)$=0$. By surgery, we may assume that W is 6-connected. Let W_1 be the cobordism between M_1 and Σ given in the proof of lemma 2, which is 7-connected. We construct a 15-manifold whose boundary is a disjoint union of $M_1\#M_s$ and $\Sigma\#M_s$ as follows. Choose an embedding $\bar{f}: I\longrightarrow W_1$ so that $\bar{f}(0)\in M_1$ and $\bar{f}(1)\in\Sigma$. Since $\bar{f}$ has a trivial normal bundle, we have an embedding $f: I\times D^{14}\longrightarrow W_1$ so that $f(0\times D^{14})\in M_1$ and $f(1\times D^{14})\in\Sigma$. Let x be a point of M_s and D is a disc neighborhood of x in M_s. Define

$$W_2=(W_1-f(I\times int\, D^{14}))\cup(M_s\times I-int\, D\times I)$$

by identifying the points $f(t, s)$ and (t, s), where $t\in I, s\in\partial D$. Clearly $\partial W=M_1\#M_s\cup(-(\Sigma\#M_s))$. By the homology Meyer-Vietoris exact sequence, we can show that W_2 is 6-connected. According to the arguments in [5], Σ can be obtained from $\Sigma\#M_s$ by a sequence of surgeries. Let W_3 be the cobordism between $\Sigma\#M$ and Σ, which is 6-connected. By identifying the common boundary of W_2 and W_3, we can construct a 15-manifold W' whose boundary is a disjoint union of Σ and $M_1\#M_s$.

Again, by identifying the common boundary of W' and W, we obtain a 15-manifold V whose boundary is Σ. It is not difficult to see that V is 6-connected. Now we shall prove that V is 7-parallelizable, i. e. the restriction of the tangent bundle τ_V of V to the 7-skelton of V is trivial. In fact, the obstrucions to 7-parallelizability are in $H^i(V; \pi_{i-1}(SO(15)))$, $i=1, 2, \cdots\cdots, 7$. Sinee V is 6-connected and $\pi_6(SO(15))=0$, there are no obstructions. By a theorem of Wall [7], Σ bounds a contractible manifold, and hence Σ is diffeomorphic to the standard 14-sphere S^{14}.

Thus we have proved

THEOREM 3. *Let M be an (H)-manifold which bounds a manifold W with $w_2(W)=p_1(W)=0$. Then M can be written as a connected sum*

$$M=M_1\#(S^7\times S^7)\#\cdots\cdots\#(S^7\times S^7),$$

where M_1 can be obtained from the standard 14-sphere S^{14} by surgery on a disjoint set of embeddings $g_i : S\times D^7 \longrightarrow S^{14}(1\leqq i\leqq k)$; M_1 is boundary of a handlebody $W\in\mathcal{H}(15, k, 8)$.

In next section, we show that M is framed cobordant to zero, then M_1 is boundary of a parallelizable habdlebedy.

4. Invariants.

In his paper [6], Wall has proved the following

THEOREM. *Diffeomorphism classes of elements of $\mathcal{H}(15, k, 8)$ correspound bijectively to isomorphism classes of structures of invariants;*

a free abelian group H

a symmetric bilinear map $\lambda : H\times H \longrightarrow \pi_8(S^7)$

a map $\alpha : H \longrightarrow \pi_7(SO(7))$

subject to; for $x, y\in H$

i) $\lambda(x, x)=S\pi\alpha(x)$

ii) $\alpha(x+y)=\alpha(x)+\alpha(y)+\partial\lambda(x, y)$,

where π is the homorphism induced by $SO(7) \longrightarrow S^6$ and ∂ the boundary homomorphism of the fibring $SO(7) \longrightarrow SO(8) \longrightarrow S^7$.

We recall the definition of invariants H, α and λ. Let W be an element of $\mathcal{H}(15, k, 8)$; $W=D^{15}\cup\bigcup_{i=1}^{k} D^8\times D_i^7$, where a handle $D^8\times D_i^7$ is attached by an embedding $g_i : S^7\times D^7 \longrightarrow S^{14}$, $1\leqq i\leqq k$. The handles have homology classes in $H_8(W, D^{15})=H_8(W)$; denote these classes in $H_8(W)$ by e_i. Then H is the group $H_8(W)$. Let $\bar{g_i}=g_i/S^7\times 0$, and we have a link in the sense of Haefliger [1]. Let $\lambda_{ij}(i\leqq j)$ be linking numbers. Then the map λ is given by the formula;

$$\lambda(e_i, e_j) = S\lambda_{ij}.$$

Let FC_7^7 be the group of isotopy classes of embeddings $g : S^7\times D^7 \longrightarrow S^{14}$. In his paper [3], Haefliger has obtained an isomorphism $\tau : \pi_7(SO(7)) \longrightarrow FC_7^7$, where τ is the map which twists the tubular neighborhood of g. Then the map α is defined by the formula;

$$\alpha(e_i) = \tau^{-1}[g_i],$$

where $[g_i]$ denotes the isotopy class of g_i.

It can be shown that a handlebody W is parallelizable if and only if the boundary of W is s-parallelizable. In fact, let F be a trivialization of the stable tangent bundle of ∂W. It is sifficient to show that F can be extend over W. Obstructions to the extending F over W are in $H^i(W, \partial W; \pi_{i-1}(SO))$. By straightforward calculations, these groups are zero. Hence τ_W is stable trivial and then W is parallelizable.

Now we shall seek the condition under which a handlebody $W\in\mathcal{H}(15, k, 8)$ is parallelizable. Let $W=D\cup\bigcup_{i=1}^{k}D^8\times D^7$ and h_i an embedding $D^8\times D^7 \longrightarrow W$ so that

$$h_i(S^7\times D^7) = g_i(S^7\times D^7)$$

and T, D the natural trivialization of $\tau_{D^8\times D^7}$ and $\tau_{D^{15}}$, respectively. Define a map $\varphi_i : S^7 \longrightarrow SO(15)$ by the formula;

$$\varphi_i(x) = \langle D, h_i'(T)\rangle g_{(x,0)} \quad x\in S^7$$

Thus we have an element $o\in H^8(W, D^{15}; \pi_7(SO(15)))$ so that $\langle o, e_i\rangle = [\varphi_i]$. It is clear that $[\varphi_i] = j_*\alpha(e_i)$, where j_* is the homomorphism: $\pi_7(SO(7)) \longrightarrow \pi_7(SO(15))$ induced by the inclusion $SO(7) \longrightarrow SO(15)$. Since W is parallelizable if and only if $o=0$, W is parallelizable if and only if $\alpha=0$.

We have shown that an (H)-manifold M which bounds a 7-parallelizable manifold can be written as a connected sum $M_1 \# M_s$, where M_1 is a boundary of a handlebody $W\in\mathcal{H}(15, k, 8)$. It is clear that two diffeomophic (H)-manifolds which bound 7-parallelizable manifolds determine diffeomorphic handlebodies and two diffeomorphic handlebodies determine diffeomorphic (H)-manifolds which bound 7-parllelizable manifolds. From the arguments above and the theorem of Wall quated above, we have

THEOREM 4. *Diffeomorphism classes of (H)-manifolds which bound 7-parallelizable manifolds correspond bijectively to isomorphism classes of structures of invariants $\{H, G, \alpha, \lambda\}$, where $\{H, \alpha, \lambda\}$ is given in the theorem of Wall and G is a free abelian group.*

COROLLARY. *Diffeomorphism classes of (H)-manifolds which bound parallelizable manifolds correspond bijectively to isomorphism classes of invariant $\{H, G, \alpha, \lambda\}$ with $\alpha=0$.*

5. Embeddings.

In this section, we shall consider the embedding problem of (H)-manifolds and closed 5-dimensional s-parallelizable manifolds.

Let M be an (H)-manifold which bounds a parallelizable manifold. Then, by a result of DeSapio [4], M can be embedded in R^{17}. We shall show that M can be embedded in R^{16}. We can assume that M is boundary of a handlebody $W\in\mathcal{H}(15, k, 8)$. We construct a new handlebody W' by giving framed links $g_i':S^7\times D^8\longrightarrow S^{15}$, the suspenion of earlier link $g_i:S^7\times D^7\longrightarrow S^{14}$. Then $W'\in\mathcal{H}(16, k, 8)$. Clearly W' is obtained from $W\times I$ by rounding the corneres. Since W is parallelizable, W' is also parallelizable. Since all the 7-spheres lie in the equater S^{14}, they are unliked. Thus the invariants α and λ are zero. Hence W' is diffeomorphic to a boundary-connected sum of copies of trivial D^8-bundles over S^8. Clearly W' embeds in R^{16}, and hence M embeds in R^{16}. It is not difficult to see that an (H)-manifold M which bounds 7-parallelizable manifold embeds in R^{16} if and only if M bounds a parallelizable manifold.

Next we shall consider embeddability of a cloed 5-dimensional s-parallelizable manifold M in R^8. By a result of [4], M can be embedded in R^9. We shall show that if $H_1(M)$ is free, then M can be embedded in R^8. The proof is a straightforward application of the following theorem of Wall [8].

Theorem of Wall.

Let M be a closed simply connected 6-manifold with torsion free homology and vanishing first Pontrjagin class. Then M embeds in R^8.

Let M be a closed s-parallelizable 5-manifold. Then there exists a parallelizable 6-manifold W whose boundary is M. We may assume that W is 2-connected. Let $\tilde{W}$ be the double of W. It is known that $\tilde{W}$ is a simply connected s-parallelizable 6-manifold. We shall show that $H_*(\tilde{W})$ has no torsion. By the homology exact sequence of the pair $(\tilde{W}, W)$, we have $H_2(\tilde{W})=H_2(\tilde{W}, W)$, which is isomorphic to $H_2(W, M)$. Similarly $H_2(W, M)$ is isomorphic to $H_1(M)$. By the assumption $H_1(M)$ is free and hence $H_2(\tilde{W})$ is free. Since $\tilde{W}$ is simply connected, $H_*(\tilde{W})$ has no torsion. By the theorem of Wall above, $\tilde{W}$ embeds in R^8 and hence M also embeds in R^8. This completes our assertion.

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