Bundles contained in fibre bundles

By

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In the present paper we define microbundles and vector bundles admitted by cross-sections of fibre bundles, and study relations between properties of fibre bundles, cross-sections and their admitted bundles.

Milnor has defined piecewise linear microbundles and obtained theorems analogous to those about vector bundles, and applied them to the smoothing problem of piecewise linear manifolds in [3]. As he noted, the definitions and many of the theorems make sense in the category of topological spaces and maps. Letely their detailes are stated in [6].

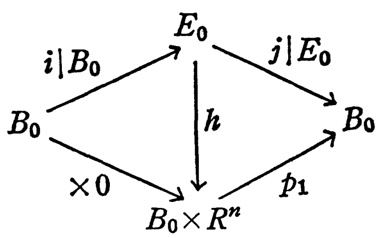
We recall in §1 the definitions required in the later, study about microbunles admitted by cross-sections in §2, study about vector bundles admitted by cross-sections in §3.

1. Microbundles

DEFINITION 1.1. A *microbundle* $\mathfrak{x}$ of dimension n is a diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

where B, E are topological spaces and i, j are continuous maps; such that the following local triviality condition is satisfied. For each $b \in B$, there should exist neighborhoods B_0 of b, E_0 of $i(b)$ and a homeomorphism $h: E_0 \longrightarrow B_0 \times R^n$ so that the diagram



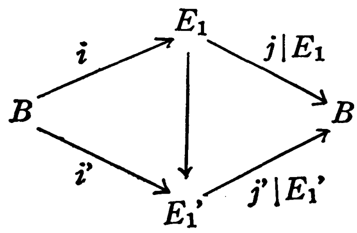
is commutative. Here the notation $\times 0$ stands for the map $b \longrightarrow (b, 0)$, p_1 denotes the projection into the first factor, and R^n denotes Euclidean n-space.

It is easily seen that $j \circ i : B \longrightarrow B$ is the identity map of B.

DEFINITION 1.2. A second microbundle

$$\mathfrak{x}' : B \xrightarrow{i'} E' \xrightarrow{j'} B$$

over the same base space is *isomorphic* to $\mathfrak{x}$ (witten $\mathfrak{x}' \approx \mathfrak{x}$) if there exist neighborhoods E_1 of $i(B)$ and E_1' of $i'(B)$ and a homeomorphism $: E_1 \longrightarrow E_1'$ so that the diagram



is commutative.

The trivial microbundle $\mathfrak{e}^n_B$ means the diagram

$$B \xrightarrow{\times 0} B \times R^n \xrightarrow{p_1} B.$$

Any bundle isomorphic to $\mathfrak{e}^n_B$ is also called a trivial microbundle.

DEFINITION 1.3. The *tangent microbundle* $\mathfrak{t}_M$ of a manifold M means the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$$

where Δ denotes the diagonal map.

DEFINITION 1.4. Given two microbundles

$$\mathfrak{x}_\alpha : B \xrightarrow{i_\alpha} E_\alpha \xrightarrow{j_\alpha} B, \quad \alpha = 1, 2,$$

over the same base space, the *whitney sum* $\mathfrak{x}_1 \oplus \mathfrak{x}_2$ is the diagram

$$B \xrightarrow{i} E \xrightarrow{j} B,$$

where $E \subset E_1 \times E_2$ is the set of (e_1, e_2) with $j_1 e_1 = j_2 e_2$, and $ib = (i_1 b, i_2 b)$, $j(e_1, e_2) = j_1 e_1$.

DEFINITION 1.5. Given a microbundle

$$\mathfrak{x} : B \xrightarrow{i} E \xrightarrow{j} B$$

and a map $f : B_1 \longrightarrow B$, the *induced microbundle* $f^* \mathfrak{x}$ is the diagram

$$B_1 \xrightarrow{i_1} E_1 \xrightarrow{j_1} B_1$$

where $E_1 \subset B_1 \times E$ is the set of (b_1, e) with $f(b_1) = j(e)$, and $i_1(b_1) = (b_1, if(b_1))$, $j_1(b_1, e_1) = b_1$. If f is an inclusion, $f^* \mathfrak{x}$ is called the *restriction* of $\mathfrak{x}$ over B_1.

When the Whitney sum of two microbundles is trivial, we say that they are *inverse* to each other. And we have

PROPOSITION. *Every microbundle over a finite dimensional complex has an inverse.*

DEFINITION 1.6. We say that two microbundles $\mathfrak{x}$ and $\mathfrak{x}'$ over B belong to the *same s-class* if $\mathfrak{x}\oplus\mathfrak{e}_B^q$ is isomorphic to $\mathfrak{x}'\oplus\mathfrak{e}_B^r$ for some q and r.

By above Proposition, the s-classes of microbundles over a finite dimensional complex B form an abelian group under the composition operation induced by Whitney sum. This group is denoted by $k_{top}(B)$.

2. Microbundles contained in fibre bundles

Let $\mathfrak{B}=(E,B,p)$ be a fiber bundle with the structural group G [5, p. 9] having a cross-section σ.

DEFINITION 2.1. When the diagram

$$\mathfrak{x}: B \xrightarrow{\sigma} E \xrightarrow{p} B$$

is a microbundle, $\mathfrak{x}$ is said to be a *microbundle contained in* $\mathfrak{B}$ *and admitted by* σ, and is denoted by $(\mathfrak{B},\sigma)$.

THEOREM 2.1. *If the fibre* M *of* $\mathfrak{B}=(E,B,p)$ *is a manifold, then any cross-section* σ *admits a microbundle.*

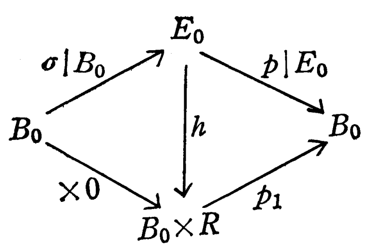
Proof. For each $b\in B$, let V_j be a coordinate neighborhood containing b, $\phi_j: V_j\times M\longrightarrow p^{-1}(V_j)$ be the corresponding coordinate function, and $\phi_{j,b}: M\longrightarrow p^{-1}(b)$ be defined by setting $\phi_{j,b}(y)=\phi_j(b,y)$. Let $N(b)$ be a neighborhood of $\phi_{j,b}{}^{-1}(\sigma(b))$ in M which is homeomorphic to R^n. ($n=\dim M$). Then $V_j\times N(b)$ is a neighborhood of $\phi_{j,b}{}^{-1}(\sigma(b))$ in $V_j\times M$, and there is a neighborhood B_0 of b in B such that

$$\{\phi_j{}^{-1}\sigma(x): x\in B_0\}\subset B_0\times N(b).$$

Now let $\lambda: B_0\times N(b)\longrightarrow B_0\times R^n$ be the canonical homeomorphism, and let $p_2: B_0\times R^n\longrightarrow R^n$ be the projection into the second factor. Then $E_0=\phi_j(B_0\times N(b))$ is a neighborhood of $\sigma(b)$, and $h: E_0\longrightarrow B_0\times R^n$ defined by setting

$$h(z)=(p(z),\ p_2\lambda\phi_j{}^{-1}(z)-p_2\lambda\phi_j{}^{-1}(\sigma p(z)))$$

is a homeomorphism. The diagram



is commutative. Thus the diagram $B \xrightarrow{\sigma} E \xrightarrow{p} B$ is a microbundle.

A fibre bundle may contain various isomorphism classes of microbundles.

EXAMPLE 2.1. Let $\mathfrak{B} = (S^2 \times S^2, S^2, p_1)$ be a trivial 2-sphere bundle, and define cross-sections σ and σ' by setting

$$\sigma(x, y) = ((x, y), (p, q)),$$

$$\sigma'(x, y) = ((x, y), (x, y)),$$

where (p, q) is a fixed point of S^2. Then σ admits a trivial microbundle and σ' admits a tangent microbundle of S^2 which is not trivial.

When two fibre bundles $\mathfrak{B} = (E, B, p)$ and $\mathfrak{B}' = (E', B, p')$ are isomorphic each other [5, p.11], if a cross-section σ of $\mathfrak{B}$ admits a microbundle $\mathfrak{x}$, then $\mathfrak{B}'$ contains a microbundle which is isomorphic to $\mathfrak{x}$. In fact, for the homeomorphism $H: E \longrightarrow E'$ which induces the isomorphism between $\mathfrak{B}$ and $\mathfrak{B}'$, the microbundle $\mathfrak{x}'$ which is admitted by the cross-section $\sigma' = H \circ \sigma$ is the required one.

The converse does not hold as seen in the following example.

EXAMPLE 2.2. Let $T = S^1 \times S^1$ and represent its point by (x, y) where x, y are real numbers modulo 1. Let $\mathfrak{B} = (E, S^1, p)$ be a bundle over S^1 with the fibre T and the structural group $G = O(2) \times O(2) \times Z_2$, where Z_2 is the group which is consist of exchange of two S^1 in T and identity. Take coordinate neighborhoods V_1 and V_2 such that $V_1 \cup V_2 = S^1$ and $V_1 \cap V_2$ is the sum of two open arcs I_1 and I_2 disjoint each other, and assume that the coordinate transformation

$$g_{21} : V_1 \cap V_2 \longrightarrow G$$

is given by

$$g_{21}(b)(x, y) = \begin{cases} (x, y) & \text{if } b \in I_1 \\ (y, -x) & \text{if } b \in I_2. \end{cases}$$

Then $\mathfrak{B}$ is not trivial. But the cross-section σ, which is given by $\sigma(b) = \phi_{j,b}(0, 0)$ for $j = 1$ or 2, admits a trivial microbundle, where ϕ_j is a coordinate function of $\mathfrak{B}$. $\sigma(b)$ is well defined because $\phi_{1,b} = \phi_{2,b}$ at $(0, 0)$. Thus $\mathfrak{B}$ contains a common microbundle with the trivial bundle $T \times S^1$.

As seen in Example 2.1, a sphere bundle may contain various microbundles, but we have;

THEOREM 2.2. *Let $\mathfrak{x}$ be a microbundle contained in an n-sphere bundle $\mathfrak{B} = (E, B, p)$, and let $Y = (\overline{E}, B, \overline{p})$ be the associated $(n+1)$-dimensional vector bundle of $\mathfrak{B}$, and $\mathfrak{y}$ be the underlying microbundle of Y. Then $\mathfrak{x} \oplus \mathfrak{e}^1_B$ is isomorphic to $\mathfrak{y}$.*

Proof. Let $\mathfrak{x}$ be admitted by a cross-section σ, then the diagram of $\mathfrak{x} \oplus \mathfrak{e}^1_B$ is

$$B \xrightarrow{\sigma \times 0} E \times R^1 \xrightarrow{p'} B$$

where $\sigma\times 0$ and p' are defined by setting

$$(\sigma\times 0)(b) = (\sigma(b), 0)$$

and

$$p'(x, y) = p(x).$$

The diagram of $\mathfrak{y}$ is

$$B \xrightarrow{\ i\ } \overline{E} \xrightarrow{\ \overline{p}\ } B$$

where i is the zero-cross-section of Y. (Cf. [6]). $\mathfrak{B}$ and Y have common families of coordinate neighborhoods and coordinate transformations, denoted by $\{V_j\}$ and $\{g_{ji}\}$ respectiely. We denote the families of coordinate functions of $\mathfrak{B}$ and Y by $\{\phi_j\}$ and $\{\psi_j\}$ respectively.

Let $\mathfrak{A}(\sigma(B))$ be the set of $x\in E$ such that $\phi_{j,b}{}^{-1}(\sigma(b))$ and $\phi_{j,b}{}^{-1}(x)$ are mutually antipodal in S^n, where $b=p(x)$ and $V_j\ni b$. An unique x corresponds to each b, i.e. independently to the choice of j, because $g_{ji}(b)\in O(n+1)$. $E_0=E-\mathfrak{A}(\sigma(b))$ is a neighborhood of $\sigma(B)$ in E, accordingly $E_0\times R^1$ is that of $\sigma(B)\times 0$ in $E\times R^1$.

For a point x on a standard n-sphere S^n in R^{n+1}, let $A(x)$ be the antipodal point of x on S^n, R^n_x be the tangent n-plane of S^n at x and let α_x be the projection of $S^n-A(x)$ onto R^n_x from $A(x)$. Let β_x be a translation of R^n_x in R^{n+1} which carries x onto the origin of R^{n+1}, and for each $z\in R^1$ let $\gamma_{x,z}$ be a translation of this image with the same direction as β_x by the distance z.

Now we define a fibre preserving homeomorphism

$$h: E_o\times R^1 \longrightarrow \overline{E}$$

by setting

$$h(y, z) = \psi_{j,b}\gamma_{x,z}\beta_x\alpha_x\phi_{j,b}{}^{-1}(y)$$

where $b=p(y)$, $x=\phi_{j,b}{}^{-1}(\sigma(b))$. When $b\in V_i\cap V_j$, denoting $x(i)=\phi_{j,b}{}^{-1}(\sigma(b))$ and $x(j)=\phi_{j,b}{}^{-1}(\sigma(b))$,

$$\begin{aligned}
&\psi_{i,b}\gamma_{x(i),z}\beta_{x(i)}\alpha_{x(i)}\phi_{i,b}{}^{-1}(y)\\
&=\psi_{j,b}g_{ji}(b)\gamma_{x(i),z}\beta_{x(i)}\alpha_{x(i)}g_{ji}(b)^{-1}\phi_{j,b}{}^{-1}(y)\\
&=\psi_{j,b}\gamma_{x(j),z}\beta_{x(j)}\alpha_{x(j)}\phi_{j,b}{}^{-1}(y).
\end{aligned}$$

Hence the definition of $h(y, z)$ is independent of the choice of j, accordingly h is well defined. This homeomorphism induces the isomorphism between $\mathfrak{x}\oplus\varepsilon^1_B$ and $\mathfrak{y}$.

Let $\mathfrak{H}$ be a space of all homeomorphisms of S^n provided with the compact open topology, then we have, THEOREM 2.3. *Every microbundle of dimension n over a locally finite finite-dimensional complex B is isomcrphic to some microbundle which is contained in a fibre bundle over B with the fibre S^n and the structural group $\mathcal{H}$.*

To prove this theorem we recall Kister's theorem.

DEFINITION 2.2. An n-dimensional microbundle

$$\mathfrak{x} : B \xrightarrow{i} E \xrightarrow{j} B$$

is said to *admit* a fibre bundle $\eta = (E_1, B, p)$ with fibre R^n, if E_1 is a neighborhood of zero-cross-section $i(b)$ in E and $p = j | E_1$.

Let $\mathcal{H}_0$ be a space of all origin-preserving homeomorphisms of R^n provided with the compact open topology. Kister has proved in [2],

LEMMA 2.4. *Every microbundle of dimension n over a locally finite finite-dimensional complex admits a fibre bundle with fibre R^n and structural group $\mathcal{H}_0$ unique up to isomorphism.*

Proof of Theorem 2.3. Let

$$\mathfrak{x} : B \xrightarrow{i} E \xrightarrow{j} B$$

be the n-dimensional microbundle. Let $\eta = (E_1, B, p)$ be a fibre bundle admitted by $\mathfrak{x}$ with fibre R^n and structural group $\mathcal{H}_o$, and let the family of coordinate neighborhoods, that of coordinate functions and that of coordinate transformations be $\{V_j\}$, $\{\phi_j\}$ and $\{g_{ji}\}$ respectively. The existence and uniqueness of η is seen by Lemma 2.4.

Let R^n be tangent to S^n at x_o, and $R^n \ni 0 \leftrightarrow x_o \in S^n$. We define the projection α_{xo} of $S^n - A(x_o)$ onto R^n as in the proof of Theorem 2.2, and define a homeomorphism $\overline{g}_i(b) : S^n \longrightarrow S^n$ for $b \in V_i \cap V_j$ by setting

$$\overline{g}_{ji}(b)(z) = \alpha_{xo}^{-1} g_{ji}(b) \alpha_{xo}(z) \quad \textit{for} \quad z \neq A(x_o)$$

and

$$\overline{g}_{ji}(b)(A(x_o)) = A(x_o).$$

Let $\mathfrak{B} = (\overline{E}, B, \overline{p})$ be the fibre bundle with th fibre S^n, the group $\mathcal{H}$, and the family of coordinate transformations $\{\overline{g}_{ji}\}$. (Cf. [5, p. 14]). We denote the family of coordinate functions of $\mathfrak{B}$ by $\{\psi_j\}$, and define a cross-section $\sigma : B \longrightarrow \overline{E}$ by setting $\sigma(b) = \psi_{j,b}(x_o)$. This is well defined because $\psi_{j,b}^{-1}\psi_{i,b}(x_o) = x_o$ for $b \in V_i \cap V_j$

Let $\overline{\mathfrak{A}}$ $(\sigma(b))$ and $\overline{E_o}$ be defined as $\mathfrak{A}(\sigma(b))$ and E_o in the proof of Theorem 2.2 using $\{\psi_{j,b}\}$ in place of $\{\phi_{j,b}\}$, then we can define a homeomorphism

$$h : \overline{E}_o \longrightarrow E_1$$

by setting

$$h(z) = \phi_{j,b}\alpha_{xo}\psi_{j,b}^{-1}(z)$$

where $b=\bar{p}(z)$ and $b\in V_j$. This is well defined because, for $b\in V_i\cap V_j$

$$\phi_{j,b}^{-1}\circ\phi_{i,b} = g_{ji}(b)$$

and

$$\psi_{j,b}^{-1}\circ\psi_{i,b} = \bar{g}_{ji}(b) = \alpha_{xo}^{-1}\circ g_{ji}(b)\circ\alpha_{xo},$$

accordingly

$$\phi_{j,b}\circ\alpha_{xo}\circ\psi_{j,b}^{-1} = \phi_{i,b}\circ\alpha_{xo}\cdot\psi_{i,b}^{-1}.$$

h induces the isomorphism between $(\mathfrak{B},\sigma)$ and $\mathfrak{x}$.

Remark. An equivalent theopem to our Theorem 2.3 is reported in [7].

THEOREM 2.5. *If $\mathfrak{B} = (E, B, p)$ contains a microbundle $\mathfrak{x}$ and $f: X \longrightarrow B$ is a map, then the induced bundle $f^*\mathfrak{B}$ contains $f^*\xi$.*

Proof. Let $\mathfrak{x} = (\mathfrak{B},\sigma)$ and $f^*\mathfrak{B}=(E, X, p)$, then $\bar{E} = \{(x, e) : x\in X, e\in E, f(x)=pe\}$. We define a cross-section $\bar{\sigma} : X\longrightarrow\bar{E}$ by setting $\bar{\sigma}(x)=(x, \sigma f(x))$, then $(f^*\mathfrak{B}, \bar{\sigma})=f^*\mathfrak{x}$.

3. Standard admissible vector bundles

DEFINITION 3.1. When a fibre bundle $\mathfrak{B} = (E, B, p)$ contains a microbundle $\mathfrak{x} = (\mathfrak{B},\sigma)$ and $\mathfrak{x}$ admits a vector bundle η, η is called a vector bundle *contained* in $\mathfrak{B}$ and *admitted* by σ.

A cross-section of a sphere bundle may admit various vector bnudles. We define a standard one in them as follows.

Let $\tau(S^n) = (E_\tau, S^n, \pi)$ be the tangent vector bundle of S^n, and denote the family of coordinate neighborhoods and that of coordinate functions by $\{U_l\}$ and $\{\lambda_l\}$ respectively. Here we assume that the structural group of $\tau(S^n)$ is $O(n)$ in place of $GL(n, R)$, using 12.9 Theorem of [5]. Let $\mathfrak{B} = (E, B, p)$ be an n-sphere bundle with the family of coordinate neighbordhoods $\{V_j\}$ and that of coordinate functions $\{\phi_j\}$, and assume that $\mathfrak{B}$ has a cross-section $\sigma: B\longrightarrow E$.

Now we define $W_{(j,l)}$ by

$$W_{(j,l)} = V_j\cap(\sigma^{-1}\phi_j p_2^{-1}(U_l))$$

where $p_2: V_j\times S^n\longrightarrow S^n$ is the projection into the second factor, and define

$$g^*_{(j,l)(i,k)} : W_{(i,k)}\cap W_{(j,l)}\longrightarrow O(n)$$

by setting for $b\in W_{(i,k)}\cap W_{(j,l)}$

$$g^*_{(j,l)(i,k)}(b) = \lambda_{l,y}^{-1}\circ\alpha_y\circ\phi_{j,b}^{-1}\circ\phi_{i,b}\circ\alpha_x^{-1}\circ\lambda_{k,x}$$

where $y = \phi_{j,b}^{-1}(\sigma(b))$, $x = \phi_{i,b}^{-1}(\sigma(b))$, α_x and α_y are defined as in the proof of Theorem 2.2.

Then the n-dimensional vector bundle $\eta=(E^*, B, p^*)$ with the family of coordinate neighborhoods $\{W_{(j,l)}\}$ and that of coordinate transformations $\{g^*_{(j,l)(i,k)}\}$ is admitted by σ.

DEFINITION 3.2. η is called the *standard admissible vector bundle* of σ, and denoted by $s.a.v.b.$ of σ.

THEOREM 3.1. *The definition of* $s.a.v.b.$ *of* σ *is independent of the choice of* $\{U_l\}$ and $\{\lambda_l\}$ of $\tau(S^n)$ *up to isomorphism.*

Proof. Let another family of coordinate neighborhoods be $\{U'_m\}$ and the corresponding family of coordinate functions be $\{\lambda'_m\}$. Then our construction gives an n-dimensional vector bundle $\eta'=(E', B, p')$ with the family of coordinate neighborhoods $\{W'_{(t,m)}\}$ and that of coordinate transformations $\{g'_{(t,m)(s,p)}\}$ where

$$W'_{(t,m)}=V_t\cap(\sigma^{-1}\phi_t p_2{}^{-1}(U'_m))$$

and

$$g'_{(t,m)(s,p)}(b)=\lambda'_{m,y}{}^{-1}\circ\alpha_y\circ\phi_{t,b}{}^{-1}\circ\phi_{s,b}\circ\alpha_x{}^{-1}\circ\lambda'_{p,x}$$

with

$$y=\phi_{t,b}{}^{-1}(\sigma(b)) \text{ and } x=\phi_{s,b}{}^{-1}(\sigma(b)).$$

Let the family of coordinate functions of η be $\{\phi^*_{(j,l)}\}$ and

$$\phi^*_{(j,l),b}: R^n \longrightarrow p^{*-1}(b)$$

be defined by setting

$$\phi^*_{(j,l),b}(y)=\phi^*_{(j,l)}(b, y),$$

and similarly $\{\phi'_{(t,m)}\}$ and $\{\phi'_{(t,m),b}\}$ be defined for η'.

We define $h: E^*\longrightarrow E'$ by setting

$$h(z)=\phi'_{(t,m),b}\lambda_{m,x}{}^{-1}\alpha_x\phi_{t,b}{}^{-1}\phi_{j,b}\alpha_y{}^{-1}\lambda_{l,y}\phi^*_{(j,l),b}{}^{-1}(z) \quad \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

where $b=p^*(z)\in W_{(j,l)}\cap W'_{(t,m)}$, $x=\phi_{t,b}{}^{-1}(\sigma(b))$ and $y=\phi_{j,b}{}^{-1}(\sigma(b))$.

1°. This is well defined because the the right side of (1) is independent of the choice of j, l, t and m as seen below. We assume that $b\in W_{(j,l)}\cap W_{(i,k)}$. Let $\tilde{y}=\phi_{i,b}{}^{-1}(\sigma(b))$, then we have

$$\begin{aligned}\phi^*_{(j,l),b}{}^{-1}\circ\phi^*_{(i,k),b}&=g_{(j,l)(i,k)}(b)\\ &=\lambda_{l,y}{}^{-1}\circ\alpha_y\circ\phi_{j,b}{}^{-1}\circ\phi_{i,b}\circ\alpha_{\tilde{y}}{}^{-1}\circ\lambda_{k,\tilde{y}},\end{aligned}$$

accordingly,

$$\phi_{j,b}\circ\alpha_y{}^{-1}\circ\lambda_{l,y}\circ\phi^*_{(j,l),b}{}^{-1}=\phi_{i,b}\circ\alpha_{\tilde{y}}{}^{-1}\circ\lambda_{k,\tilde{y}}\circ\phi^*_{(i,k),b}{}^{-1}.$$

Hence $\phi_{j,b}\circ\alpha_y{}^{-1}\circ\lambda_{l,y}\circ\phi^*_{(j,l),b}{}^{-1}$ is independent of the choice of j and l. Similarly $\phi'_{(t,m),b}\circ\lambda'_{m,x}\circ\alpha_x\circ\phi_{t,b}{}^{-1}$ is independent of the choice of t and m. Thus 1° is proved.

2°. h is a bunndle map which induces the identity map of B as seen below. Let $h_b : p^{*-1}(b) \longrightarrow p'^{-1}(b)$ be the map induced by h, then

$$\phi'_{(t,m)} \circ h_b \circ \phi^*_{(j,l)} = \lambda'_{m,x}{}^{-1} \circ \alpha_x \circ \phi_{t,b}{}^{-1} \circ \phi_{j,b} \circ \alpha_y{}^{-1} \circ \lambda_{l,y}.$$

This is an orthogonal transformation of R^n, because $\phi_{t,b}{}^{-1} \circ \phi_{j,b}$ is an orthogonal transformation of S^n which sends y to x. The proof of other properties of h to be a bundle map is trivial.

1° and 2° complete the proof of the theorem.

THEOREM 3.2. *When two n-sphere bundles $\mathfrak{B} = (E, B, p)$ and $\mathfrak{B}' = (E', B, p')$ are isomorphic each other, if a cross-section σ of $\mathfrak{B}$ admits a s.a.v.b. η, then $\mathfrak{B}'$ has a cross-section σ' whose s.a.v.b. is isomorphic to η.*

Proof. For the homeomorphism $H : E \longrightarrow E'$ which induces the isomorphism between $\mathfrak{B}$ and $\mathfrak{B}'$, let $\sigma' = H \circ \sigma$ and η' be the s.a.v.b. of σ'. Let the families of coordinate functions of $\mathfrak{B}$ and $\mathfrak{B}'$ be $\{\phi_j\}$ and $\{\psi_i\}$ respectively, and those of $\eta = (E^*, B, p^*)$ and $\eta' = (\bar{E}, B, \bar{p})$ be $\{\phi^*_{(j,b)}\}$ and $\{\psi^*_{(i,k)}\}$ respectively. For $b \in W_{(j,l)} \cap W'_{(i,k)}$, where $\{W_{(j,l)}\}$ and $\{W'_{(i,k)}\}$ be the families of coordinate neighborhoods of η and η' respectively, we define a homeomorphism

$$h : E^* \longrightarrow \bar{E}$$

by setting

$$h(z) = \psi^*_{(i,k),b} \lambda_{k,y}{}^{-1} \alpha_y \psi_{i,b} H \phi_{j,b} \alpha_x{}^{-1} \lambda_{l,x} \phi^*_{(j,l),b}{}^{-1}(z),$$

where $x = \phi_{j,b}{}^{-1}\sigma(b)$ and $y = \psi_{i,b}{}^{-1}\sigma'(b)$. Then h induces the isomorphism between η and η'.

The converse of this theorem holds in the sense of Corollary 3.5.

THEOREM 3.3. *Every n-dimensional vector bundle over a C_σ-space B is isomorphic with a s.a.v.b. of some cross-section of an n-sphere bundle.*

Proof. Let $\xi = (E, B, p)$ be an n-dimensional vector bundle with the family of coordinate neighborhoods $\{V_j\}$ and that of coordinate transformations $\{g_{ji}\}$. Here we may assume that the structural group of ξ is $O(n)$ because B is a C_σ-space, using 12.9 theorem of [5]. We define an n-sphere bundle $\mathfrak{B} = (\bar{E}, B, \bar{p})$ with the same family of coordinate neighborhoods $\{V_j\}$ and the family of coordinate transformations $\{\bar{g}_{ji}\}$, where

$$\bar{g}_{ji} : V_i \cap V_j \longrightarrow O(n+1)$$

is defined as follows. Let x be a base point of S^n and $A(x)$ be the antipodal point of x, and define α_n as in the proof of Theorem 2.2. For $b \in V_i \cap V_j$,

$$\overline{g}_{ji}(b) : S^n \longrightarrow S^n$$

is defined by setting

$$\overline{g}_{ji}(b)(z) = \begin{cases} \alpha_x^{-1} g_{ji}(b) \alpha_x(z) & \text{if } z \neq A(x), \\ A(x) & \text{if } z = A(x). \end{cases}$$

Let the families of coordinate functions of ξ and $\mathfrak{B}$ be $\{\phi_j\}$ and $\{\psi_j\}$ respectiveiy. We define a cross-section $\sigma : B \longrightarrow \overline{E}$ by setting

$$\sigma(b) = \psi_{j,b}(x)$$

with j such that $b \in V_j$. This is well defined because,

$$\psi_{j,b}^{-1} \psi_{i,b}(x) = x \text{ if } b \in V_i \cap V_j.$$

Let the s. a. v. b. of σ be $\eta = (E^*, B, p^*)$, the families of coordinate neighborhoods and coordinate transformations be $\{W_{(j,l)}\}$ and $\{g^*_{(j,l)(i,k)}\}$ respectively. Here we assume without loss of generality that only one coordinate neighborhood U_l of $\tau(S^n)$ contains x. Then $W_{(j,l)} = V_j$ and $W_{(j,k)}$ is empty for $k \neq l$. Accordingly if there is a point $b \in W_{(i,k)} \cap W_{(j,l)}$, then $k=l$ and

$$\begin{aligned} g^*_{(j,l)(i,l)}(b) &= \lambda_{l,x}^{-1} \circ \alpha_x \circ \overline{g}_{ji}(b) \circ \alpha_x^{-1} \circ \lambda_{l,x} \\ &= \lambda_{l,x}^{-1} \circ g_{ji}(b) \circ \lambda_{l,x} \\ &= g_{ji}(b). \end{aligned}$$

Thus ξ and η have cnmmon families of coordinate neighborhoods and coordinate transformations, accordingly are isomorphic.

THEOREM 3. 4. *Let $\mathfrak{B} = (E, B, p)$ be the associated n-sphere bundle of an $(n+1)$-dimensional vector bundle $Y = (\overline{E}, B, \overline{p})$ and let $\xi = (E^*, B, p^*)$ be the s. a. v. b. of a cross-section σ of $\mathfrak{B}$. Then $\xi \oplus \varepsilon^1_B$ is isomorphic with Y, where ε^1_B is a trivial 1-dimensional vector bundle over B.*

Proof. Let the families of coordinate neighborhoods and coordinate functions of $\mathfrak{B}$ be $\{V_j\}$ and $\{\phi_j\}$ respectively, and let those of ξ be $\{W_{(j,l)}\}$ and $\{\phi^*_{(j,l)}\}$ respectively. Then the families of coordinate neighborhoods of $\xi \oplus \varepsilon^1_B = (E', B, p')$ is also $\{W_{(j,l)}\}$ and that of coordinate functions is $\{\phi'_{(j,l)}\}$, where $E' = E^* \times R$ and

$$\phi'_{(j,l)} : W_{(j,l)} \times R^{n+1} \longrightarrow p'^{-1}(W_{(j,l)}) = p^{*-1}(W_{(j,l)}) \times R$$

is defined by setting for $b \in W_{(j,l)}$ and $(y, z) \in R^n \times R = R^{n+1}$

$$\phi'_{(j,l),b}(y, z) = (\phi^*_{(j,l),b}(y), z).$$

Let the family of coordinate functions of Y be $\{\psi_j\}$. We define a homeomorphism

$$h : E' = E^* \times R \longrightarrow \overline{E}$$

by setting

$$h(y, z) = \phi_{j,b}\gamma_{x,z}\beta_x\lambda_{l,x}\phi^*_{(j,l),b}{}^{-1}(y),$$

where $b = p'(y, z) = p^*(y) \in W_{(j,l)}$, $x = \phi_{j,b}{}^{-1}(\sigma(b))$, λ_l is a coordinate function of the tangent bundle $\tau(S^n)$ used in the definition of ξ, β_x is a translation of R^n_x in R^{n+1} which carries x onto the origin of R^{n+1} and $\gamma_{x,z}$ is also a translation with the same direction as β_x by the distance z. (Cf. Proof of Theorem 2. 2). This definition is independent of the choice of j and l as seen below. Let $b \in W_{(i,k)} \cap W_{(j,l)}$ and $\bar{x} = \phi_{i,b}{}^{-1}(\sigma(b))$, and let the common family of cordinate transformations of $\mathfrak{B}$ and Y be $\{g_{ji}\}$ and that of ξ be $\{g^*_{(j,l)(i,k)}\}$. Then

$$(\phi_{j,b}\circ\gamma_{x,z}\circ\beta_x\circ\lambda_{l,x}\circ\phi^*_{(j,l),b}{}^{-1})^{-1}\circ(\phi_{i,b}\circ\gamma_{\bar{x},z}\circ\beta_{\bar{x}}\circ\lambda_{k,\bar{x}}\circ\phi^*_{(i,k),b}{}^{-1})$$
$$= \phi^*_{(j,l),b}\circ\lambda_{l,x}{}^{-1}\circ\beta_x{}^{-1}\circ\gamma_{x,z}{}^{-1}\circ\phi_{j,b}{}^{-1}\circ\phi_{i,b}\circ\gamma_{\bar{x},z}\circ\beta_{\bar{x}}\circ\lambda_{k,\bar{x}}\circ\phi^*_{(i,k),b}{}^{-1}$$
$$= \phi^*_{(j,l),b}\circ\lambda_{l,x}{}^{-1}\circ\beta_x{}^{-1}\circ\gamma_{x,z}{}^{-1}\circ g_{ji}(b)\circ\gamma_{\bar{x},z}\circ\beta_{\bar{x}}\circ\lambda_{k,\bar{x}}\circ\phi^*_{(i,k),b}{}^{-1}$$
$$= \phi^*_{(j,l),b}\circ\lambda_{l,x}{}^{-1}\circ g_{ji}(b)\circ\lambda_{k,\bar{x}}\circ\phi^*_{(i,k),b}{}^{-1}$$
$$= \phi^*_{(j,l),b}\circ g^*_{(j,l)(i,k)}(b)\circ\phi^*_{(i,k),b}{}^{-1}$$
$$= 1.$$

Thus h is well defined. This homeomorphism induces the isomorphism between $\xi\oplus\varepsilon^1_B$ and Y.

COROLLARY 3. 5. *If two sphere bundles $\mathfrak{B}$ and $\mathfrak{B}'$ have cross-sections σ and σ' respectively and the s. a. v. b. of σ is isomorphic with that of σ', then $\mathfrak{B}$ is isomorphic with $\mathfrak{B}'$.*

When n-dimensional vector bundles ξ and η over a C_σ-space B are s. a. v. b. s of cross-sections of a same n-sphere bundle, we say that they are c-equivalent. When ξ is isomorphic with ξ', η is isomorphic with η' and ξ' is c-equivalent with η', we also say that ξ is c-equivalent with η. By Theorem 3. 3 and Corollary 3. 5 c-equivalence is an equivalence relation. Let $\mathrm{CO}^n(B)$ be the set of above c-equivalence classes, and define

$$h : \mathrm{CO}^n(B) \longrightarrow \mathrm{CO}^{n+1}(B)$$

by setting $h(\{\xi\}) = \{Y\}$, where $\{\ \}$ denotes a c-equivalence class and Y is an $(n+1)$-dimensional vector bundle such that ξ is isomorphic with the s. a. v. b. of a cross-section of the n-sphere bundle which is associated with Y. Then the limit set $\underrightarrow{\mathrm{Lim}}\,\mathrm{CO}^n(B)$ is nothing but $\mathrm{ko}(B)$ as a set. (Cf. [1]).

On microbundles we cannot consider as above because we don't know the theorem corresponding to Theorem 3. 4, i. e. we don't know whether the structural group $O(n+1)$ of $\mathfrak{B}$ in Theorem 2. 2 can or cannot be replaced by $\mathcal{H}$.

THEOREM 3. 6. *Let $\mathfrak{B} = (E, B, p)$ be an n-sphere bundle with cross-section σ, ξ be* *the s. a. v. b. of σ and let $f: X \longrightarrow B$ be a map. Then $f^*\xi$ is the s. a. v. b. of some cross-section of $f^*\mathfrak{B}$.*

Proof. About $\mathfrak{B}$ and ξ, we use notations given in the definition of s. a. v. b. Let $f^*\mathfrak{B} = (\overline{E}, B, \overline{p})$, then $\overline{E} = \{(x, e) : x \in X,\ e \in E,\ f(x) = pe\}$, and $\overline{p}(x, e) = x$. The families of coordinate neighborhoods are $\{f^{-1}V_j\}$, the coordinate function $\overline{\phi_j}$ and the coordinate transformation $\overline{g_{ji}}$ are given by

$$\overline{\phi_j}(x, y) = (x, \phi_j(f(x), y))$$

and

$$\overline{g_{ji}}(x) = g_{ji}(f(x))$$

respectively. We define a cross-section $\overline{\sigma} : X \longrightarrow \overline{E}$ by setting

$$\overline{\sigma}(x) = (x, \sigma f(x)).$$

Then the s. a. v. b. of $\overline{\sigma}$, denoted by $\overline{\xi}$, is isomorphic to $f^*\xi$ as seen below.

The family of coordinate neighborhoods $\{W'_{(j,l)}\}$ of $\overline{\xi}$ is given by

$$W'_{(j,l)} = (f^{-1}V_j) \cap (\overline{\sigma}^{-1}\overline{\phi}_j\overline{p}_2^{-1}(U_l))$$

where $\overline{p}_2 : (f^{-1}V_j) \times S^n \longrightarrow S^n$ is the projection into the second factor, and as easily seen,

$$W'_{(j,l)} = f^{-1}(W_{(j,l)}) \quad \cdots\cdots\cdots\cdots\cdots\cdots (1).$$

The coordinate transformation $g'_{(j,l)(i,k)}$ of $\overline{\xi}$ is given by setting, for $x \in W'_{(i,k)} \cap W'_{(j,l)}$,

$$g'_{(j,l)(i,k)}(x) = \lambda_{l,t}^{-1} \circ \alpha_t \circ \overline{\phi}_{j,x}^{-1} \circ \overline{\phi}_{i,x} \circ \alpha_s^{-1} \circ \lambda_{k,s}$$

where $t = \overline{\phi}_{j,x}^{-1}(\overline{\sigma}(x))$, $s = \overline{\phi}_{i,x}^{-1}(\overline{\sigma}(x))$. By definitions,

$$(x, \phi_j(f(x), t)) = \overline{\phi}_j(x, t) = \overline{\sigma}(x) = (x, \sigma f(x)).$$

Hence

$$\phi_j(f(x), t) = \sigma f(x), \text{ i. e. } t = \phi_{j, f(x)}^{-1}(\sigma(f(x))).$$

Similarly

$$s = \phi_{i, f(x)}^{-1}(\sigma(f(x))).$$

Therefore

$$g'_{(j,l)(i,k)}(x) = \lambda_{l,t}^{-1} \circ \alpha_t \circ \phi_{j,f(x)}^{-1} \circ \phi_{i,f(x)} \circ \alpha_s^{-1} \circ \lambda_{k,s}$$
$$= g^*_{(j,l)(i,k)}(x) \quad \cdots\cdots\cdots\cdots\cdots\cdots (2).$$

By (1) and (2), $\overline{\xi}$ is isomorphic to $f^*\xi$.

We say that two cross-sections σ and σ' of a fibre bundle $\mathfrak{B}$ are sec-homotopic if there is a homotopy between σ and σ' which is a cross-section of $\mathfrak{B}$ in each stage.

THEOREM 3.7. *Let $\mathfrak{B}=(E, B, p)$ be an n-sphere bundle over a compact base space B with cross-sections σ and σ', and let ξ and ξ' be the s.a.v.b.s of them respectively. If σ and σ' are sec-homotopic, then ξ and ξ' are isomorphic.*

Proof. Let the family of coordinate neighborhoods and that of coordinate functions of $\mathfrak{B}$ be $\{V_j\}$ and $\{\phi_j\}$ respectively.

1°. We assume that σ is near enough to σ' in the following sense; let $x=\phi_{j,b}^{-1}(\sigma(b))$ and $y=\phi_{j,b}^{-1}(\sigma'(b))$ and regard S^n to be the standard n-shere in R^{n+1}, then $\angle xOy$ is smaller than π for every j and $b \in V_j$, where O is origin of R^{n+1} and $\angle xOy$ is regarded to be non-negative. Let R^2 be the plane generated by $\overrightarrow{Ox}$ and $\overrightarrow{Oy}$, and R^{n-1} be the $(n-1)$-space which contains O and orthogonal to R^2. Let $\theta_j(b)$ be the element of $O(n+1)$ which fixes R^{n-1} and rotates R^2 around O by $\angle xOy$ such as $\theta_j(b)(x)=y$, then the induced function $\theta_j : V_j \longrightarrow O(n+1)$ is a map. Now we define a map $f: E \longrightarrow E$ by setting

$$f(e)=\phi_{j,b}\theta_j(b)\phi_{j,b}^{-1}(e),$$

where $b=p(e) \in V_j$. When $b \in V_i \cap V_j$, we have

$$\begin{aligned}&(\phi_{j,b}\circ\theta_j(b)\circ\phi_{j,b}^{-1})^{-1}\circ(\phi_{i,b}\circ\theta_i(b)\circ\phi_{i,b}^{-1})\\ &=\phi_{j,b}\circ\theta_j(b)^{-1}\circ g_{ji}(b)\circ\theta_i(b)\circ\phi_{i,b}^{-1}\\ &=\phi_{j,b}\circ g_{ji}(b)\circ\phi_{i,b}^{-1}=1.\end{aligned}$$

Thus f is well defined. This map induces an isomorphism of $\mathfrak{B}$ with itself, and $\sigma'=f\circ\sigma$. This fact means that ξ is isomorphic with ξ' as seen in the proof of Theorem 3.2.

2°. In general cace, let the sec-homotopy between σ and σ' be h_t $(0 \leqq t \leqq 1)$. Then by the compactness of B, we have a finite decomposition of $[0,1]$,

$$0=t(0)<t(1)< \cdots\cdots\cdots\cdots <t(n-1)<t(n)=1,$$

such that $h_{t(i-1)}$ is near enough to $h_{t(i)}$ in the sense of 1° for $i=1,2,\cdots\cdots,n$. Accordingly we have maps $f_i : E \longrightarrow E$ which induces isomorphism of $\mathfrak{B}$ with itself and $h_{t(i)}=f_i\circ h_{t(i-1)}$ for each i. The map $f=f_n\circ f_{n-1}\circ\cdots\cdots\circ f_2\circ f_1$ induces the isomorphism of $\mathfrak{B}$ with itself, and $\sigma'=f\circ\sigma$. Therefore ξ is isomorphic with ξ'.

The converse of this theorem does not hold as follows.

EXAMPLE 3.1. Let $T=S^1\times S^1$ and $p: T \longrightarrow S^1$ be the projection into the first factor, and let $\mathfrak{B}=(T, S^1, p)$ be the trivial sphere bundle. Then the set of sec-homotopy classes of cross-sections corresponds one to one with Z, but the s.a.v.b. of any cross-section of $\mathfrak{B}$ is trivial.

We say a bundle $\eta=(\overline{E}, \overline{B}, \overline{p})$ to be an extension of $\xi=(E, B, p)$ over $\overline{B}$, when

$B \subset \overline{B}$ and ξ is the restriction of η over B.

Let B be a locally finite $(n+1)$-dimensional complex, $B^{(n)}$ be its n-skeleton, and ξ be a vector bundle over $B^{(n)}$. The extension of ξ over B don't always exist. For example a Möbius band $\xi = (M, S^1, p)$ over $S^1 = \dot{D}^2$ cannot be extended over D^2.

THEOREM 3.8. *Let B and $B^{(n)}$ be as above, $\mathfrak{B} = (E, B, p)$ be an n-sphere bundle and let $\mathfrak{B}|B^{(n)}$ be the restriction of $\mathfrak{B}$ over $B^{(n)}$. Assume that $\mathfrak{B}|B^{(n)}$ has a cross-section $f : B^{(n)} \longrightarrow E|B^{(n)}$ and n is even, and let ξ be the s.a.v.b. of f. Then f is extendable to a cross-section of $\mathfrak{B}$ if and only if ξ is extendable over B.*

To prove this theorem we prepare two lemmas.

LEMMA 3.9. *Let $\mathfrak{B}$ be a k-sphere bundle over B with a cross-section f, $\mathfrak{B}|B^{(n)}$ be the restriction over $B^{(n)}$ and let $\bar{f}$ be the restriction of f over $B^{(n)}$. Then the s.a.v.b. of $\bar{f}$ is the restriction of the s.a.v.b. of f over $B^{(n)}$.*

This lemma is trivial by the definition of s.a.v.b.

LEMMA 3.10. *Let $\mathfrak{B} = (D^{n+1} \times S, D^{n+1}, p_1)$ be a trivial n-sphere bundle, $\mathfrak{B}|S^n$ be its restriction over $S^n = \dot{D}^{n+1}$, f be a cross-section of $\mathfrak{B}|S^n$ and let ξ be the s.a.v.b. of f, where n is even. Then the following four statements are equivalent.*

① *ξ has a non-zero cross-section.*

② *f is extendable to a cross-section of $\mathfrak{B}$.*

③ *ξ is extendable over D^{n+1}.*

④ *ξ is trivial.*

Proof. Let $p_2 : S^n \times S^n \longrightarrow S^n$ be the projection into the second factor, then $p_2 \circ f$ is a map of S^n into S^n. By the trivialness of $\mathfrak{B}$ two cross-sections f and f' are sec-homotopic if and only if $p_2 \circ f$ and $p_2 \circ f'$ are homotopic. Let ξ' be the s.a.v.b. of f', then ξ is isomorphic with ξ' if f is sec-homotopic to f' by Theorem 3.7. So it suffices to consider in each case when the degree of $p_2 \circ f$ is r.

(a). In the case of $r = 0$, ξ is trivial.

(b). In the case of $r=1$, ξ is isomorphic with $\tau(S^n)$. Because Euler characteristic of $S^n = 2$, ξ has not any non-zero cross-section. (Cf. [4, Theorem 16 and 22]).

(c). In the case of $r>1$, ξ is isomorphic with $f^*(\tau(S^n))$ and has not any non-zero cross-section, because $\bar{c}(\xi) = f^*\bar{c}(\tau(S^n)) = r \cdot \bar{c}(\tau(S^n)) = 2r \neq 0$. (Cf. [5, 35.7 Theorem]).

By (a), (b) and (c), ξ has a non-zero cross-section if and only if the degree of $p_2 \circ f$ is zero. Hence if ① holds then ② holds. If ② holds then ③ holds by Lemma 3.9. If ③ holds then ④ holds becase D^{n+1} is contractible. It is trivial that if ④ holds then ① holds. Thus Lemma 3.10 is proved.

When n is an odd number, various cases can be considered about the maximum number of independent non-zero cross-sections of ξ, hence we cannot obtain the analogous result as Lemma 3.10.

Proof of Theorem 3.8. Let σ be an $(n+1)$-simplex of B, then $\mathfrak{B}|\sigma$ is trivial, and $f|\dot{\sigma}$ is a sross-section of $\mathfrak{B}|\dot{\sigma}$ where $\dot{\sigma}$ is the boundary of σ. f is extendable to a cross-section of $\mathfrak{B}$ if and only if $f|\dot{\sigma}$ is extendable to a cross-section of $\mathfrak{B}|\sigma$ for every $(n+1)$-simplex σ of B, and ξ is extendable over B if and only if $\xi|\dot{\sigma}$ is extendable over σ for every $(n+1)$-simplex σ of B. Hence Theorem 3.8 follows to Lemma 3.10.

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