On extreme points of the unit ball of a non-commutative L^p-space with $0<p\leq 1$

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1. Introduction

The notion of an extreme point is playing an important role in the theory of topological vector spaces, in particular, that of Banach spaces. Some properties of a linear map can be stated in terms of extreme points of a certain convex set of linear maps. For example, a non-zero representation of a C^*-algebra is irreducible if and only if it is spatially equivalent to a GNS-representation associated with a pure state (that is an extreme point of the state space). R. Kadison gave a characterization of extreme points of the unit ball of a C^*-algebra, and he applied it to classify isometries between C^*-algebras. Thus it is fundamental to characterize extreme points of the unit ball of a (quasi-) Banach space associated with an operator algebra. For a commutative L^p-space $L^p(X, \mu)$ of measurable functions, the results are well-known. In case of $1<p<\infty$, the unit sphere is precisely the set of all extreme points of the unit ball. If $p=\infty$, the set of all extreme points is exactly the unitary group of $L^\infty(X, \mu)$ ([**6**, Chapter I, Lemma 10.11]). For $0<p\leq 1$, there exists an extreme point if and only if the measure space (X, μ) has an atom.

Our first aim in this paper is to give a necessary and sufficient condition for the existence of an extreme point of the closed unit ball of a non-commutative L^p-space. Secondarily, we shall determine the form of each extreme point completely. However, when $1<p<\infty$, it is shown that the Clarkson-McCarthy's inequality holds for non-commutative L^p-spaces associated with von Neumann algebras (see [**1**], [**3**], [**7**]). Therefore they are uniformly convex, in particular, strictly convex. Thus the set of all extreme points coincides with the unit sphere. For $p=\infty$, L^∞-space was defined to be the von Neumann algebra itself and so it is a C^*-algebra. If S is the unit ball of a C^*-algebra A, the following facts are well-known; (i) there exists an extreme point x in S if and only if A is unital; and (ii) when A is unital, then $x\in S$ is extreme if and only if x^*x is a projection such that $(1-x^*x)A(1-xx^*)=\{0\}$ ([**6**, Chapter I, Theorem 10.2]). Therefore, we may concentrate our attention to the case of $0<p\leq 1$. Finally we also consider

an example for $p=1$ which satisfies the same consequence as the Klein-Milman's Theorem.

2. Preliminaries

In this section we recall some basic results as well as definitions of non-commutative L^p-spaces associated with a von Neumann algebra which is not necessarily semifinite. Let M be an arbitrary von Neumann algebra with a faithful normal semifinite weight φ_0. Denote by N the crossed product $M \rtimes_{\sigma^{\varphi_0}} \boldsymbol{R}$ determined by M and the modular automorphism group $\{\sigma_t^{\varphi_0}\}_{t\in\boldsymbol{R}}$ with respect to φ_0. Then there exists a canonical faithful normal semifinite trace τ on N satisfying $\tau\circ\theta_s=e^{-s}\tau$, $s\in\boldsymbol{R}$, where $\{\theta_s\}_{s\in\boldsymbol{R}}$ is the dual action of $\{\sigma_t^{\varphi_0}\}_{t\in\boldsymbol{R}}$. Also, we denote by $\widetilde{N}$ the set of all τ-measurable operators (affiliated with N). For $0<p\leq\infty$, the Haagerup's L^p-space $L^p(M)$ is defined by

$$L^p(M)=\{a\in\widetilde{N};\ \theta_s(a)=e^{-s/p}a,\ s\in\boldsymbol{R}\}.$$

For each $\varphi\in M_{*,+}$, a unique $h_\varphi\in\widetilde{N}_+$ is given by $\widetilde{\varphi}=\tau(h_\varphi\cdot)$ where $\widetilde{\varphi}$ is the dual weight of φ. The mapping $\varphi\longrightarrow h_\varphi$ is extended to a linear order isomorphism from M_* onto $L^1(M)$, and so the linear functional tr on $L^1(M)$ is defined by $tr(h_\varphi)=\varphi(1)$, $\varphi\in M_*$. For $0<p<\infty$, the (quasi-) norm of $L^p(M)$ is defined by

$$\|a\|_p=tr(|a|^p)^{1/p},\qquad a\in L^p(M).$$

When $1\leq p<\infty$, $L^p(M)$ is a Banach space with the norm $\|\cdot\|_p$ and its dual Banach space is $L^q(M)$ where $1/p+1/q=1$ by the following duality;

$$(a,b)=tr(ab)=tr(ba),\qquad a\in L^p(M),\ b\in L^q(M).$$

The space $L^p(M)$ is independent of the choice of φ_0 up to isomorphism. Furthermore, if M is semifinite with a faithful normal semifinite trace τ_0, the Haagerup L^p-space constructed by τ_0 can be identified with the classical non-commutative L^p-space $L^p(M,\tau_0)$.

3. Main Theorem

Let M be an arbitrary von Neumann algebra and let M_1 (resp. M_2) be the discrete (resp. continuous) direct summand of M. Fix any $0<p\leq 1$ and fix a faithful normal semifinite trace τ on M_1. We denote by S, S_1, S_2 the unit ball of $L^p(M)$, $L^p(M_1,\tau)$, $L^p(M_2)$, respectively, and we denote by $\mathrm{Ext}(S)$ the set of all extreme points of S. Then the main result of this note is

THEOREM 1. Keep the situations and notations as above. Then (i) $\mathrm{Ext}(S)$ is not empty if and only if M has a minimal projection. (ii) If M has a minimal projection, then $\mathrm{Ext}(S)=\mathrm{Ext}(S_1)\oplus 0$. For $x\in L^p(M_1,\tau)$ with its polar decomposition $x=u|x|$, $x\in\mathrm{Ext}(S_1)$ if and only if $e=u^*u$ is minimal and $|x|=\tau(e)^{-1/p}e$.

To prove this theorem, we need some lemmas.

LEMMA 2. If there exists an element x in Ext(S), then the support projection $s(|x|)$ of $|x|$ is minimal in M.

PROOF. Suppose that $s(|x|)$ is not minimal in M. Thus, there exists a projection e in M such that $0<e<s(|x|)$. Putting $f=s(|x|)-e$, we have $1=\|x\|_p^p=\||x|^p\|_1=\||x|^{p/2}\cdot e|x|^{p/2}\|_1+\||x|^{p/2}f|x|^{p/2}\|_1$. If $\lambda=\||x|^{p/2}e|x|^{p/2}\|_1=0$ (resp. 1), then it is easy to see that $e=0$ (resp. $s(|x|)$). Hence we have $0<\lambda<1$. By Hölder's inequality, we have

$$\||x|^{1/2}e|x|^{1/2}\|_p\leq\||x|^{\frac{1}{2}-\frac{p}{2}}\|_\alpha\||x|^{\frac{p}{2}}e|x|^{\frac{p}{2}}\|_1\||x|^{\frac{1}{2}-\frac{p}{2}}\|_\alpha\leq\lambda,$$

where $\alpha=\dfrac{2p}{1-p}$. Thus we obtain a convex combination of two elements in S:

$$|x|=\lambda\frac{|x|^{1/2}e|x|^{1/2}}{\lambda}+(1-\lambda)\frac{|x|^{1/2}f|x|^{1/2}}{1-\lambda}.$$

Let $x=u|x|$ be the polar decomposition of x and let $|x|=\int_0^\infty t\,de_t$ be the spectral decomposition of $|x|$. Then we clearly have

$$x=\lambda\frac{u|x|^{1/2}e|x|^{1/2}}{\lambda}+(1-\lambda)\frac{u|x|^{1/2}f|x|^{1/2}}{1-\lambda}.$$

Since $x\in$ Ext (S), we conclude that

$$x=\frac{u|x|^{1/2}e|x|^{1/2}}{\lambda}=\frac{u|x|^{1/2}f|x|^{1/2}}{1-\lambda}.$$

Therefore, we have that $\lambda|x|=|x|^{1/2}e|x|^{1/2}$. Multiplying on the left and right side by $\int_{1/n}^\infty t^{-1/2}\,de_t$, we get that $\lambda E_n=E_neE_n$ with $E_n=\int_{1/n}^\infty de_t$. Since $\{E_n\}$ converges to $s(|x|)$ strongly, it follows that $\lambda s(|x|)=s(|x|)es(|x|)=e$. This contradiction completes the proof.

We denote by z the unique central projection in M such that $M_1=Mz$ is discrete and $M_2=M(1-z)$ is continuous. Then $L^p(M)$ is isometricaly isomorphic to the l^p-direct sum $L^p(Mz,\tau)\oplus L^p(M(1-z))$, where τ is a faithful normal semifinite trace on Mz. Lemma 2 shows that Ext $(S_2)=\phi$. Moreover we have the following lemma.

LEMMA 3. Ext$(S)=\phi$ if and only if Ext$(S_1)=\phi$. If Ext$(S)\neq\phi$, then Ext$(S)=$Ext $(S_1)\oplus 0$.

PROOF. Suppose that $x\in$ Ext (S). Then it follows from Lemma 2 that $s(|x|)$ is a minimal projection in Mz satisfying $s(|x|)\leq z$. This implies that $x=xz\in$ Ext(S_1). Conversely, suppose that $x_1\in$ Ext(S_1). If $x_1\oplus 0=\frac{1}{2}y+\frac{1}{2}y'$ for some elements y, y' in S, then we have that $x_1=\frac{1}{2}yz+\frac{1}{2}y'z$. It follows from the assumption that $x_1=yz=y'z$

and $1=\|x_1\|_p=\|yz\|_p$. Since $1\geq\|y\|_p^p=\|yz\|_p^p+\|y(1-z)\|_p^p$, we have that $\|y(1-z)\|_p=0$. Consequently, we conclude that $y=yz\oplus 0=x_1\oplus 0$ and that $x_1\oplus 0\in \mathrm{Ext}(S)$.

Therefore, to prove Theorem 1, it is sufficient to consider the case of discrete von Neumann algebras.

LEMMA 4. Let M be a von Neumann algebra of discrete type, and let τ be a faithful normal semifinite trace on M. If e is a minimal projection in M, then $\tau(e)^{-1/p}e\in\mathrm{Ext}(S)$.

PROOF. Since e is a minimal projection in M, we have $0<\tau(e)<\infty$ by the semifiniteness of τ. Suppose that there exists two elements x_1 and x_2 in S such that $\tau(e)^{-1/p}e=\frac{1}{2}x_1+\frac{1}{2}x_2$. From the minimality of e, there exist scalars λ_1, λ_2 satisfying that $ex_1e=\lambda_1 e$ and $ex_2e=\lambda_2e$, so $\tau(e)^{-1/p}e=\frac{1}{2}(\lambda_1+\lambda_2)e$. Since $|\lambda_1|^p\tau(e)=\|ex_1e\|_p^p\leq\|x_1\|_p^p\leq 1$, we have $|\lambda_1|$, $|\lambda_2|\leq\tau(e)^{-1/p}$. It follows that $1=\|\frac{1}{2}(\lambda_1+\lambda_2)e\|_p=\frac{1}{2}|\lambda_1+\lambda_2|\tau(e)^{1/p}\leq\frac{1}{2}(|\lambda_1|+|\lambda_2|)\tau(e)^{1/p}\leq 1$, so that $|\lambda_1|=|\lambda_2|=\tau(e)^{-1/p}$. Let $\lambda_1=\exp(i\theta_1)|\lambda_1|=\tau(e)^{-1/p}\exp(i\theta_1)$ be the polar form of the complex number λ_1. Then we have $e=\frac{1}{2}(\exp(i\theta_1)+\exp(i\theta_2))e$, and we have $1=\exp(i\theta_1)=\exp(i\theta_2)$. Consequentry we conclude that $\tau(e)^{-1/p}e=ex_1e=ex_2e$. If $(1-e)x_1e\neq 0$, then we have

$$\begin{aligned}(x_1e)^*(x_1e)&=e(ex_1+(1-e)x_1)^*(ex_1+(1-e)x_1)e\\&=ex_1^*ex_1e+ex_1^*(1-e)x_1e\\&=\tau(e)^{-2/p}e+ex_1^*(1-e)x_1e.\end{aligned}$$

Since $ex_1^*(1-e)x_1e$ is positive and nonzero, we have $|x_1e|-\tau(e)^{-1/p}e$ is positive and nonzero, which implies that $\|x_1e\|_p^p\gneqq 1$. This contradicts the choice of x_1, hence we have $(1-e)x_1e=0$. Similarly, $ex_1(1-e)=(1-e)x_2e=ex_2(1-e)=0$. If $(1-e)x_1(1-e)\neq 0$, then we have

$$\begin{aligned}(1-e)x_1^*(1-e)x_1(1-e)&=(x_1^*-\tau(e)^{-1/p}e)(1-e)(x_1-\tau(e)^{-1/p}e)\\&=x_1^*(1-e)x_1.\end{aligned}$$

Thus $x_1^*(1-e)x_1$ is a nonzero positive operator. Hence $x_1x_1^*=x_1ex_1^*+x_1(1-e)x_1^*\gneqq x_1ex_1^*$, which implies that $\|x_1\|_p^p=\|x_1^*\|_p^p\gneqq\|x_1ex_1^*\|_{p/2}^{p/2}=\|ex_1^*x_1e\|_{p/2}^{p/2}=\|\tau(e)^{-2/p}e\|_{p/2}^{p/2}=1$. This is a contradiction, hence we have $(1-e)x_1(1-e)=0$. Similarly, $(1-e)x_2(1-e)=0$. Finally we have $x_1=ex_1=ex_1e=\tau(e)^{-1/p}e=ex_2e=x_2$, and we conclude that $\tau(e)^{-1/p}e\in\mathrm{Ext}(S)$.

PROOF of THEOREM 1. (i) It immediately follows from Lemmas 2, 3 and 4.

(ii) The first statement is precisely Lemma 3. If $x\in\mathrm{Ext}(S_1)$, then, by Lemma 2, we have that $|x|=\alpha e$ for some $\alpha>0$, thus $1=\alpha^p\tau(e)$. Conversely, suppose that e is minimal in M and that $|x|=\tau(e)^{-1/p}e$. If there exist $x_1, y_1\in S_1$ such that $x=\frac{1}{2}(x_1+y_1)$, then we have $\tau(e)^{-1/p}e=\frac{1}{2}(u^*x_1+u^*y_1)$. It follows from Lemma 4 that $\tau(e)^{-1/p}e=u^*x_1=u^*x_1=u^*y_1$. Putting $f=uu^*$, we have $f(x_1-y_1)=0$. Since f is also minimal in M, and since $x^*=\frac{1}{2}(x_1^*+y_1^*)$, we similarly obtain that $\tau(f)^{-1/p}f=|x^*|=ux_1^*=uy_1^*$ which implies that $x_1u^*=y_1u^*=\tau(f)^{-1/p}f$. It follows that the range projection of x_1 is a subprojection of f. Thus we conclude that $(1-f)x_1=(1-f)y_1=0$, and so $x_1=y_1$. This completes the proof.

4. An example

In this section, we give a familiar example for $p=1$ which satisfies the same consequence as the Klein-Milman's Theorem. Let $M=B(H)$, the set of all bounded operators on a Hilbert space H, and let $\tau=Tr$, the canonical trace. Then it is well-known that $L^p(B(H), Tr)$ is the trace ideal C^p, where C^p consists of compact operators which the sum of *p-th* power of its singular number is finite. We denote by S^p the unit ball of C^p. Of course, unless H is finite dimensional, S^1 is not compact in the L^1-norm topology, so that the Klein-Milman's Theorem is not applicable.

PROPOSITION 5. S^1 is precisely the L^1-norm closure of $\mathrm{Conv}(\mathrm{Ext}(S^p))$ for all $0<p\leq 1$.

PROOF. For two vectors ξ, η in H, we donote by $t_{\xi,\eta}$ the operator $t_{\xi,\eta}(\zeta)=(\zeta\mid\eta)\xi$. Note that the set of all minimal projections in $B(H)$ consists of $|t_{\xi,\eta}|=t_{\eta,\eta}$, where ξ and η are unit vectors in H, and note that $Tr(t_{\eta,\eta})=1$. Hence for any $0<p\leq 1$, it follows from Theorem 1 that $\mathrm{Ext}(S^p)$ coincides with the set of all $t_{\xi,\eta}$, where $\xi, \eta\in H$ and $\|\xi\|=\|\eta\|=1$. Thus $\mathrm{Conv}(\mathrm{Ext}(S^p))$ is included in S^1, because C^p is contained in C^1. Any $x\in S^1$ has the canonical expansion $x=\sum_{n=1}^{\infty}\mu_n(x)t_{\xi_n,\eta_n}$, where $\{\xi_n\}$ and $\{\eta_n\}$ are suitable orthonormal sets and $\mu_n(x)$ is the *n-th* singular number of x as a compact operator (that is the *n-th* large eigenvalue of $|x|$ with the multiplicity counted). Then by $\sum_{n=1}^{\infty}\mu_n(x)=\|x\|_1=1$, it is easy to see that the above expansion converges in L^1-norm. Put $x_n=\sum_{k=1}^{n}\mu_k(x)t_{\xi_k,\eta_k}$ and $\widetilde{x_n}=x_{n-1}+(1-\sum_{k=1}^{n-1}\mu_k(x))t_{\xi_n,\eta_n}$, then it is clear that $\widetilde{x_n}\in\mathrm{Conv}(\mathrm{Ext}(S^p))$. Since $\|\widetilde{x_n}-x_n\|_1=\|(\sum_{k=n+1}^{\infty}\mu_k(x))t_{\xi_n,\eta_n}\|_1=\sum_{k=n+1}^{\infty}\mu_k(x)\to 0$ as $n\to\infty$, we conclude that $\|x-x_n\|_1\to 0$ and that x is in the L^1-norm closure of $\mathrm{Conv}(\mathrm{Ext}(S^p))$ as desired.

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