Normal Structure of Weakly Compact Sets of Banach Spaces

By
Hideyuki FUJIHIRA and Tetsuo KANEKO

(Received November 16, 1987)
(Revised December 10, 1987)

Abstract. We obtain the geometrical conditions for weakly compact sets to be normal.

1. Introduction

A subset C of a Banach space E is said to be normal if there exists x in C such that

$$\sup\{\|x-y\|: y\in C\}<\operatorname{diam} C,$$

and a convex set $D\subset E$ is said to have normal structure if every nontrivial convex subset of D is normal. A Banach space E is said to have weak normal structure if every weakly compact convex subset of it has the normal structure.

The concept "normal structure" has relations with existence of fixed points of nonexpansive maps. Browder [1] proved that a nonexpansive self-map on a bounded closed convex subset of a uniformly convex Banach space has a fixed point. Bounded closed convex subsets of a uniformly convex Banach space are weakly compact, and uniformly convex Banach spaces have the normal structure. It is known that if a Banach space E has the weak normal structure, then every nonexpansive self-map on a weakly compact convex subset of E has a fixed point ([4]).

A Banach space is said to be uniformly convex (UC) if for each $\varepsilon>0$ there exists $\delta>0$ such that

$$\|x\|\leqq 1,\ \|y\|\leqq 1,\ \|x-y\|>\varepsilon \Rightarrow (\|x+y\|)/2<1-\delta.$$

$B_r(x)$ denotes the closed ball of radius r with center x.

Sufficient conditions of a Banach space to have the weak nermal structure were obtained successively as follows:

(1) nearly uniformly convex (NUC): for each $\varepsilon>0$ there exists $\delta>0$ such that

$$\|x_n\|\leqq 1,\ \inf\{\|x_n-x_m\|: n\neq m\}\geqq\varepsilon \Rightarrow \operatorname{co}\{x_n\}\cap B_{1-\delta}(0)\neq\phi,$$

where $\operatorname{co}\{x_n\}$ denotes the covex hull of $\{x_1, x_2, \dots\}$ ([3]),

(2) uniformly Kadec-Klee (UKK): for each $\varepsilon>0$ there exists $\delta>0$ such that

$$\|x_n\| \leqq 1,\ x_n \to x \text{ (weakly)},\ \inf\{\|x_n - x_m\| : n \neq m\} \geqq \varepsilon \Rightarrow \|x\| < 1-\delta \text{ ([3])},$$

(3) weakly uniformly Kadec-Klee (WUKK): there exist $\varepsilon < 1$ and $\delta > 0$ such that

$$\|x_n\| \leqq 1,\ x_n \to x \text{(weakly)},\ \inf\{\|x_n - x_m\| : n \neq m\} \geqq \varepsilon \Rightarrow \|x\| < 1-\delta \text{ ([2])}.$$

It is seen that UC $\Rightarrow$ NUC $\Rightarrow$ UKK $\Rightarrow$ WUKK.

In this paper we concern with the conditions for weakly compact (not necessarily convex) sets to be normal. Our results have not inclusion relation with the preceding results.

2. Statements of theorems

Let E be a Banach space. B denotes the closed unit ball in E, (x, y) denotes the open line segment joining x and y for two points x, y in E.

THEOREM 1. *Let E be a real Banach space which satisfies the following condition: for each continuous linear functional f and $\alpha > 0$,*

$$\operatorname{diam}(B \cap \{x : f(x) \geqq \alpha\}) < 2.$$

Let S be a weakly compact subset of E. If there exist x_1, x_2 and y_1, y_2 in S such that $(x_1, x_2) \cap (y_1, y_2)$ consists of one point and this point is in S, then S is normal.

PROOF. We suppose that there exists a weakly compact set S satisfying the condition of the theorem which is not normal. We may assume that

$$\operatorname{diam} S = 1,\ \{0\} = (x_1, x_2) \cap (y_1, y_2)\ (0 \in S) \text{ and } \|x_1\| \leqq \|x_2\|,\ \|y_1\| \leqq \|y_2\|.$$

Since S is not normal, there exists a sequence $\{z_n\}$ in S such that $\|z_n\| \to 1$ $(n \to \infty)$. By the assumption of the theorem we have $z_n \in B_1(x_1)$ for all n. We have also $\|z_n\| \leqq 1$ and $\|x_2 - z_n\| \leqq 1$, hence $\|x - z_n\| \leqq 1$ for any point $x \in (0, x_2)$. Thus we have $\|-x_1 - z_n\| \leqq 1$ as $-x_1 \in (0, x_2]$, and $-z_n \in B_1(x_1)$ for all n. In a similar way we can see that z_n and $-z_n$ belong to $B_1(y_1)$ for all n. Since S is weakly compact, the sequence $\{z_n\}$ has a weak accumulating point z_0 in S. At first, we assume that z_0 does not belong to the line joining x_1 and x_2. Then by the Hahn-Banach theorem, there exist a continuous linear functional f and $\alpha > 0$ such that

$$[z_0, -z_0] \subset \{x : f(x) < \alpha\} \text{ and } x_1 \in \{x : f(x) > \alpha\}.$$

Since for infinitely many integer n, both of z_n and $-z_n$ belong to $\{x : f(x) < \alpha\} \cap B_1(x_1)$, we obtain that

$$\operatorname{diam}(\{x : f(x) < \alpha\} \cap B_1(x_1)) = 2,$$

i.e., $\operatorname{diam}(\{x : -f(x) > f(x_1) - \alpha\} \cap B) = 2$. This contradicts to the assumption of the theorem.

Next, we assume that z_0 belongs to the line joining x_1 and x_2. If $z_0 \in (-x_1, x_1)$, then by the same way, the contradiction is deduced. In the other case, there exist a continuous linear functional g and $\beta > 0$ such that

$$\operatorname{diam}(\{x: g(x) < \beta\} \cap B_1(y_1)) = 2,$$

and this is a contradiction. Thus we obtain the theorem.

The condition on E mentioned in the above theorem is equivalent to that

$$\|x_n\| \leqq 1,\ \|y_n\| \leqq 1,\ \|x_n - y_n\| \to 2 (n \to \infty) \Rightarrow 0 \in \overline{\mathrm{co}}(\{x_n\} \cup \{y_n\}),$$

where $\overline{\mathrm{co}}\, A$ denotes the closed convex hull of A.

Under the stronger condition, weakly compact sets of wider class are normal.

THEOREM 2. Let E be a real Banach space which satisfies that if $\|x_n\| \leqq 1$, $\|y_n\| \leqq 1$, $\|x_n - y_n\| \to 2$ then the set $\overline{\mathrm{co}}(\{(x_n + y_n)/2\})$ contains 0. Let S be a weakly compact subset of E. If there exist x and y in S such that $(x, y) \cap S \neq \phi$, then S is normal.

PROOF. Let S be a weakly compact set satisfying the condition of the theorem which is not normal. In the same way as the proof of theorem 1, we may assume that $0 \in (x, y) \cap S$, $\|x\| \leqq \|y\|$ and $\operatorname{diam} S = 1$. Let $\{z_n\}$ be a sequence in S such that $\|z_n\| \to 1$ $(n \to \infty)$. Since $0 \in (x, y)$, we have $\|z_n\| \leqq \frac{1}{2}(\|z_n - x\| + \|z_n - y\|) \leqq 1$, hence $\|x - z_n\| \to 1$ $(n \to \infty)$, $\|z_n - y\| \to 1 (n \to \infty)$. Then, since $\frac{x}{2} \in (x, y)$, we have $\|z_n - \frac{x}{2}\| \to 1$ $(n \to \infty)$, thus we have $\|z_n - (x - z_n)\| \to 2$ $(n \to \infty)$. But $\overline{\mathrm{co}}(\{(z_n + (x - z_n))/2\})$ cannot contain 0 contradicting to the assumption on E. Thus we completed the proof.

The condition for E statsd in theorem 2 is weaker than the following condition;

(a) $\|x_n\| \leqq 1$, $\|y_n\| \leqq 1$, $\|x_n - y_n\| \to 2 \Rightarrow (x_n + y_n)/2 \to 0$ (weakly).

Consider the additional condition;

(b) $\|x_n\| \leqq 1$, $\|x\| = 1$, $x_n \to x$ (weakly) $\Rightarrow x_n \to x$ (strongly).

Then it is easily seen that (a) and (b) is equivalent to UC.

Of course, convex sets are in conformity with the conditions for weakly compact set mentioned in theorem 1 and 2.

THEOREM 3. Let E be a real Banach space which satisfies the condition in theorem 1. Then, every weakly compact convex subset of E has the normal structure.

PROOF. Note that a line segment is a normal convex set. If there exists a convex subset of a weakly compact convex set which is not normal, then we can deduce a contradiction as the same way to the proof of theorem 1.

References

[1] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. 54 (1965), 1041–1044.

[2] D. van Dulst and B. Sims, *Fixed points of nonexpansive mappings and Chebyshev centers in Banach spaces with normes of type* (*KK*), Lecture Notes in Math. 991, Springer-Verlag, Berlin, Heidelberg, New York (1981), 35–42.

[3] R. E. Huff, *Banach spaces spaces which are nearly uniformly convex*, Rocky Mountain J. Math. 10 (1980), 743–749.

[4] W. A. Kirk, *Fixed point theory for nonexpansive mappings II*, Contemporary Math., vol. 18, Amer. Math. Soc. (1983), 121–140.

[5] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly 72 (1965), 1004–1006.

[6] S. Swaminathan, *Normal structure in Banach spaces and its generalizations*, Contemporary Math., vol. 18, Amer. Math. Soc. (1983), 201–215.

Hideyuki FUJIHIRA
Department of Mathematics
Utsunomiya University
Utsunomiya 321, Japan

Tetsuo KANEKO
Derartment of Mathematics
Niigata University
Niigata 950–21, Japan