A Remark on the Analyticity of Spectral Functions for Some Exterior Boundary Value Problems

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Abstract

We give some improvement to the results on the analytical properties with respect to a spectral parameter of solutions to the exterior boundary value problems for elastic equations; we remove a restriction on the Gaussian curvatures of the slowness surfaces and prove that a meromorphic extension of the resolvent remains holomorphic on $\boldsymbol{R}\setminus\{0\}$.

Introduction

The present paper is concerned with the study of the analytical dependence on a spectral parameter k of solutions to the exterior boundary value problems

(0. 1) $\qquad (H-k^2)u=f \qquad \text{in } \Omega,$

$$Hu=-\sum_{m,n=1}^{d}\partial_m(A_{mn}(x)\partial_n u), \qquad \partial_m=\partial/\partial x_m,$$

with homogeneous boundary condition of Dirichlet or Neumann type. Here $u\in\boldsymbol{C}^d$, and $A_{mn}(x)$ are $d\times d$ real matrices whose (p, q)-elements $a_{mpnq}(x)$ are C^∞-functions of $x\in\boldsymbol{R}^d$ and take constant values a^0_{mpnq} outside of a large ball, say for $|x|>b$. The systems of elastic equations

$$Lu=-\{\mu\Delta u+(\lambda+\mu)\,\mathrm{grad}\,(\mathrm{div}\,u)\} \qquad \text{in } \Omega\subset\boldsymbol{R}^3,$$

where the Lamé constants λ and μ satisfy $\mu>0$ and $3\lambda+2\mu>0$ come under the theory developed in this paper.

In our previous paper [5] we have shown that the resolvent $(H-k^2)^{-1}$, $\mathrm{Im}\,k<0$, admits an extension $R(k)$ as a meromorphic function of k to the entire region D (see (1. 2) for the definition) and that $R(k)$ is holomorphic on the real axis except the origin, in particular, when H has constant coefficients; the study of the behaviour of $R(k)$ near $k=0$ is one of the main results in [5] although we do not refer to it in this paper. In proving that $\Lambda\cap(\boldsymbol{R}\setminus\{0\})=\phi$ with Λ being the set of all poles of $R(k)$, we assumed as one of the hypotheses that

(0. 2) the slowness surfaces of $A(\xi)=\sum_{m,n=1}^{d} A_{mn}^{0}\xi_m\xi_n$ with $\xi\in\boldsymbol{R}^d$ and $A_{mn}^{0}=(a_{mpnq}^{0})$ never have vanishing Gaussian curvatures at any point.

Using this assumption we investigated the asymptotic behaviour as $|x|\to\infty$ of the Green function for H_0-k^2 where $H_0=A(-i\partial)$ in $\boldsymbol{R}^d$, and formulated outgoing and incoming radiation conditions to verify Rellich's uniqueness theorem.

In this paper we shall remove the condition (0. 2) which does not contribute to the other results in [**5**] and show that $R(k)$ remains holomorphic on $\boldsymbol{R}\backslash\{0\}$. The strategy to the proof is the very one to the limiting absorption principle for the selfadjoint extension of H and we shall adopt the same approach as in [**4**] based on the commutator method due to Mourre [7].

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1. Assumptions and Results

Let Ω be an unbounded domain in $\boldsymbol{R}^d$ having a smooth and compact boundary $\Gamma\subset B_{b-1}=\{x\in\boldsymbol{R}^d;\ |x|<b-1\}$. Consider the boundary value problem (0. 1) with boundary condition

(1. 1) $\qquad Bu=0 \qquad$ on Γ,

where Bu denotes either of the following two types:

$$Bu=u,$$
$$=\sum_{m,n=1}^{d}\nu_m(x)A_{mn}(x)\partial_n u,$$

$\nu(x)={}^t(\nu_1(x),\ldots,\nu_d(x))$ being the unit outward normal to Γ at $x\in\Gamma$. We assume that the dimension d satisfies

(A. 1) $\quad d\geq 3$.

The following assumptions are imposed upon the coefficients $A_{mn}(x)$.

(A. 2) $\quad a_{mpnq}(x)=a_{pmnq}(x)=a_{nqmp}(x),\ x\in\boldsymbol{R}^d$.

(A. 3) *There exists a constant* $C>0$ *such that the inequality*

$$\sum_{m,p,n,q=1}^{d} a_{mpnq}(x)s_{nq}\bar{s}_{mp}\geq C\sum_{m,p=1}^{d}|s_{mp}|^2$$

holds for any $x\in\boldsymbol{R}^d$ *and* $d\times d$ *Hermitian matrix* $s=(s_{mp})$.

(A. 4) *The characteristic roots of* $A(\xi)$ *are of constant multiplicity for all* $\xi\in\boldsymbol{R}^d\backslash\{0\}$.

In order to state the results, we introduce the notation and functional spaces. Set

(1.2) $D=\boldsymbol{C}$ when d is odd,

$=\left\{k\in\boldsymbol{C}\setminus\{0\}\,;\ -\frac{3}{2}\pi<\arg k<\frac{\pi}{2}\right\}$ when d is even,

and $D_-=\{k\in D;\ \mathrm{Im}\,k<0\}$. For a domain G of $\boldsymbol{R}^d$ we set

$$L^2_a(G)=\{u\in L^2(G;\boldsymbol{C}^d);\ u(x)=0 \text{ for } |x|\geq a\},\ a>0;$$

$$L^{2,s}(G)=\{u;\ \langle x\rangle^s u\in L^2(G;\boldsymbol{C}^d)\},\ \langle x\rangle=(1+|x|^2)^{1/2};$$

$$H^m_e(G)=\{u;\ \exp(-|x|^2)\partial^\alpha u\in L^2(G;\boldsymbol{C}^d),\ |\alpha|\leq m\},\quad \text{an integer } m\geq 0.$$

Let $a>0$ be a fixed number so that $b<a$, and let $\boldsymbol{B}(E,F)$ denote the totality of bounded operators of E into F.

THEOREM 1.1 ([**5**]). *Suppose that* (A. 1)—(A. 4) *are valid. Then there exists an operatar* $R(k)\in\boldsymbol{B}(L^2_a(\Omega),\ H^2_e(\Omega))$ *such that* $R(k)$ *depends meromorphically on the parameter* $k\in D$ *and satisfies the following properties: Let* Λ *denote the set of all poles of* $R(k)$ *in* D. *Then,*

(i) Λ *is discrete.*

(ii) *If* $k\in D\setminus\Lambda$ *and* $f\in L^2_a(\Omega)$, *then* $u=R(k)f$ *solves the problem* (0. 1) *with* (1. 1).

(iii) $\Lambda\cap D_-=\phi$.

(iv) *If* $k\in D_-$, *then* $R(k)\in\boldsymbol{B}(L^2_a(\Omega),\ H^2(\Omega))$.

REMARK. The behaviour of the operator $R(k)$ near $k=0$ is analysed in Theorem 1.2 of [**5**].

The purpose of this paper is to prove the following

THEOREM 1.2. *Assume that* (A. 1)—(A. 4) *hold. If* $k\in\Lambda\cap(\boldsymbol{R}\setminus\{0\})$, *then there exists a non-trivial function* $u(x)\in C^\infty_0(\overline{\Omega}_b;\boldsymbol{C}^d)$, $\Omega_b=\{x\in\Omega;\ |x|<b\}$, *satisfying*

$$(1.3)\qquad \begin{aligned}(H-k^2)u&=0 &&\text{in } \Omega,\\ Bu&=0 &&\text{on } \Gamma.\end{aligned}$$

As a corollary of Theorem 1.2 we obtain by the unique continuation theorem

COROLLARY 1.3. *Let* (A. 1)—(A. 4) *be satisfied and* $A_{mn}(x)\equiv A^0_{mn}$. *Then,* $\Lambda\cap(\boldsymbol{R}\setminus\{0\})=\phi$.

2. Preliminaries

This section is devoted to the investigation of the resolvent $G_0(z)=(H_0-z)^{-1}$ extended to the real axis. For the operator H_0, we use only the property that the symmetric matrix $A(\xi)$ is positive definite for $|\xi|=1$ which is an immediate consequence of the assumptions (A. 2) and (A. 3). This property implies that the domain $\mathcal{D}(H_0)$ of H_0 coincides with $H^2(\boldsymbol{R}^d; \boldsymbol{C}^d)$, the Sobolev space of order two.

We now let $A=-i\sum_{j=1}^{d}(x_j\partial_j+\partial_j x_j)$ in $\boldsymbol{R}^d$, the generator of a dilation unitary group. The commutator form $i[H_0, A]=i(H_0A-AH_0)$ defined on $\mathcal{D}(A)\cap\mathcal{D}(H_0)$ is calculated as $i[H_0, A]=H_0$ and has the selfadjoint extension $i[H_0, A]^0=H_0$. For each $\lambda\in(0, \infty)$, choose $f\in C_0^\infty(0, \infty)$ so that $0\leq f\leq 1$ and $f=1$ near λ. Then the operator $M\equiv f(H_0)\cdot i[H_0, A]^0 f((H_0)$ is bounded and positive in the sense that $M\geq\alpha f(H_0)^2$, where $\alpha=\inf\operatorname{supp} f>0$. It hence follows that the inverse $G_0(\varepsilon, z)=(H_0-i\varepsilon M-z)^{-1}$ exists and is bounded for $z\in\boldsymbol{C}$ with $\operatorname{Re} z=\lambda$ and $\pm\operatorname{Im} z>0$, and $\pm\varepsilon\geq 0$. By analysing $G_0(\varepsilon, z)$ (cf. Mourre [7]), we have

THEOREM 2. 1. *Let I be a compact interval in $(0, \infty)$ and $s>1/2$.*

(i) *The inequality*

$$\|\langle x\rangle^{-s}G_0(z)\langle x\rangle^{-s}\|\leq C$$

holds for any $z\in\boldsymbol{C}$ with $\operatorname{Re} z\in I$ and $\operatorname{Im} z\neq 0$.

(ii) *For each $\lambda\in I$, the norm limits exist:*

$$\langle x\rangle^{-s}G_0(\lambda\pm i0)\langle x\rangle^{-s}=\lim_{\kappa\downarrow 0}\langle x\rangle^{-s}G_0(\lambda\pm i\kappa)\langle x\rangle^{-s}.$$

The following proposition is proved in the same way as in Tamura [8] by using the fact

$$G_0(\lambda\pm i0)=\lim_{\pm\varepsilon\downarrow 0}G_0(\varepsilon, \lambda) \text{ in } \boldsymbol{B}(L^{2,1}(\boldsymbol{R}^d), L^{2,-1}(\boldsymbol{R}^d)).$$

PROPOSITION 2. 2. *If $\psi\in L^{2,1}(\boldsymbol{R}^d)$ and $\operatorname{Im}(\psi, G_0(\lambda\pm i0)\psi)=0$, where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\boldsymbol{R}^d)$, then $G_0(\lambda\pm i0)\psi\in L^{2,-\delta}(\boldsymbol{R}^d)$ for any $\delta>0$.*

REMARK. We should note that one can employ Agmon's method [1] in place of the commutator method since the system H_0 satisfies the assumption (A. 4).

3. Proof of Theorem 1. 2

We shall verify Theorem 1. 2 by using the results in the preceding section.

RROPOSITION 3. 1. *Let $k\in\Lambda\cap(\boldsymbol{R}\backslash\{0\})$. Then there exists a non-trivial $C^\infty(\bar{\Omega})$-function u such that $u\in L^{2,-\delta}(\Omega)$ for arbitrary $\delta>0$, and u satisfies (1. 3).*

PROOF. It suffices to verify the proposition in the case $k\in\Lambda\cap(0, \infty)$ since the other case can be similarly treated. Let $k_0\in\Lambda\cap(0, \infty)$ and suppose that the operator $R(k)$ has a pole of order $j>0$ at the point k_0. By (i) of Theorem 1. 1, we can find a neighbourhood U of k_0 in D such that U does not contain any other point of the set Λ. Put

$U_- = U \cap D_-$. Under the assumption made, there exists a function $f \in L^2_a(\Omega)$ such that the limit

$$\lim_{U_- \ni k \to k_0} (k-k_0)^j R(k) f = u$$

exists in $H^2_e(\Omega)$ and $u \not\equiv 0$. By (ii) of Theorem 1.1, $R(k)f$ satisfies (0.1) and (1.1) for $k \in U_-$, so the limit u solves the homogeneous problem (1.3). Then we also see that $u \in C^\infty(\bar{\Omega})$ since H is elliptic.

It remains to show that $u \in L^{2,-\delta}(\Omega)$ for any $\delta > 0$. Since $R(k)f \in H^2_e(\Omega)$ for $k \in U_-$, we can employ the Lions method to construct an extension $\tilde{u}(k) = \tilde{u}(x; k) \in H^2_e(\boldsymbol{R}^d)$ of $R(k)f$ such that

$$\|\tilde{u}(k)\|_{H^2(B_b)} \leq C \|R(k)f\|_{H^2(\Omega_b)},$$

where C is independent of k. We may assume that $\tilde{u}(k)$ is an $H^2_e(\boldsymbol{R}^d)$-valued meromorphic function of $k \in U$ and has the only pole of order j at $k = k_0$. Let

$$H_1 = -\sum_{m,n=1}^{d} \partial_m (A_{mn}(x) \partial_n \cdot) \qquad \text{in } \boldsymbol{R}^d,$$

(3.1) $$\tilde{f}(k) = (H_1 - k^2) \tilde{u}(k) \qquad \text{in } \boldsymbol{R}^d.$$

Then $\tilde{f}(k) = f$ in Ω and $\tilde{f}(k)$ has the only possible pole of order at most j at $k = k_0$. These imply that the limits

(3.2) $$\begin{aligned} \tilde{u} &= \lim_{U_- \ni k \to k_0} (k-k_0)^j \tilde{u}(k) && \text{in } H^2_e(\boldsymbol{R}^d), \\ \tilde{f} &= \lim_{U_- \ni k \to k_0} (k-k_0)^j \tilde{f}(k) && \text{in } L^2_a(\boldsymbol{R}^d) \end{aligned}$$

exist and satisfy

(3.3) $$\begin{aligned} &(H_1 - k_0^2) \tilde{u} = \tilde{f} && \text{in } \boldsymbol{R}^d, \\ &\tilde{u} = u, \ \tilde{f} = 0 && \text{in } \Omega. \end{aligned}$$

For $k \in U_-$ we rewrite (3.1) as

$$(H_0 - k^2) \tilde{u}(k) = \tilde{f}(k) - (H_1 - H_0) \tilde{u}(k),$$

and by noting that

(3.4) $$H_1 \equiv H_0 \qquad \text{for } |x| > b,$$

we have

(3.5) $$\tilde{u}(k) = G_0(k^2) [\tilde{f}(k) - (H_1 - H_0) \tilde{u}(k)].$$

Theorem 2.1 entails the fact that

(3.6) $$G_0(k^2) \to G_0(k_0^2 - i0) \qquad \text{in. } \boldsymbol{B}(L^2_a(\boldsymbol{R}^d), H^0_e(\boldsymbol{R}^d))$$

as $k \in U_-$ tends to k_0. Multiply the both sides of (3. 5) by $(k-k_0)^j$ and let $k \in U_-$ tend to k_0. Then it follows from (3. 2), (3. 5), and (3. 6) that

$$\tilde{u} = G_0(k_0^2 - i0)[\tilde{f} - (H_1 - H_0)\tilde{u}].$$

Clearly, $\tilde{f} - (H_1 - H_0)\tilde{u} \in L^{2,1}(\boldsymbol{R}^d)$. Taking into account the fact that $\tilde{u}$ satisfies the same boundary condition on Γ as for u, we can use (3. 3), (3. 4), and the symmetry of $H_1 - H_0$ to calculate with $G = \boldsymbol{R}^d \setminus \Omega$,

$$\begin{aligned}
&(\tilde{f} - (H_1 - H_0)\tilde{u}, \tilde{u}) \\
&= \int_G (H_1 - k_0^2)\tilde{u} \cdot \overline{\tilde{u}}\, dx - ((H_1 - H_0)\tilde{u}, \tilde{u}) \\
&= \int_G \tilde{u} \cdot \overline{(H_1 - k_0^2)\tilde{u}}\, dx - (\tilde{u}, (H_1 - H_0)\tilde{u}) \\
&= (\tilde{u}, \tilde{f} - (H_1 - H_0)\tilde{u}).
\end{aligned}$$

Hence, we are led to the fact that $\operatorname{Im}(\tilde{f} - (H_1 - H_0)\tilde{u}, \tilde{u}) = 0$. By applying Proposition 2. 2 we obtain $\tilde{u} \in L^{2,-\delta}(\boldsymbol{R}^d)$ for any $\delta > 0$ and thus $u \in L^{2,-\delta}(\Omega)$. Q.E.D.

The following theorem is due to Hörmander [**2**].

THEOREM 3. 2. *Let $P(D)$ be a partial differential operator in $\boldsymbol{R}^d$ with constant coefficients such that $P = CP_1^{m_1} P_2^{m_2} \ldots P_\ell^{m_\ell}$), where C is a constant and for each j, $P_j(\xi)$, $\xi \in \boldsymbol{R}^d$, is a real and irreducible polynomial such that* grad $P_j(\xi) \neq 0$ *where $P_j(\xi) = 0$. Let $u \in \mathcal{S}'(\boldsymbol{R}^d) \cap L^2_{loc}(\boldsymbol{R}^d)$ such that $P(D)u \in \mathcal{E}'(\boldsymbol{R}^d)$. If u has the asymptotic property*

$$\lim_{R \to \infty} R^{-1} \int_{R \leq |x| \leq 2R} |u(x)|^2\, dx = 0,$$

then $u \in \mathcal{E}'(\boldsymbol{R}^d)$. Furthermore, supp u *is contained in the convex hull of* supp $P(D)u$.

Theorem 1. 2 is now an immediate consequence of Proposition 3. 1 and Theorem 3. 2.

PROOF OF THEOREM 1. 2. Let u be the function specified in Proposition 3. 1. Take a number $R < b$ so that $\boldsymbol{R}^d \setminus \Omega \subset B_R$. Choose $\varphi \in C^\infty(\boldsymbol{R}^d)$ so that $\varphi = 1$ for $|x| \geq R$ and $=0$ in a neighbourhood of $\boldsymbol{R}^d \setminus \Omega$. Then $v = \varphi u$ satisfies

$$(3.7) \qquad \begin{aligned}
&(H_0 - k_0^2)v = g; \\
&g = -(H_1 - H_0)v - (H_1\varphi)u + \sum_{m,n=1}^{d} (A_{mn}(x) + A_{nm}(x))\partial_n \varphi \partial_m u.
\end{aligned}$$

We denote by $Q_c(k)$ and $Q_d(k)$ the differential operators whose symbols are the cofactor matrix and the determinant of the matrix $A(\xi) - k^2 I$, respectively. Then it follows from (3. 7) that $Q_d(k)v = Q_c(k)g$. Note that supp $g \subset \bar{B}_b$ and hence supp $Q_c(k)g \subset \bar{B}_b$. By applying Theorem 3. 2 to v, we arrive at the theorem. Q.E.D.

4. Remarks

In this section we give some characterization to $R(k)f$ for $k \in \boldsymbol{R} \setminus (\Lambda \cup \{0\})$.

Let $k \in \boldsymbol{R} \setminus \Lambda$, $k > 0$, and $f \in L^2_a(\Omega)$. Then $R(k)f$ is said to be *k^2–incoming* in the following sense (see Agmon [**1**]). Let $G_0(\lambda \pm i0)$ be the operators introduced in Section 2. There exists $f_- \in L^2_a(\boldsymbol{R}^d)$ such that

$$R(k)f = G_0(k^2 - i0)f_- \qquad \text{in } \Omega.$$

For $k \in \boldsymbol{R} \setminus \Lambda$ and $k < 0$, $R(k)f$ is said to be *k^2–outgoing* in the sense that there exists $f_+ \in L^2_a(\boldsymbol{R}^d)$ such that

$$R(k)f = G_0(k^2 + i0)f_+ \qquad \text{in } \Omega.$$

In fact, these are verified similarly as in Section 3 and the classes defined above correspond to $W_k^-(\Omega)$ and $W_k^+(\Omega)$ in [**5**]. Characterizations of $G_0(\lambda \pm i0)$ are given in Agmon [**1**], Hörmander [**3**], and Jensen-Mourre-Perry [**6**].

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