A remark on the existence of geometries on 4-dimensional aspherical Seifert fiber spaces

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(Received November 4, 1987)
(Revised December 23, 1987)

Introduction

In this note, we shall consider the existence of geometries on 4-dimensional Seifert fiber spaces. First we shall prove the followings.

THEOREM A. *Let M be a 4-dimensional Seifert fiber space with typical fiber a surface F. Then F is aspherical and the homomorphism $i_*: \pi_1(F) \to \pi_1(M)$ induced by the inclusion is injective.*

THEOREM B. *If F is a toruse, then M or its finite covering is an injective Seifert fiber space.*

In section 1, we shall define an injective Seifert fiber space ([LR]). As in the 3-dimensional case, R. P. Filipkiewicz has classified the geometries on 4-dimensional manifolds (see [W_1], [W_2]).

The main results of this note are the followings.

THEOREM C. *Let M be an injective Seifert fiber with typical fiber a torus. Suppose that exact sequence*

$$1 \to \pi_1(T^2) \to \pi_1(M) \to Q \to 1$$

is a central extension. Then M adimits a geometry.

THEOREM D. *Let F be a surface except a torus. If M admits a geometry, then M is a product manifold up a finite covering.*

Throughout this note, we shall work in the smooth category and a manifold means a closed connected manifold.

1. Preliminaries

DEFINITION. A closed manifold M is a Seifert fiber space if M is a union of a collection $\{F_\alpha\}$ of pairwise disjoint 2-dimensional manifold F_α (called fibers) such that for each α, there is a closed neighborhood V of F_α with a covering map $p: D^2\times F\to V$ satisfying
(1) p maps each $\{x\}\times F$ $(x\in D^2)$ to some F_β,
(2) $p^{-1}(F_\alpha)$ is connected, and
(3) the covering transformation group G is a subgroup of $O(2)$ and G acts on F freely.
Here F is called a typical fiber.

DEFINITION Let X be a space, G and π subgroups of $H(X)$ (=the group of all homeomorphisms of X). Suppose the followings are satisfied,
(1) G is a Lie group acting freely on X as subgroup of $H(X)$ such that (G, X) is equivariantly isomorphic to $(G, G\times W)$, where $W=X/G$ and G acts on $G\times W$ by the left translation.
(2) G is normalized by π.
Put $\Gamma=G\cap\pi$. Then Γ is a normal subgroup in π, and $Q=\pi/\Gamma$ acts on W naturally.
(3) Q acts properly discontinuously on W.

It follows from (3) that π acts on X properly discontinously. Put $B=W/\Gamma$ and $E=X/\pi$. E is called an injective Seifert fiber space with typical fiber G/Γ, B is called the base space and the natural map $E\to B$ is called the injective Seifert fibering.

Next we shall describe a construction of an injective Seifert fiber space.
(1) Give a pair (W, Q), where a discrete group Q acts on W properly discontinuously, W/Q is compact and W is contractible.
(2) $X=G\times W$, where Lie group G acts on $G\times W$ by the left translation. We denote this action by l_G and also $l_G(G)$ by l_G.
(3) $D=$ Maps $(W, G)=\{f: W\to G$ continuous$\}$. Define a multiplication on D by $(f*f')(w)=f'(w)\cdot f(w)$.
(4) Define an action of Aut $(G)\times H(W)$ on D by $(g, h)f=g\cdot f\cdot h^{-1}$.
(5) Construct a semidirect product $D\circ$ (Aut $(G)\times H(W))$. This group acts on $G\times W$ by $(f, g, h)(x, w)=(g(x)fh(w), h(w))$.
(6) Let $H^F(G\times W)$ be the normalizer of l_G in $H(G\times W)$.
Then the followings are proved.

THEOREM 1 ([LR]). *$H^F(G\times W)=D\circ(Aut\ (G)\times H(W))$.*

THEOREM 2 ([LR]). *Assume $G=R^n$. Then for any axact sequece $1\to\Gamma\to\pi\to Q\to 1$, there exists a homomorphism $\psi:\pi\to H^F(G\times W)$ such that the following diagram is commutative;*

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & \pi & \to & Q & \to & 1 \\ & & \downarrow & & \downarrow \Psi & & \downarrow & & \\ 1 & \to & D \circ Inn(G) & \to & H^F(G \times W) & \to & Out(G) \times H(W) & \to & 1 \end{array}$$

Then it is clear that $G/\Gamma \to (G \times W)/\pi \to W/Q$ is an injective Seifert fiber space. We shall restrict ourselves to the case where $G = R^n$ and hence $\Gamma = Z^n$. Let U be a Lie subgroup of $H^F(R^n \times W)$ such that $K = U \cap D$ contains l_{R^n}. Let $S = U/K$ be the quotient. Then the following diagram is commutative;

$$\begin{array}{ccccccccc} 1 & \to & K & \to & U & \to & S & \to & 1 \\ & & \cap & & \cap & & \cap & & \\ 1 & \to & D & \to & H^F(R^n \times W) & \to & GL(n, R) \times H(W) & \to & 1. \end{array}$$

Let $\rho : Q \to S$ be a homomorphism defining a properly discotinuous action of Q on W. We have the following

THEOREM 3 ([LR]). *The following statements are equivalent.*
(1) *There exists a homomorphism* $\Psi : \pi \to U$ *such that the diagram*

$$\begin{array}{ccccccccc} 1 & \to & Z^n & \to & \pi & \to & Q & \to & 1 \\ & & \downarrow \varepsilon & & \downarrow \Psi & & \downarrow \rho & & \\ 1 & \to & K & \to & U & \to & S & \to & 1 \end{array}$$

is commutative, where $\varepsilon : Z^n \to R^n \subset K$ *is an inclusion.*
(2) $\varepsilon_*[\pi] = \rho^*[U]$ in $H^2(Q; K)$.
NOTE (1). If $U = \mathrm{Iso}_\circ(R^n \times W)$ (=the identity component of the isometry group of $R^n \times W$), then $R^n \times W/\pi$ admits a geometry modelled on $(R^n \times W, U)$ in the sense of Wall ([W_1]).
(2). We note that the extension $1 \to Z^n \to \pi \to Q \to 1$ is central.

DEFINITION. A Riemannian manifold M is said to have a geometry modelled on (X, G_X) if M is diffeomorphic to $\Gamma \backslash G_X / K_X$.

We needs the following results on fundamental group of the Seifert fiber space.

THEOREM 4 ([V]). *Let M be a Seifert manifold with typical fiber a surface F. Then $\pi_1(M)$ has one of the following two presentations*

(A) *The space B of fibers is orientable of genus g;*
Generators;
$(s_1, t_1, \ldots, s_g, t_g, q_1, \ldots, q_m, e_1, \ldots, e_m, c_{1,1}, \ldots, c_{n,m_n+1}) = E$
and generators of $\pi_1(F)$.
Relations:

$cgc^{-1}=A(c)(g)$, *g a generators of* $\pi_1(F)$, $\quad c\in E$

$c_{i,j}{}^2=g_{i,j}$

$c_{i,1}\,e_i\,c_{i,m_i+1}\,e_i{}^{-1}=f_{i,m_i+1}$

$q_i{}^{h_i}=f_i$

$(c_{i,j}\,c_{i,j+1})^{h_{i,j}}=f_{i,j},\ i=1,\ldots,n,\ j=1,\ldots,m_i$.

$(*)=[s_1,t_1]\ldots[s_g,t_g]q_1\ldots q_m\,e_1\ldots e_n=f_\circ$ *where the* $f_i, f_{i,j}, g_{i,j}$ *are all in* $\pi_1(F)$ *and relations for* $\pi_1(F)$.

(B) *B is non orientable of genus g*

Generators:

$(v_1,\ldots,v_g,q_1,\ldots,q_m,e_1,\ldots,e_n,c_{1,1},\ldots,c_{n,m_n+1})=E$

and generators of $\pi_1(F)$.

Relations : Asi n (A), one just has to replace $(*)$ *by* $v_1^2\ldots v_g^2 q_1\ldots q_m\,e_1\ldots e_n$.

2. Proofs of theorems

In this section, we shall prove the following theorems.

THEOREM A. *Let M be a 4-dimensional aspherical Seifert fiber space with typical fiber a surface F. Then we have*

(1) F is aspherical

(2) $j_*:\pi_1(F)\to\pi_1(M)$ *injective, where* $j:F\to M$ *is an inclusion.*

PROOF. From results in [V], we have a presentation of $\pi_1(M)$. Put Γ be the image of j_*. Let $\overline{M}$ be the covering space of M associated to Γ. It is clear that $\overline{M}$ is also a Seifrt fiber space with the components of the inverse images of the fibers of M as fibers. Let $\overline{B}$ be the space of fibers and $\overline{f}:\overline{M}\to\overline{B}$ the natural projection. We shall show that $\overline{f}$ is a fibration. In fact, let F_i be a fiber in M and F_i' a component of the inverse image F_i. Let p denote the projection $p:\overline{M}\to M$. It is clear that the map $i\circ q:F\to M$, where $q:E\to F_i$ is the natural map, lifts a map $q':F\to\overline{M}$ and is factored by $r:F\to F_i'$. Then we have the following commutative diagram;

$$\begin{array}{ccccccc} & & F_i' & \subset & \overline{M} & \xrightarrow{\overline{f}} & \overline{B} \\ & {}^{r}\nearrow\ \nearrow{}^{q'} & & & \downarrow p & & \downarrow \\ F & \xrightarrow{q} & F_i & \subset & \underset{1}{M} & \xrightarrow{f} & B \end{array}$$

We note that

(1) any loop in the neighborhood U of F_i' in $\overline{M}$ is homotopic to a loop x, $q_i^\alpha\,x$ or $c_{i,j}^\beta\,x$, where x is a loop in F.

(2) any loop in U is in the kernel of $\pi_1(M)\to Q(=\pi_1(M)/\Gamma)$.

(3) any element of finite order in Q is conjugate to q_i or $c_{i,j}$. It follows that $\alpha=\beta=0$, which means $F'_i=F$. Thus the fiber of $\overline{M}\to\overline{B}$ is all typical and hence $\overline{M}$ is a locally trivial fiber space. So we have an injection $p_*\circ q'_* : \pi_1(F)\to\pi_1(M)$.

Next we shall prove that Q is an infinite group. Assume Q is finite. Then B is a sphere S^2 or a real projective plane P^2. We may assume that B is S^2. In fact, if not, consider the fiber space $M'\to S^2$ induced from $S^2\to P^2$. Then M' is also a Seifert fiber space and is aspherical. Let $\widetilde{M}$ be the universal covering space of M. Then the natural map $\widetilde{M}\to S^2$ is a fiber space with F as fibers. We have the following Wang exact sequence

$$\dots\to H_i(\widetilde{M})\to H_{i-2}(F)\to H_{i-1}(F)\to H_{i-1}(\widetilde{M})\to\dots$$

Since $\widetilde{M}$ is contractible, we have a contradiction. Thus Q is infinite and hence B is homeomorphic to R^2. It is clear that F is aspherical. QED.

REMARK 1. We have just obtained an exact sequence;

$$1\to\pi_1(F)\to\pi_1(M)\to Q\to 1,$$

where $Q=\pi_1(M)/\pi_1(F)$ is a planar discontinuous group in the sense of [ZVC]. It follows from a result in [ZVC] (Theorem 4. 10. 1 in [ZVC]) that Q contnins a torsionfree subgroup Q_1 of finite index. Let M_1 be the finite covering of M corresponding to the subgroup of $\pi_1(M)$ which is the inverse image of Q_1. It is clear that M_1 is a Seifert fiber space which is locally product. Since the existence of a geometry is not changed by taking a finte covering, we may assume that the structure of Seifert fiber space of M is locally product. Moreover we may assume the base space is orientable, if necessary, by taking the oriented double covering. If M is not orientable, then the oriented double of M is also a Seifert fiber space whose base space is orientable since it is a finte covering of B. Thus we may assume M is also orientable. Then typical fiber F is also orientable In fact, there is a relation of the tangent bundles;

$$\tau_M=\widehat{\tau_F}+p\tau^*B,\qquad \widehat{\tau_F}\text{ is a bundle along the fiber.}$$

Then, considering the first Stiefel Whitney classes, we obtain $w_1(F)=0$. Thus F is orientable.

THEOREM B. *Let M aud F be as in Thorem A. If F is a torus, then M is an injective Seifert fiber space up to a finite covering.*

PRUUF. Let B be the space of fibers (= base space). By the proof of Theorem A, we have the exact sequence

$$1\to\pi_1(T^2)\to\pi_1(M)\to Q\to 1.$$

Q acts on $\overline{B}$ properly discontinuously and its quotient space is B. Then Q contains a normal subgroup Q_1 of finite index that is torsionfree ([Z]). Let π be the inverse image of Q_1 and M_1 the finite covering space of M associated to π. Then M_1 is an aspherical fiber space with fiber T^2, that is to say, $T^2 \to M_1 \to B_1$. Since Q_1 is a finitely generated discontinuous group of the plane, we obtain an injective Seifert fiber space $M(\pi)$ by an in jective Seifert fiber space construction.

Case 1. The base space B_1 is not a torus.
In this case, we can apply the classification Theorem in [V, (7. *i*)]. Since $2=\text{rank}\ \pi_1(T^2) < J(M_1)$, every isomorphism $\pi_1(M_1) \to \pi_1(M(\pi))$ is induced by an Seifert fiber space isomorphism $M_1 \to M(\pi)$.

Case 2. The base space B_1 is a torus.
In this case, we can apply the theorem in [SF]; the total spaces of two T^2-bundles over T^2 are diffeomorphic if and only if their fundamental groups are isomorphic. Thus we obtain a diffeomorphism $M_1 \to M(\pi)$. QED.

THEOREM C. *Let M be an injective Seifert fiber space with typical fiber T^2. Then M admits a geometry.*

PROOF. We have the following central exact sequece

$$1 \to \pi_1(T^2) \to \pi \to Q \to 1,$$

where $\pi=\pi_1(M)$ and $Q=\pi/\pi_1(T^2)$.

Let $\chi(Q)$ denote Euler characteristic of Q and the above exact sequence represents an element $[\pi]$ of $H^2(Q; Z^2)$.

Case 1. $\chi(Q) < 0$.

Subcase 1. $[\pi]$ has finite order in $H^2(Q; Z^2)$.

Let $i: Z^2 \to R^2$ be the inclusion. From the assumption we have $i_*[\pi]=0$, where $i_*: H^2(Q; Z^2) \to H^2(Q; R^2)$. We shall consider the following commutative diagram;

$$\begin{array}{ccccccccccl}
1 & \to & Z^2 & \to & \pi & \to & Q & \to & 1 & & [\pi] \in H^2(Q; Z^2) \\
 & & \downarrow & & \downarrow & & \downarrow = & & & & \\
1 & \to & R^2 & \to & \bar{\pi} & \to & Q & \to & 1 & & i_*[\pi]=\rho_*[U] \\
 & & \downarrow = & & \downarrow & & \downarrow \rho & & & & \\
1 & \underset{\to}{\downarrow} & R^2 & \to & R^2 \times PSL_2R & \to & PSL_2R & \to & 1 & & \rho^*[U] \in H^2(Q; R^2), \\
 & & & & \| & & & & & & \\
 & & & & U & & & & & &
\end{array}$$

where ρ is an embedding as a cocompact discrete subgroup. This diagram exists from the theorem 3 ([LR]), since $i_*[\pi]=\rho^*[U]=0$. Then we have an injection $\pi \to U=Iso_\circ(R^2 \times H^2)$, thus M admits a geometry modelled on $R^2 \times H^2$, where H^2 denotes the hyperbolic 2-space.

Subcase 2. $[\pi]$ has infinite order in $H^2(Q; Z^2)$. Since $i_*[\pi]$ is nonzero, we can define $\bar{Q}$ such that $1 \to \boldsymbol{R}^2 \to \bar{Q} \to Q \to 1$ is an exact sequence and $[\bar{Q}]$ is nonzero. In fact, we have the following exact sequences;

$$\begin{array}{ccccccccc} 1 & \to & R & \to & \mathrm{Iso}_\circ(\widetilde{PSL}_2) & \overset{k}{\to} & \mathrm{Iso}_\circ(H^2) & \to & 1, \\ 1 & \to & R \times R & \to & R \times Iso_\circ(\widetilde{PSL}_2R) & \to & Iso_\circ(H^2) & \to & 1, \end{array}$$

where $P\widetilde{S}R_2R$ is a universal covering of PSL_2R. Put $Q'=k^{-1}(Q)$ and $\bar{Q}=R \times Q'$. Then we have the commutative diagram;

$$\begin{array}{ccccccccc} 1 & \to & R^2 & \to & \bar{Q} & \to & Q & \to & 1 \\ & & \downarrow pr & & \downarrow pr & & \| & & \\ 1 & \to & R & \to & Q' & \to & Q & \to & 1, \end{array}$$

where pr is a projection on the second factor.
It is clear that the homomorphism $pr_* : H^2(Q\,;R^2) \to H^2(Q;R)$ maps $[\bar{Q}]$ to $[Q']$. Since $[Q']$ is nonzero (see [KLR]), $[\bar{Q}]$ Is nonzero.

On the other hand, $i_*[\pi]$ is nonzero in $H^2(Q;R^2)$. Since $H^2(Q;R^2) \simeq R^2$ ([KLR]), there exists a linear homomorphism $\varepsilon : R^2 \to R^2$ such that $\epsilon_* i_*[\pi]=[\bar{Q}]$. Then we have the commutative diagram;

$$\begin{array}{ccccccccc} 1 & \to & Z^2 & \to & \pi & \to & Q & \to & 1 \\ & & \downarrow \varepsilon\circ i & & \downarrow & & \| & & \\ 1 & \to & R^2 & \to & \bar{Q} & \to & Q & \to & 1 \\ & & \| & & \downarrow & & \downarrow \rho & & \\ 1 & \to & R^2 & \to & R \times \mathrm{Iso}_\circ(\widetilde{PSL}_2R) & \to & \mathrm{Iso}_\circ(H^2) & \to & 1. \end{array}$$

Thus we have an injection $\pi \to \bar{Q} \subset R \times Iso_\circ(\widetilde{PSL}_2R)=Iso_\circ(R \times \widetilde{PSL}_2R)$ by the theorem 3 ([LR]). In other words, M admits a geometry modelled on $R \times \widetilde{PSL}_2R$.

Case 2. $\chi(Q)=0$.

This case is very similar to the proof of the cace 1. When $[\pi]$ has finite order in $H^2(Q;Z^2)$, we have the following commutative diagram;

$$\begin{array}{ccccccccc} 1 & \to & Z^2 & \to & \pi & \to & Q & \to & 1 \\ & & \downarrow i & & \downarrow & & \| & & \\ 1 & \to & R^2 & \to & \bar{\pi} & \to & Q & \to & 1 \\ & & \| & & \downarrow & & \downarrow \rho & & \\ 1 & \to & R^2 & \to & R^2 \times Iso_\circ(R^2) & \to & Iso_\circ(R^2) & \to & 1. \end{array}$$

Since $i_*[\pi]=\rho^*[U]=0$, this diagram exists and the homomorphism $\pi \to U \subset Iso_\circ(R^2 \times R^2)$ is injective. Thus M admits a geometry modelled on R^4.

Next we consider the case $[\pi]$ has infinite order in $H^2(Q;Z^2)$. In the exact sequence

$1 \to R \to Iso_\circ(Nil) \to Iso_\circ(R^2) \to 1$, let Q' be the inverse image of Q. Then an exact sequence $1 \to R \to Q' \to Q \to 1$ represents a nonzero element $[Q'] \in H^2(Q; R)$ $(\simeq R)$ ([KLR]). We have the commutative exact sequences;

$$\begin{array}{ccccccccc} 1 & \to & R^2 & \to & R\times Iso_\circ(Nil) & \to & Iso_\circ^{\,!}(R^2) & \to & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \to & R^2 & \to & R\times Q' & \to & Q & \to & 1 \\ & & \downarrow pr & & \downarrow pr & & \| & & \\ 1 & \to & R & \to & Q' & \to & Q & \to & 1. \end{array}$$

The homomorphism $pr_* : H^2(Q; R^2) \to H^2(Q; R)$ maps $[\bar{Q}]$ to $[Q']$, where $\bar{Q} = R\times Q'$. Since $[Q_\prime]$ is nonzero, $[\bar{Q}]$ is so.

On the other hand, since $[\pi]$ has infinite order, $i_*[\pi]$ is nonzero. Therefore there exists a linear homomorphism $\varepsilon : Q^2 \to R^2$ such that $\varepsilon_* \circ i_* [\pi] = [\bar{Q}]$, because $H^2(Q; R^2) \simeq R^2$ ([KLR]). Thus the commutative diagram is obtained;

$$\begin{array}{ccccccccc} 1 & \to & Z^2 & \to & \pi & \to & Q & \to & 1 \\ & & \downarrow \varepsilon\circ i & & \downarrow & & \| & & \\ 1 & \to & R^2 & \to & \bar{Q} & \to & Q & \to & 1 \\ & & \| & & \cap & & \downarrow p & & \\ 1 & \to & R^2 & \to & R\times Iso_\circ(Nil) & \to & Iso_\circ(R^2) & \to & 1. \end{array}$$

Since the homomorphism $\pi \to R\times Iso_\circ(Nil) = Iso_\circ(R\times Nil)$ is injective, M admits a geometry modelled on $R\times Nil$.

Case 3. $\chi(Q) > 0$.

In this case, since $\rho : Q \to Iso_\circ(S^2)$ is an embedding, Q is a finite group. Then we have $i_*[\pi] = \rho^*[U] = 0$, so the following commutative diagram is obtained by the Theorem 3 ([LR]);

$$\begin{array}{ccccccccc} 1 & \to & Z^2 & \to & \pi & \to & Q & \to & 1 \\ & & \downarrow i & & \downarrow & & \downarrow \rho & & \\ 1 & \to & R^2 & \to & R^2\times Iso_\circ(S^2) & \to & Iso_\circ(R^2\times R^2) & \to & 1 \\ & & & & \| & & & & \\ & & & & U & & & & . \end{array}$$

Thus we have an injection $\pi \to U \subset Iso_\circ(R^2\times S^2)$ and M admits a geometry modelled on $R^2\times S^2$.

QED.

THEOREM D. *Let M be a 4-dimensional aspherical Seifert fiber space wieh typical fiber* *a surface F except for a torus. If M admits a geometry, then M is a product manifold up to a finite covering.*

PROOF. It follows from Remark 1 that up to a finite covering M is a fiber space over B and both M and B are orientable. Then F is also orientable.
From the bundle relation $\tau_M=\widehat{\tau_F+p*\tau_B}$, the signature of M is always zero.

Case 1. B is not a torus.

We have the exact sequence; $1\to\pi_1(F)\to\pi\to Q\to 1$, where $\pi=\pi_1(M)$ and $Q=\pi_1(B)$. We apply the following results of [S]; let Γ be a discrete group and Γ' its subgroup, if $X=K(\Gamma, 1)$, $Y=K(\Gamma', 1)$ and $Z=K(\Gamma/\Gamma', 1)$ are finite complexes, then (i) $\chi(\Gamma)=\chi(\Gamma')\chi(\Gamma/\Gamma')$ (ii) $\chi(\Gamma)=\chi(X)$ and so the others. Thus we obtain $\chi(M_1)=\chi(\pi)=\chi(\pi_1(F))\chi(Q_1)>0$. By the results ([$W_1$]) on the characteristic numbers of closed oriented geometric 4-manifolds, M admits a geometry on modelled on $H^2\times H^2$. So π is contained in $Iso_\circ(H^2\times H^2)=PSL_2R\times PSL_2R$. There exists a subgroup π' in π with a finite index such $\pi'=\pi_1\times\pi_2\subset PSL_2R\times PSL_2R$([R, Theorem 5.22]), where $\pi_i\subset PSL_2R$ for $i=1, 2$. Let M' be a finite covering of M associated to π'. Thus this M' is a product manifold $H^2/\pi_1\times H^2/\pi_2$.

Case 2. B is a torus.

In this case we have a fiber bundle $F\to M\xrightarrow{p}T^2$. Since $p*\tau_{T^2}$ is trivial, τ_M has a nonzero cross-section. Thus $\chi(M)$ is zero. Applying the results of Wall ([W_1]), possible geometries on M are $R\times\widetilde{PSL}_2R$, $R^2\times H^2$ or $R\times H^3$, because $\pi_1(M)$ is not solvable.

First we consider the case of $R\times\widetilde{PSL}_2R$. π is a cocompact discrete subgroup of $Iso_\circ(R\times\widetilde{PSL}_2R)$, we have the exact sequence;

$$\begin{array}{ccccccccc} 1 & \to & R^2 & \to & Iso_\circ(R\times\widetilde{PSL}_2R) & \to & Iso_\circ(H^2) & \to & 1\\ & & \cup & & \cup & & \cup & & \\ 1 & \to & R^2\cap\pi & \to & \pi & \to & Q & \to & 1. \end{array}$$

We may assume Q is torsionfree. If $R^2\cap\pi=\{1\}$ or Z, then cohomological dimension of π is smaller than 4, which is a contradiction. Thus we have $R^2\cap\pi=Z^2$. We have the following diagram;

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & \downarrow & & & \\ & & & Z^2 & & & \\ & & & \downarrow j & & & \\ 1 & \to & \pi_1(F) & \xrightarrow[i]{} & \pi_1(M_1) & \to Z^2 \to & 1 \\ & & & \downarrow & & & \\ & & & Q & & & \\ & & & \downarrow & & & \\ & & & 1 & & & \end{array}$$

Since $\pi_1(F)$ does not contnin Z as a normal subgroup by the assumption of F, $i(\pi_1(F)) \cap j(Z^2)=\{1\}$. The direct product $i(\pi_1(F))\times j(Z^2)$ is a subgroup of $\pi_1(M)$ with a finite index. Let M' be a finite covering of M associated to this subgroup. It follows that $H_1(M)\simeq H_1(F)+Z^2$. Thus the rank of $H_1(M')$ is even. On the other hand, 1-st Betti number of a manifold that admits a geometry modelled on $R\times \widetilde{PSL_2}\, R$ is odd ([W_1]). So we have a contradiction.

Next we consider the case of $R\times H^3$. Since π is a cocompact discrete subgroup of $Iso_\circ(R\times H^3)$, we have the exact sequence;

$$\begin{array}{ccccccccc} 1 & \to & R & \to & Iso_\circ(R\times H^3) & \to & Iso_\circ(H^3) & \to & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \to & R\cap\pi & \to & \pi & \to & Q & \to & 1. \end{array}$$

As before, we obtain $R\cap\pi=Z$. The following commutative diagram is obtained;

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & Z & \searrow & & & \\ & & & & \downarrow j & & & & \\ 1 & \to & \pi_1(F) & \xrightarrow{i} & \pi_1(M_1) & \to & Z^2 & \to & 1 \\ & & & s\searrow & \downarrow & & & & \\ & & & & Q & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

where there exist injectons $r: Z\to Z^2$ and $s:\pi_1(F)\to Q$ because of $j(Z)\cap i(\pi_1(F))=\{1\}$. Thus there exists a subgroup Q_1 of Q with a finite index such that $1\xrightarrow{s}\pi_1(F)\to Q_1\to Z\to 1$ is an exact sequence. By results of ([H. Chap. 11]), H^3/Q_1 is diffeomorphic to a fiber bundle over S^1 with fiber a surface, because $Q_1=\pi_1(H^3/Q_1)$. In fact, it was proved that there exist hyperbplic 3-manifolds which admits the structure of bundle over S^1 ([J]). Thus this case can happen and M is a product manifold $S^1\times N$ up to a finite covering, where N is a hyperbolic 3-manifold.

In the case of $R^2\times H^2$, it is clear that M has a product structure $F\times B$ up to a finite covering, where F and B are surfaces.

QED.

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