Classifications of Kaehler manifolds satisfying some curvature conditions

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1. Introduction

A Riemannian manifold[(**)] is called *locally flat* if its Riemann-Christoffel curvature tensor R vanishes identically, and it is called *locally conformally flat* if its Weyl conformal curvature tensor C vanishes identically when dimension $\geqslant 4$. In the following, the Ricci tensor of a Riemannian manifold will be denoted by S, and the Ricci endomorphism will be denoted by Q: for all tangent vector fields X and Y one has $S(X, Y) = \langle QX, Y \rangle$, where $\langle , \rangle$ is the Riemannian metric. A Kaehler manifold is said to be *Bochner flat* or is said to be a *Bochner–Kaehler manifold* if its Bochner curvature tensor B vanishes identically. We recall that the Bochner curvature tensor of a Kaehler manifold was introduced as a complex version of the Weyl conformal curvature tensor of a Riemannian manifold. For results on Bochner-Kaehler manifolds, we refer to M. Kon [8], M. Matsumoto and S. Tanno [9], S. Tachibana and R. S. Liu [18], S. Yamaguchi and S. Sato [25].

R. L. Bishop and S. I. Goldberg [1] and K. Sekigawa and H. Takagi [16], classified the locally conformally flat spaces which satisfy the curvature condition $R \cdot S = 0$, or equivalently $R \cdot Q = 0$, where the first tensor acts on the second one as a derivation. Concerning curvature conditions on Kaehler manifolds, we recall the following results.

THEOREM A. (K. Yano [26]). *A locally conformally flat Kaehler manifold of complex dimension $n \geqslant 3$ is locally flat.*

THEOREM B. (H. Takagi and Y. Watanabe [19]). *A Bochner-Kaehler manifold of complex dimension $n \geqslant 2$ satisfies the curvature condition $R \cdot Q = 0$ if and only if either* (i) *M is a space $M^n(c)$ of constant holomorphic sectional curvature c, or* (ii) *for some strictly positive real number c and some $m, m' \in \boldsymbol{N}_0$ with $m + m' = n$, M is locally holomorphically isometric to a product $M^m(c) \times M^{m'}(-c)$. A Bochner-Kaehler manifold is semi-symmetric, i.e.*

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(**) All manifolds considered in this paper are assumed to be connected.

it satisfies the curvature condition $R \cdot R=0$, *if and only if it is of type* (i) *or* (ii).

In this paper, we study Bochner-Kaehler manifolds which satisfy one of the curvature conditions $Q \cdot R=0$, $Q \cdot C=0$, $C \cdot Q=0$, $R \cdot C=0$, $C \cdot R=0$ or $C \cdot C=0$. More precisely, we will prove the following results.

THEOREM 1. *A Bochner-Kaehler manifold M of complex dimension* $n \geqslant 2$ *satisfies* $R \cdot C=0$ *if and only if* $n=2$ *or M is one of the manifolds from Theorem B.*

THEOREM 2. *For a Bochner-Kaehler manifold M of complex dimension* $n \geqslant 2$, *the following assertions are equivalent:*

(i) $Q \cdot R=0$,

(ii) $Q \cdot C=0$,

(iii) $C \cdot C=0$,

(iv) $C \cdot R=0$,

(v) $C=0$,

(vi) *M is locally flat or* $n=2$ *and for some strictly positive real number c, M is locally holomorphically isometric to a product* $M(c) \times M(-c)$, *where* $M(c)$, *respectively* $M(-c)$, *denotes a complex 1-dimensional space of constant Gauss curvature c, respectively* $-c$.

REMARK. See S. Tanno [21] for $n=2$. Furthermore, we see from Theorem F and Theorem 3 of [9] that a Bochner-Kaehler manifold M of complex dimension $n \geqq 2$ and with Levi Civita connection ∇ satisfies $\nabla C=0$ if and only if M is one of the manifolds from Theorem B.

THEOREM 3. *A Bochner-Kaehler manifold M of complex dimension* $n \geqslant 2$ *satisfies* $C \cdot Q=0$ *if and only if either* (i) *M is a space* $M^n(c)$ *of constant holomorphic sectional curvature c, or* (ii) *n is even,* $n=2m$, *and for some strictly positive real number c, M is locally holomorphically isometric to a product* $M^m(c) \times M^m(-c)$.

The following theorem was proven by B. Y. Chen, A. Derdzinski and J. P. Bourguignon.

THEOREM C. *Let M be a (complex) 2-dimensional compact Bochner-Kaehler manifold. Then M is locally symmetric, and hence a space* $M^2(c)$ *of constant holomorphic sectional curvature* $c \in \boldsymbol{R}_0^+$.

Thus, from Theorem 1 and Theorem C, it follows that a compact Bochner-Kaehler manifold of complex dimension $n \geqslant 2$ satisfying the condition $R \cdot C=0$ is a space $M^n(c)$ of constant holomorphic sectional curvature $c \in \boldsymbol{R}_0^+$.

For literature on Riemannian manifolds, on hypersurfaces of real space forms and on complex hypersurfaces of complex space forms which satisfy the type of curvature conditions studied in this paper, we refer to Z. I. Szabó [17], D. E. Blair, P. Verheyen and L. Verstraelen [2], J. Deprez [4] [5], J. Deprez, P. Verheyen and L. Verstraelen [6] [7], Y. Matsuyama [10], I. Mogi and H. Nakagawa [11], K. Nomizu [12], P. J. Ryan [13] [14] [15], S. Tanno and T. Takahashi [22], T. Takahashi [20], P. Verheyen and L. Verstraelen [23] [24], etc.

2. Basic formulas

Let $(M, <,>, J)$ be a Kaehler manifold of complex dimension n with complex structure J, Kaehler metric $<,>$, Levi Civita connection ∇, Riemann-Christoffel curvature tensor R, Ricci endomorphism Q and scalar curvature τ. In the following, we will delete the complex structure and the metric in the notation. For vectors X and Y tangent to M at a point p of M, let $X \wedge Y$ denote the endomorphism $T_pM \longrightarrow T_pM : Z \longrightarrow (X \wedge Y)Z := <Y, Z> X - <X, Z> Y$. The *Weyl conformal curvature tensor* C of M is defined by

$$C(X, Y) := R(X, Y) - \frac{1}{2(n-1)}(QX \wedge Y + X \wedge QY) + \frac{\tau}{2(n-1)(2n-1)} X \wedge Y, \tag{2.1}$$

and the *Bochner curvature tensor* B of M is defined by

$$\begin{aligned} B(X, Y) := R(X, Y) &- \frac{1}{2(n+2)}(QX \wedge Y + X \wedge QY + QJX \wedge JY \\ &+ JX \wedge QJY - 2<QJX, Y> J - 2<JX, Y> QJ) \\ &+ \frac{\tau}{4(n+1)(n+2)}(X \wedge Y + JX \wedge JY - 2<JX, Y> J), \end{aligned} \tag{2.2}$$

for all vectors X and Y tangent to M at the same point. The following properties are well-known:

$$\begin{aligned} & R(JX, JY) = R(X, Y), \qquad R(X, Y)J = JR(X, Y), \\ & QJ = JQ, \\ & C(JX, JY) = -JC(X, Y)J. \end{aligned} \tag{2.3}$$

Moreover, it is easy to verify the following equalities:

$$\begin{aligned} [X \wedge Y, Z \wedge W] &= <Y, Z> X \wedge W + <X, W> Y \wedge Z \\ &\quad - <X, Z> Y \wedge W - <Y, W> X \wedge Z, \\ [J, X \wedge Y] &= JX \wedge Y + X \wedge JY, \\ [QJ, X \wedge Y] &= QJX \wedge Y + X \wedge QJY. \end{aligned}$$

In the sequel, we assume that M is a *Bochner-Kaehler* manifold, i.e., $B=0$, or equivalently

$$R(X, Y) = \frac{1}{2(n+2)}(QX \wedge Y + X \wedge QY + QJX \wedge JY + JX \wedge QJY$$

$$-2<QJX, Y>J-2<JX, Y>QJ)$$
$$-\frac{\tau}{4(n+1)(n+2)}(X\wedge Y+JX\wedge JY-2<JX, Y>J).$$

The endomorphisms $R(X, Y)$, $C(X, Y)$ and Q of TM act as derivations on the algebra of tensor fields of M. For instance:

$$\begin{aligned}(Q\cdot R)(Z, V)W&=Q(R(Z, V)W)-R(QZ, V)W-R(Z, QV)W\\&\quad -R(Z, V)QW\\&=[Q, R(Z, V)]W-R(QZ, V)W-R(Z, QV)W,\end{aligned}$$

and

$$\begin{aligned}(R(X, Y)\cdot Q)Z&=R(X, Y)(QZ)-Q(R(X, Y)Z)\\&=[R(X, Y), Q]Z,\end{aligned}$$

for all vectors X, Y, Z, V, W tangent to M at the same point. By $R\cdot C=0, \dots,$ we will always mean that $R(X, Y)\cdot C=0, \dots,$ for all X, Y. The derivations $R\cdot$ and $C\cdot$ commute with contractions. For every tensor T on M, we have that $R(X, Y)\cdot T=\nabla_X\nabla_Y T-\nabla_Y\nabla_X T-\nabla_{[X, Y]}T$. This implies that a locally symmetric space (i.e. a Riemannian manifold for which $\nabla R=0$) satisfies the condition $R\cdot R=0$, that a Ricci parallel space (i.e. a Riemannian manifold for which $\nabla Q=0$) satisfies $R\cdot Q=0$, etc.

For every point p in M, Q is a symmetric endomorphism of T_pM. Furthermore, if X is an eigenvector of Q acting on T_pM, then, by (2.3), also JX is an eigenvector of Q with the same eigenvalue as X. Thus, we can choose an orthonormal basis $\{e_1, \dots, e_n, e_{1*}, \dots, e_{n*}\}$ for T_pM such that $e_{i*}=Je_i$, and such that

$$Qe_i=\lambda_i e_i, \qquad Qe_{i*}=\lambda_i e_{i*}, \tag{2.5}$$

for some real numbers λ_i, ($i\in\{1, 2, \dots, n\}$). In particular, we then have:

$$\tau=2\sum_{i=1}^{n}\lambda_i. \tag{2.6}$$

Using (2.1), (2.4) and (2.5), we obtain that

$$\begin{aligned}&R(e_i, e_j)=a_{ij}(e_i\wedge e_j+e_{i*}\wedge e_{j*}), \qquad (i\neq j),\\&R(e_i, e_{j*})=a_{ij}(e_i\wedge e_{j*}-e_{i*}\wedge e_j)+c_{ij}J-\frac{\delta_{ij}}{n+2}QJ,\\&C(e_i, e_j)=(a_{ij}+b_{ij})e_i\wedge e_j+a_{ij}e_{i*}\wedge e_{j*}, \qquad (i\neq j),\\&C(e_i, e_{j*})=(a_{ij}+b_{ij})e_i\wedge e_{j*}-a_{ij}e_{i*}\wedge e_j+c_{ij}J-\frac{\delta_{ij}}{n+2}QJ,\\&C(e_{i*}, e_{j*})=a_{ij}e_i\wedge e_j+(a_{ij}+b_{ij})e_{i*}\wedge e_{j*}, \qquad (i\neq j),\\&a_{ij}=\frac{2(n+1)(\lambda_i+\lambda_j)-\tau}{4(n+1)(n+2)},\end{aligned} \tag{2.7}$$

(2.7) $$b_{ij}=\frac{-(2n-1)(\lambda_i+\lambda_j)+\tau}{2(n-1)(2n-1)},$$

$$c_{ij}=\frac{-2(n+1)\lambda_i+\tau}{2(n+1)(n+2)}\delta_{ij},$$

for all $i, j\in\{1, 2, \dots, n\}$,

Next, we compute the numbers λ_i, a_{ij}, b_{ij}, c_{ij} and τ for the manifolds M appearing in Theorem B. For a Kaehler manifold M of constant holomorphic sectional curvature $c\in\boldsymbol{R}$, the tensor R is given by

$$R(X, Y)=\frac{c}{4}(X\wedge Y+JX\wedge JY-2<JX, Y>J).$$

With respect to an orthonormal basis $\{e_1,\dots, e_n, e_{1*},\dots, e_{n*}\}$ for T_pM, $p\in M$, one easily finds that

$$\lambda_i=\frac{n+1}{2}c,$$

$$\tau=n(n+1)c,$$

$$a_{ij}=\frac{c}{4},$$

$$b_{ij}=-\frac{n+1}{2(2n-1)}c,$$

$$c_{ij}=-\frac{c}{2(n+2)}\delta_{ij}.$$

For a product manifold $M=M^m(c)\times M^{m'}(-c)$ of a Kaehler manifold $M^m(c)$ of constant holomorphic curvature $c\in\boldsymbol{R}_0^+$ and a Kaehler manifold $M^{m'}(-c)$ of constant holomorphic curvature $-c$, the tensor R is given by

$$R(X, Y)=\frac{c}{4}(X\wedge Y+JX\wedge JY-2<JX, Y>J),$$

$$R(X, Y')=0,$$

$$R(X', Y')=-\frac{c}{4}(X'\wedge Y'+JX'\wedge JY'-2<JX', Y'>J),$$

for all vectors X, Y tangent to $M^m(c)$ and X', Y' tangent to $M^{m'}(-c)$. Let $p=(p_1, p_2)\in M^m(c)\times M^{m'}(-c)$, $(m+m'=n)$, and choose an orthonormal basis $\{e_1,\dots, e_m, e_{m+1},\dots, e_n, e_{1*},\dots, e_{n*}\}$ for T_pM such that $T_{p_1}M^m(c)$ is spanned by $\{e_1,\dots, e_m, e_{1*},\dots, e_{m*}\}$ and $T_{p_2}M^{m'}(-c)$ by $\{e_{m+1},\dots, e_n, e_{(m+1)*},\dots, e_{n*}\}$. Then, for all $i, j\in\{1,\dots, m\}$ and $x, y\in\{m+1,\dots, n\}$, one finds that

$$\lambda_i=\frac{m+1}{2}c,\ \lambda_x=-\frac{m'+1}{2}c,$$

$$\tau=(m-m')(m+m'+1)c,$$

$$a_{ij}=\frac{c}{4},\ a_{ix}=0,\ a_{xy}=-\frac{c}{4},$$

(2.9)
$$b_{ij}=\frac{-m^2-2mm'-m'^2-3m'+1}{2(m+m'-1)(2m+2m'-1)}c,$$

$$b_{ix}=\frac{3(m-m')}{4(m+m'-1)(2m+2m'-1)}c,$$

$$b_{xy}=\frac{m'^2+2mm'+m^2+3m-1}{2(m+m'-1)(2m+2m'-1)}c,$$

$$c_{ii}=-\frac{m'+1}{2(m+m'+2)}c,\quad c_{xx}=\frac{m+1}{2(m+m'+2)}c.$$

For later use, we now give a necessary and sufficient condition for a Bochner-Kaehler manifold to satisfy the condition $R\cdot Q=0$. First, using (2. 3), we observe that $R\cdot Q=0$ if and only if

$$(R(e_i, e_j)\cdot Q)e_k=0$$

and

$$(R(e_i, e_{j*})\cdot Q)e_k=0$$

for all $i, j, k\in\{1, 2, \dots, n\}$. Furthermore, (2. 7) implies that

$$(R(e_i, e_j)\cdot Q)e_k=a_{ij}\{\delta_{kj}(\lambda_k-\lambda_i)e_i-\delta_{ki}(\lambda_k-\lambda_j)e_j\}$$

and

$$(R(e_i, e_{j*})\cdot Q)e_k=-a_{ij}\{\delta_{ki}(\lambda_k-\lambda_i)e_{i*}+\delta_{kj}(\lambda_k-\lambda_j)e_{j*}\}.$$

Examining all possible choices for $i, j, k\in\{1, 2, \dots, n\}$, we thus conclude that M satisfies $R\cdot Q=0$ if and only if M satisfies the condition

$$(*)\qquad \forall i, j\in\{1, 2, \dots, n\}: a_{ij}(\lambda_i-\lambda_j)=0.$$

3. Proof of Theorem 1

By (2. 3), we find that M satisfies $R\cdot C=0$ if and only if

$$(R(e_i, e_j)\cdot C)(e_k, e_l)=0,$$

$$(R(e_i, e_{j*})\cdot C)(e_k, e_l)=0,$$

$$(R(e_i, e_j)\cdot C)(e_k, e_{l*})=0$$

and

$$(R(e_i, e_{j*})\cdot C)(e_k, e_{l*})=0,$$

for all $i, j, k, l\in\{1, 2, \dots, n\}$. From (2.7), we have

(3.1)
$$\begin{aligned}(R(e_i, e_j)\cdot C)(e_k, e_l)=a_{ij}\{&\delta_{il}[(a_{kl}+b_{kl})-(a_{kj}+b_{kj})]e_j\wedge e_k\\ &+\delta_{jk}[(a_{kl}+b_{kl})-(a_{il}+b_{il})]e_i\wedge e_l\\ &-\delta_{ik}[(a_{kl}+b_{kl})-(a_{jl}+b_{jl})]e_j\wedge e_l\\ &-\delta_{jl}[(a_{kl}+b_{kl})-(a_{ki}+b_{ki})]e_i\wedge e_k\end{aligned}$$

$$+\delta_{il}(a_{kl}-a_{kj})e_{j*}\wedge e_{k*}$$
$$+\delta_{jk}(a_{kl}-a_{il})e_{i*}\wedge e_{l*}$$
$$-\delta_{ik}(a_{kl}-a_{jl})e_{j*}\wedge e_{l*}$$
$$-\delta_{jl}(a_{kl}-a_{ki})e_{i*}\wedge e_{k*}\},$$

(3.2)
$$\begin{aligned}(R(e_i, e_{j*})\cdot C)(e_k, e_l)=a_{ij}\{&\delta_{jl}[(a_{kl}+b_{kl})-(a_{ik}+b_{ik})]e_{i*}\wedge e_k\\ &-\delta_{jk}[(a_{kl}+b_{kl})-(a_{il}+b_{il})]e_{i*}\wedge e_l\\ &+\delta_{il}[(a_{kl}+b_{kl})-(a_{jk}+b_{jk})]e_{j*}\wedge e_k\\ &-\delta_{ik}[(a_{kl}+b_{kl})-(a_{jl}+b_{jl})]e_{j*}\wedge e_l\\ &-\delta_{jl}(a_{kl}-a_{ik})e_i\wedge e_{k*}\\ &+\delta_{jk}(a_{kl}-a_{il})e_i\wedge e_{l*}\\ &-\delta_{il}(a_{kl}-a_{jk})e_j\wedge e_{k*}\\ &+\delta_{ik}(a_{kl}-a_{jl})e_j\wedge e_{l*}\\ &+(\delta_{il}c_{jk}-\delta_{jk}c_{il}+\delta_{jl}c_{ik}-\delta_{ik}c_{jl})J\},\end{aligned}$$

(3.3)
$$\begin{aligned}(R(e_i, e_j)\cdot C)(e_k, e_{l*})=a_{ij}\{&-\delta_{jl}[(a_{kl}+b_{kl})-(a_{ik}+b_{ik})]e_{i*}\wedge e_k\\ &-\delta_{jk}(a_{kl}-a_{il})e_i\wedge e_l\\ &+\delta_{il}[(a_{kl}+b_{kl})-(a_{jk}+b_{jk})]e_{j*}\wedge e_k\\ &+\delta_{ik}(a_{kl}-a_{jl})e_{j*}\wedge e_l\\ &+\delta_{jl}(a_{kl}-a_{ik})e_i\wedge e_{k*}\\ &+\delta_{jk}[(a_{kl}+b_{kl})-(a_{il}+b_{il})]e_i\wedge e_{l*}\\ &-\delta_{il}(a_{kl}-a_{jk})e_j\wedge e_{k*}\\ &-\delta_{ik}[(a_{kl}+b_{kl})-(a_{jl}+b_{jl})]e_j\wedge e_{l*}\\ &+\frac{1}{n+2}\delta_{kl}(\lambda_i-\lambda_j)(e_{i*}\wedge e_j-e_i\wedge e_{j*})\\ &+(\delta_{ik}c_{jl}+\delta_{il}c_{jk}-\delta_{jk}c_{il}-\delta_{jl}c_{ik})J\},\end{aligned}$$

and

(3.4)
$$\begin{aligned}(R(e_i, e_{j*})\cdot C)(e_k, e_{l*})=a_{ij}\{&-\delta_{jl}[(a_{kl}+b_{kl})-(a_{ik}+b_{ik})]e_i\wedge e_k\\ &-\delta_{jk}(a_{kl}-a_{il})e_i\wedge e_l\\ &-\delta_{il}[(a_{kl}+b_{kl})-(a_{jk}+b_{jk})]e_j\wedge e_k\\ &-\delta_{ik}(a_{kl}-a_{jl})e_j\wedge e_l\\ &-\delta_{jl}(a_{kl}-a_{ik})e_{i*}\wedge e_{k*}\end{aligned}$$

$$-\delta_{jk}[(a_{kl}+b_{kl})-(a_{il}+b_{il})]e_{i*}\wedge e_{l*}$$
$$-\delta_{il}(a_{kl}-a_{jk})e_{j*}\wedge e_{k*}$$
$$-\delta_{ik}[(a_{kl}+b_{kl})-(a_{jl}+b_{jl})]e_{j*}\wedge e_{l*}$$
$$+\frac{1}{n+2}\delta_{kl}(\lambda_i-\lambda_j)(e_i\wedge e_j+e_{i*}\wedge e_{j*})\},$$

for all $i, j, k, l\in\{1, 2, \dots, n\}$. Using (3. 1), we obtain that

$$(R(e_i, e_j)\cdot C)(e_k, e_i)e_{j*}=a_{ij}(a_{ki}-a_{kj})e_j$$

for all mutually distinct $i, j, k\in\{1, 2, \dots, n\}$. This implies that M satisfies condition (*) of § 2 if $R\cdot C=0$ and $n\geqslant 3$. Conversely, the manifolds of Theorem B are locally symmetric, and so they satisfy $R\cdot R=0$. Moreover, they also satisfy $R\cdot Q=0$, such that for these manifolds we also have $R\cdot C=0$. For $n=2$, one easily verifies, using (3. 1), (3. 2), (3. 3) and (3. 4) for all $i, j, k, l\in\{1, 2\}$, that $R\cdot C=0$. This proves Theorem 1.

4. Proof of Theorem 2

The implications (v)$\Rightarrow$(i) and (v)$\Rightarrow$(iv) are trivial. The implication (iv)$\Rightarrow$(iii) holds since the derivation $C(X, Y)\cdot$ commutes with contractions. The implication (vi)$\Rightarrow$(v) follows easily. Therefore, it is sufficient to prove the implications (ii)$\Rightarrow$(i), (i)$\Rightarrow$(iv), and (iii)$\Rightarrow$(vi).

If M satisfies $Q\cdot C=0$, it follows from

$$(Q\cdot C)(e_i, e_j)e_{i*}=2a_{ij}\lambda_i e_{j*}$$

that, for all distinct $i, j\in\{1, 2, \dots, n\}$, we have

$$a_{ij}\lambda_i=0. \tag{4.1}$$

From (2. 3) we derive that $Q\cdot R=0$ if and only if

$$(Q\cdot R)(e_i, e_j)e_k=0$$

and

$$(Q\cdot R)(e_i, e_{j*})e_k=0,$$

for all $i, j, k\in\{1, 2, \dots, n\}$. On the other hand, (2. 7) implies that

$$(Q\cdot R)(e_i, e_j)e_k=a_{ij}\{-\delta_{jk}(\lambda_j+\lambda_k)e_i+\delta_{ik}(\lambda_i+\lambda_k)e_j\}$$

and

$$(Q\cdot R)(e_i, e_{j*})e_k=a_{ij}\{\delta_{jk}(\lambda_j+\lambda_k)e_{i*}+\delta_{ik}(\lambda_i+\lambda_k)e_{j*}\}$$
$$+\left(c_{ij}-\frac{1}{n+2}\delta_{ij}\lambda_k\right)(\lambda_i+\lambda_k)e_{k*},$$

for all $i, j, k\in\{1, 2, \dots, n\}$. Examining all possible choices of indices, we find that M

satisfies $Q\cdot R=0$ if and only if

$$(**)\qquad \forall i, j\in\{1, 2, \ldots, n\},\ i\neq j: a_{ij}\lambda_i=0.$$

Consequently, by (4. 1) the implication (ii)⇒(i) holds good. Furthermore, it is easy to see that M satisfies (*) if $Q\cdot R=0$. From (2. 8) and (**) it follows that a space of constant curvature satisfies $Q\cdot R=0$ only if its curvature is zero. Next, if for instance for $m\geqslant 2$ a space $M^m(c)\times M^{m'}(-c)$ satisfies $Q\cdot R=0$, then $\lambda_1 a_{12}=0$, which contradicts (2. 9). This proves the implication (ii)⇒(vi). Finally, we prove that (iii) implies (vi). From (2. 7), we obtain that

$$\begin{aligned}(4.2)\qquad (C(e_i, e_j)\cdot C)(e_k, e_l)=&(R(e_i, e_j)\cdot C)(e_k, e_l)\\ &+b_{ij}\{-\delta_{jl}[(a_{kl}+b_{kl})-(a_{ik}+b_{ik})]\,e_i\wedge e_k\\ &+\delta_{jk}[(a_{kl}+b_{kl})-(a_{il}+b_{il})]\,e_i\wedge e_l\\ &+\delta_{il}[(a_{kl}+b_{kl})-(a_{jk}+b_{jk})]\,e_j\wedge e_k\\ &-\delta_{ik}[(a_{kl}+b_{kl})-(a_{jl}+b_{jl})]\,e_j\wedge e_l\}\end{aligned}$$

and

$$\begin{aligned}(4.3)\qquad (C(e_i, e_{j*})\cdot C)(e_k, e_l)=&(R(e_i, e_{j*})\cdot C)(e_k, e_l)\\ &+b_{ij}\{\delta_{il}[(a_{kl}+b_{kl})-(a_{jk}+b_{jk})]\,e_{j*}\wedge e_k\\ &-\delta_{ik}[(a_{kl}+b_{kl})-(a_{jl}+b_{jl})]\,e_{j*}\wedge e_l\\ &-\delta_{jl}a_{kl}e_i\wedge e_{k*}+\delta_{jk}a_{kl}e_i\wedge e_{l*}\\ &-\delta_{ik}a_{jl}e_j\wedge e_{l*}+\delta_{il}a_{jk}e_j\wedge e_{k*}\\ &+(-\delta_{ik}c_{jl}+\delta_{il}c_{jk})J\\ &+\frac{1}{n+2}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})QJ\},\end{aligned}$$

for all $i, j, k, l\in\{1, 2, \ldots, n\}$. This implies that

$$(4.4)\qquad (C(e_i, e_j)\cdot C)(e_k, e_i)e_{k*}=a_{ij}(a_{ik}-a_{jk})e_{j*}$$

for all mutually distinct indices $i, j, k\in\{1, 2, \ldots, n\}$. Now, suppose that M satisfies the condition $C\cdot C=0$. Then (4. 4) yields that M satisfies (*) whenever $n\geqslant 3$. Consequently either $n=2$ or M is one of the manifolds from Theorem B. Moreover, from Theorem 1, we know that $R\cdot C=0$, which by (4. 3) implies that

$$(C(e_i, e_{j*})\cdot C)(e_i, e_j)e_i=b_{ij}\left(a_{ij}-c_{jj}+\frac{\lambda_i}{n+2}\right)e_{i*},$$

for all distinct $i, j\in\{1, 2, \ldots, n\}$. Thus we find that

$$(4.5)\qquad b_{ij}\left(a_{ij}-c_{jj}+\frac{\lambda_i}{n+2}\right)=0,$$

and

$$(4.6)\qquad b_{ij}\left(a_{ij}-c_{ii}+\frac{\lambda_j}{n+2}\right)=0$$

by interchanging the indices i and j. Adding (4. 5) and (4. 6), and using (2. 7), we obtain that

$$(4.7)\qquad a_{ij}b_{ij}=0$$

for all distinct $i, j\in\{1, 2, \dots, n\}$. From (2. 6), (2. 7) and (4. 7), we see that a 2-dimensional Bochner-Kaehler manifold satisfies $C\cdot C=0$ only if $\lambda_1+\lambda_2=0$. Consequently, such a space satisfies (*). From (2. 8) and (4. 7), it is clear that a space form $M^n(c)$, $(c\in \boldsymbol{R})$, $n\geqslant 2$, satisfies $C\cdot C=0$ only when $c=0$. On the other hand, a space $M^m(c)\times M^{m'}(-c)$, $(c\in \boldsymbol{R}_0^+)$, with for instance $m\geqslant 2$, does not satisfy $C\cdot C=0$, since for such a space, by (2.9), $a_{12}b_{12}\neq 0$. This proves the desired implication.

5. Proof of Theorem 3

From (2. 3) we know that M satisfies $C\cdot Q=0$ if and only if

$$(C(e_i, e_j)\cdot Q)e_k=0,$$

$$(C(e_i, e_{j*})\cdot Q)e_k=0,$$

and

$$(C(e_i, e_j)\cdot Q)e_{k*}=0,$$

for all $i, j, k\in\{1, 2, \dots, n\}$. From (2.7) we have

$$(C(e_i, e_j)\cdot Q)e_k=-\delta_{jk}(a_{ij}+b_{ij})(\lambda_i-\lambda_k)e_i +\delta_{ik}(a_{ij}+b_{ij})(\lambda_j-\lambda_k)e_j,$$

$$(C(e_i, e_{j*})\cdot Q)e_k=\delta_{jk}a_{ij}(\lambda_i-\lambda_k)e_{i*} +\delta_{ik}(a_{ij}+b_{ij})(\lambda_j-\lambda_k)e_{j*},$$

and

$$(C(e_i, e_j)\cdot Q)e_{k*}=-\delta_{jk}a_{ij}(\lambda_i-\lambda_k)e_{i*} +\delta_{ik}a_{ij}(\lambda_j-\lambda_k)e_{j*},$$

for all $i, j, k\in\{1, 2, \dots, n\}$. This implies that $C\cdot Q=0$ if and only if

$$a_{ij}(\lambda_i-\lambda_j)=0$$

and

$$b_{ij}(\lambda_i-\lambda_j)=0,$$

for all distinct $i, j\in\{1, 2, \dots, n\}$. Therefore it is clear that $C\cdot Q=0$ implies (*).

Conversely, from (2. 8) and (2. 9), one can see that $M^n(c)$ satisfies $C\cdot Q=0$ for all $c\in \boldsymbol{R}$, and that $M^m(c)\times M^{m'}(-c)$, where $c\in \boldsymbol{R}_0^+$ and $m, m'\in \boldsymbol{N}_0$, satisfies $C\cdot Q=0$ if and only if $m=m'$.

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