On C*-algebras Associated with Factors of Type II$_1$ Having Property Γ

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Abstract

Let M be a factor of type II$_1$ on the standard Hilbert space H. We prove that if M has property Γ, there exists a larger C^*-algebra on H than the C^*-algebra $C^*(M, M')$ which has trivial intersection with $C(H)$, the algebra of all compact operators on H.

Lat M be a factor of type II$_1$ with the faithful normal normalized trace τ. Let M act standardly on the Hibert space $H=L^2(M, \tau)$. We denote by Aut(M) (resp. Int(M)) the set of all automorphisms (resp. inner automorphisms) of M. For an automorphism $\theta \in \mathrm{Aut}(M)$, we define the unitary operator $u(\theta)$ on H by $u(\theta)\eta(x)=\eta(\theta(x))$, $x \in M$, where η is the canonical imbedding of M into H. Following [1], for a subset $G \subset \mathrm{Aut}(M)$, we write the C^*-algebra generated by the unitaries $u(\theta)$, $\theta \in G$, and M as $C^*(M, G)$. We denote by $C^*(M, M')$ the C^*-algedra generated by M and its commutant M' on H and by $C(H)$ the algebra of all compact operators on H.

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In [**3**], A. Connes has shown that a factor M of type II$_1$ has property Γ in the sense of Murray and von Neumann [**4**] if and only if $C^*(M, M') \cap C(H)=\{0\}$. In this result, we observe the fact that $C^*(M, M')=C^*(M, \mathrm{Int}(M))$ because in our context we see that $u(\theta)=vJvJ$ if $\theta=\mathrm{Ad}(v) \in \mathrm{Int}(M)$ where J is the canonical unitary involution on H. Let Cnt(M) be the set of all centrally trivial automorphisms, which clearly contains Int(M). Here an automorphism θ of M is said to be centrally trivial if for a bounded sequence (x_n) in M, such that $\|x_na-ax_n\|_2 \to 0$ for all $a \in M$, θ satisfies the condition $\|\theta(x_n)-x_n\|_2 \to 0$, where $\|x\|_2=\tau(x^*x)^{1/2}$, $x \in M$.

We then prove the following:

THEOREM 1. *If M has property Γ, then*

$$C^*(M, \mathrm{Cnt}(M)) \cap C(H)=\{0\}.$$

In general, for many factors of type II_1 having property Γ, the class of centrally trivial automorphisms of them is exactly wider than that of inner automorphisms. Hence, by the next proposition, for such a factor M the C^*-algebra $C^*(M, \mathrm{Cnt}(M))$ is exactly larger than $C^*(M, M')$. We are interested in the boundary situation of those classes G of automorphisms of M satisfying $C^*(M, G) \cap C(H)=\{0\}$.

PROOF OF THEOREM 1. Put $A=C^*(M, \mathrm{Cnt}(M))$. Since A contains $C^*(M, M')$, A is irreducible. Hence $A \supset C(H)$ or $A \cap C(H)=\{0\}$. Thus if A contains $C(H)$, the algebra A contains the one dimensional projection onto $C\eta(1)$. Then, by the proof of [**1**, Theorem 4], $\mathrm{Cnt}(M)$ acts strongly ergodically on M. Let (x_n) be a centralizing sequence in M, that is, a bounded sequence such that $\|x_n a-a x_n\|_2 \to 0$ for all $a \in M$. Then by definition,we have that $\|\theta(x_n)-x_n\|_2 \to 0$ for an automorphism $\theta \in \mathrm{Cnt}(M)$. It follows that $\|x_n-\tau(x_n)\|_2 \to 0$ by the definition of strong ergodicity. This means that any centralizing sequence is trivial. Hence M does not have property Γ ([**2**; Corollaries 3.7 and 3.8]). This is a contradiction. Therefore,

$$C^*(M, \mathrm{Cnt}(M)) \cap C(H)=\{0\},$$

which completes the proof.

As a consequence, one may easily see with Connes' result cited before that M has property Γ if and only if the above condition holds.

In order to make clear the difference between $C^*(M, \mathrm{Cnt}(M))$ and $C^*(M, M')$, we show the following proposition.

POPOSITION 2. *Let θ be an automorphism of M. Then θ is inner if and only if* $u(\theta) \in C^*(M, M')$.

PROOF. It suffices to show the if part of the statement. Thus assume $u(\theta) \in C^*(M, M')$. Since M is a factor, the canonical map

$$\sum_{i=1}^{n} a_i b_i \longrightarrow \sum_{i=1}^{n} a_i \otimes b_i, \quad a_i \in M,\ b_i \in M',\ i=1, 2, \dots n$$

from the dense subalgebra of $C^*(M, M')$ to the algebraic tensor product $M \odot M'$ may be extended to a homomorphism π from $C^*(M, M')$ onto the minimal C*-tensor product $M \otimes_{\min} M'$ (cf. [**5**]). The action Ad $u(\theta)$ induces both an automorphism of M and that of M'. Hence Ad $\pi(u(\theta))$ is of the from $\alpha \otimes \beta$ as an automorphism of $M \otimes_{\min} M'$, where α (resp. β) is an automorphosm of M (resp. M') induced by Ad $\pi(u(\theta))$. Now, by [**6**, Theorem 1], the automorphism α is inner, but it is nothing but the original automorphism θ. This completes the proof.

Noticing $C^*(M, M')=C^*(M, \mathrm{Int}(M))$, the previous proposition means the fact that the condition $\mathrm{Cnt}(M) \supsetneqq \mathrm{Int}(M)$ is equivalent to $C^*(M, \mathrm{Cnt}(M)) \supsetneqq C^*(M, M')$. With this result, we know that in many examples of factors of type II_1 having property Γ the

C*-algebra $C^*(M, \mathrm{Cnt}(M))$ is exactly larger than $C^*(M, M')$. For instance, let $R(F_2)$ be the left group von Neumann algebra constructed by the free group on 2 generators F_2 and consider an outer automorphism θ on $R(F_2)$ exchanging the 2 generators. Since $R(F_2)$ is without property Γ, θ is centrally trivial (cf. [3]). Put $M = R(F_2)\bar{\otimes} R_0$ where R_0 is the hyperfinite factor of type II_1. Then M has property Γ. On the other hand, the automorphism $\theta \otimes \mathrm{id}$ of M is in $\mathrm{Cnt}(M)$ but it is not in $\mathrm{Int}(M)$. This shows that $C^*(M, \mathrm{Cnt}(M)) \supsetneqq C^*(M, M')$.

On the contrary, in case of the hyperfinite factor of type II_1, R_0, it is well known that $\mathrm{Cnt}(R_0) = \mathrm{Int}(R_0)$, so that $C^*(R_0, \mathrm{Cnt}(R_0)) = C^*(R_0, R_0')$, and R_0 has property Γ.

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