A Note on a Linear Automorphism of R^n with the Pseudo-Orbit Tracing Property

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1. Introduction

A. Morimoto proved in [1] (Proposition 1) that for any linear automorphism f of $\boldsymbol{R}^n$, f is hyperbolic if and only if f has the pseudo-orbit tracing property (P.O.T.P.). To show that if f is not hyperbolic then f does not have the P.O.T.P., for $\delta>0$ he constructed the δ-pseudo orbit (δ-p.o.) for which there are no tracing points. But the sequence of points that he constructed is not δ-p.o. for $n\geq 2$.

To supply this gap, we show in this paper that if f has the P.O.T.P. then f is hyperbolic.

2. Definition and lemmas

Let $f: X\to X$ be a homeomorphism of a metric space (X, d). We denote by $H(X)$ the group of all homeomorphisms of X.

DEFINITION. A sequence of points $\{x_n\}_{n\in \boldsymbol{Z}}$ is called a *δ-pseudo-orbit* (δ-p.o.) of f if $d(f(x_n), x_{n+1})<\delta$ for $n\in \boldsymbol{Z}$. $\{x_n\}_{n\in \boldsymbol{Z}}$ is called to be *ε-traced* by $y\in X$ (with respect to f) if $d(f^n(y), x_n)<\varepsilon$ for $n\in \boldsymbol{Z}$. This y is called an *ε-tracing point*.

We say that f has the *pseudo-orbit tracing property* (P.O.T.P.) if for each $\varepsilon>0$ there exists $\delta>0$ such that any δ-p.o. of f can be ε-traced by some point $y\in X$.

We shall use the following lemmas given in [1] (or [2]).

LEMMA 1. *Let $h\in H(X)$ be a homeomorphism of X such that h and h^{-1} are both uniformly continuous. Take $f\in H(X)$ and put $g=h\circ f\circ h^{-1}$. Then f has the P.O.T.P. if and only if g has the P.O.T.P.*

LEMMA 2. *Let (X, d) and (X', d') be metric spaces, and let $f\in H(X)$ and $g\in H(X')$. The direct product $X\times X'$ is a metric space by the distance function $d''((x, x'), (y, y'))=Max\{d(x, y), d'(x', y')\}$ for $x, y\in X$ and $x', y'\in X'$. Put $(f\times g)(x, x')=(f(x), g(x'))$ for*

$(x, x') \in X \times X'$. Then $f \times g$ has the P.O.T.P. if and only if f and g have the P.O.T.P.

LEMMA 3. *If d and d' are uniformly equivalent metrics on X then $f \in H(X)$ has the P.O.T.P. with respect to d if and only if f has the P.O.T.P. with respect to d'.*

3. Theorem

Our object is to complete the proof of (2)→(1) in the following theorem.

THEOREM. *Let f: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ be a linear automorphism of $\boldsymbol{R}^n$, and let d be the Euclidean metric on $\boldsymbol{R}^n$. Then the following conditions are equivalent:*

(1) *f is hyperbolic, i.e., if λ is an eigenvalue of f, then $|\lambda| \neq 1$; and*

(2) *f has the P.O.T.P. with respect to d.*

PROOF OF (2)→(1). Since a linear map is uniformly continuous, it follows from Lemmas 1, 2, and 3 that f has the P.O.T.P. with respect to d if and only if every factor of the real canonical form of f has the P.O.T.P. with respect to the corresponding Euclidean metric. Let λ be an eigenvalue of f. If $\lambda \in \boldsymbol{R}$, then we put a linear automorphism g: $\boldsymbol{R}^m \to \boldsymbol{R}^m$ by

$$g = \begin{pmatrix} \lambda & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda \end{pmatrix}.$$

If $\lambda \notin \boldsymbol{R}$, that is, $\lambda = r(\cos\theta + i\sin\theta)$ $(r > 0,\ 0 < |\theta| < \pi)$, then we put a linear automorphism g: $\boldsymbol{R}^m \to \boldsymbol{R}^m$ by

$$g = \begin{pmatrix} \boldsymbol{D}_{r,\theta} & \boldsymbol{I} & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & \boldsymbol{I} \\ \mathbf{0} & & & \boldsymbol{D}_{r,\theta} \end{pmatrix},$$

where $\boldsymbol{D}_{r,\theta} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$ and $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then it suffices to show that $|\lambda| \neq 1$ if the above g has the P.O.T.P. with respect to the Euclidean metric (also denoted by d) on $\boldsymbol{R}^m$. Suppose that g has the P.O.T.P. with respect to d on $\boldsymbol{R}^m$ and $|\lambda| = 1$. Let $\varepsilon > 0$ be small enough. For this ε, choose $\delta > 0$ as in the definition of the P.O.T.P.

(a) In case of

$$g = \begin{pmatrix} \lambda & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda \end{pmatrix}, \text{ where } \lambda = \pm 1.$$

Put $x^0 = (1, 0, \dots, 0)$, and $x^n = g(x^{n-1}) + \lambda^n b$ for $n \geq 1$, where $b = (0, \dots, 0, \delta/2)$. Since for any $n \geq 0$

$$d(g(x^n), x^{n+1}) = |\lambda|^{n+1} \cdot \delta/2 = \delta/2,$$

$\{x^n\}_{n \geqq 0}$ is a δ-p.o. on $\boldsymbol{R}^m$. Let $v=(v_1, \ldots, v_m) \in \boldsymbol{R}^m$ be an ε-tracing point of $\{x^n\}_{n \geqq 0}$. However,

$$d(g^n(v), x^n) \geq |(g^n(v))_m - (x^n)_m| = |\lambda^n v_m - n \cdot \delta/2| \to \infty \qquad \text{as } n \to \infty.$$

This contradicts the definition of v.

(b) In case of

$$g = \begin{pmatrix} \boldsymbol{D}_{1,\theta} & \boldsymbol{I} & & \boldsymbol{0} \\ & \ddots & \ddots & \\ & & \ddots & \boldsymbol{I} \\ \boldsymbol{0} & & & \boldsymbol{D}_{1,\theta} \end{pmatrix}, \text{ where } 0 < |\theta| < \pi.$$

Take $M \in \boldsymbol{N}$ with $\dfrac{\pi}{M} < \text{Arccos}\left(1 - \dfrac{1}{2} \cdot \delta^2\right)$, and put $\theta' = \dfrac{\pi}{M}$. We put a linear automorphism $g' : \boldsymbol{R}^m \to \boldsymbol{R}^m$ by

$$g' = \left(\begin{array}{ccc:c} 1 & & \boldsymbol{0} & \\ & \ddots & & \boldsymbol{0} \\ \boldsymbol{0} & & 1 & \\ \hdashline & \boldsymbol{0} & & \boldsymbol{D}_{1,\theta'} \end{array}\right).$$

Put $x^0 = (0, \ldots, 0, 1, 0)$, and $x^n = g' \circ g(x^{n-1})$ for $n \geq 1$. Since for any $n \geq 0$

$$d(g(x^n), x^{n+1}) = d(g(x^n), g' \circ g(x^n)) = \{2 - 2\cos\theta'\}^{1/2}$$

$$< \left\{2 - 2\left(1 - \frac{1}{2} \cdot \delta^2\right)\right\}^{1/2} = \delta,$$

$\{x^n\}_{n \geqq 0}$ is a δ-p.o. on $\boldsymbol{R}^m$. Let $v \in \boldsymbol{R}^m$ be an ε-tracing point of $\{x^n\}_{n \geqq 0}$. Then, noticing the (m−1)-th and the m-th coordinates, $d(g^M(v), x^M) > \varepsilon$. This contradicts the definition of v. This completes the proof of (2) → (1).

References

[1] A. Morimoto, *Some stabilities of group automorphisms*, Progress in Math., 14 (1981), 283–299.

[2] A. Morimoto, The method of the pseudo-orbit tracing property and stability, Tokyo Univ. Seminar Notes, 39 (1979) (in Japanese).

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