Certain Invariant Subspace Structure of Analytic Crossed Products II

By
Tomomi KOMINATO and Kichi-Suke SAITO*

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1. Introduction

Let M be a finite von Neumann algebra on a separable Hilbert space H. Let α be a $*$-automorphism of M. Suppose that there is an α-invariant faithful normal semi-finite trace ϕ of M. Let $\mathfrak{L}_+$ be an analytic crossed product on $\boldsymbol{L}^2$ determined by M and α (see the definition to § 2). We have an interest in the invariant subspace structure of $\boldsymbol{L}^2$ with respect to $\mathfrak{L}_+$. In [**3**, **5**], McAsey introduced the notion of canonical models for invariant subspaces of $\boldsymbol{L}^2$. That is, a family of left-pure, left-full, left-invairant subspaces $\{\mathfrak{M}_i\}_{i\in I}$ constitutes a complete set of canonical models for all invariant subspaces of $\boldsymbol{L}^2$ in case (a) for no two distinct indices i and j, $P_{\mathfrak{M}_i}$ is unitary equivalent to $P_{\mathfrak{M}_j}$ by a unitary operator in $\mathfrak{R}(=\mathfrak{L}')$; and (b) for every left-pure, left-invariant subspace of $\boldsymbol{L}^2$, there is an i in I and a partial isometry V in $\mathfrak{R}$ such that $VP_{\mathfrak{M}_i}V^*=P_{\mathfrak{M}}$, so that $\mathfrak{M}=V\mathfrak{M}_i$. McAsey found the canonical model in case that $M=\ell^\infty(X)$, where X is a finite set with elements $t_0, t_1, \dots, t_{K-1}$, and the automorphism of M induced by a permutation of X. Further, in [**9**, **18**, **19**], we studied the canonical models of invariant subspaces in case that ϕ is a finite trace. On the other hand, in [**2**], we studied the canonical models when $M=L^\infty(X, \mu)$, $\mu(X)=\infty$ and α is an ergodic automorphism of M. That is, we constructed a left-pure, left-full, left-invariant subspace $\mathfrak{M}_\infty$ of $\boldsymbol{L}^2$ with the multiplicity function m of $\mathfrak{M}_\infty$ which is $m(t)=\infty$ for almost all t in X. Thus, for every left-pure, left-invariant subspace $\mathfrak{M}$ of $\boldsymbol{L}^2$, there exists a partial isometry V in $\mathfrak{R}(=\mathfrak{L}')$ such that $\mathfrak{M}=V\mathfrak{M}_\infty$. That is, the canonical model in this case is the singletone $\{\mathfrak{M}_\infty\}$.

Our aim in this note is to extend the results in [**2**]. That is, suppose that $\phi(1)=\infty$ and that α is ergodic on the center Z of M. Then we will construct a left-pure, left-full, left-invariant subspace $\mathfrak{M}_\infty$ of $\boldsymbol{L}^2$ with the multiplicity function m of $\mathfrak{M}_\infty$ which is $m(t)=\infty$ for almost everywhere t in X. Therefore, we prove that for every left-pure, left-invariant subspace $\mathfrak{M}$ of $\boldsymbol{L}^2$, there exists a partial isometry V in $\mathfrak{R}(=\mathfrak{L}')$ such that $\mathfrak{M}=V\mathfrak{M}_\infty$.

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2. Preliminaries

Let M be a finite von Neumann algebra on a separable Hilbert space H. Let α be a $*$-automorphism of M such that α is ergodic on the center Z of M. Suppose that there is an α-invariant faithful normal semifinite trace of M that is not finite i.e. $\phi(1)=\infty$. Let $L^2(M, \phi)$ be the noncommutative L^2-space associated with M and ϕ. Let ℓ_x (resp. r_x) be the left (resp. right) multiplication on $L^2(M, \phi)$; $\ell_x y=xy$ (resp. $r_x y=yx$). Put $\mathfrak{L}(M)=\{\ell_x: x\in M\}$ and $r(M)=\{r_x: x\in M\}$. Since ϕ is α-invariant, there is a unitary operator u on $L^2(M, \phi)$ induced by α. To construct a crossed product, we consider the Hilbert space $\boldsymbol{L}^2$ defined by

$$\{f: \boldsymbol{Z}\to L^2(M, \phi) \mid \sum \|f(n)\|_2^2<\infty\},$$

where $\|\cdot\|_2$ is the norm of $L^2(M, \phi)$. For $x\in M$, we define operators L_x, R_x, L_δ and R_δ on $\boldsymbol{L}^2$ by the formulae

$$(L_x f)(n)=\ell_x f(n),$$

$$(R_x f)(n)=r_{\alpha^n(x)} f(n),$$

$$(L_\delta f)(n)=uf(n-1)$$

and

$$(R_\delta f)(n)=f(n-1).$$

Put $L(M)=\{L_x: x\in M\}$ and $R(M)=\{R_x: x\in M\}$. We set $\mathfrak{L}=\{L(M), L_\delta\}''$ and $\mathfrak{R}=\{R(M), R_\delta\}''$ and define the left (resp. right) analytic crossed product $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) to be the σ-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by $L(M)$ (resp. $R(M)$) and L_δ (resp. R_δ). The automorphism group $\{\beta_t\}_{t\in\boldsymbol{R}}$ of $\mathfrak{L}$ dual to α is implemented by the unitary representation of $\boldsymbol{R}$, $\{W_t\}_{t\in\boldsymbol{R}}$, defined by the formula, $(W_t f)(n)=e^{2\pi int}f(n)$, $f\in\boldsymbol{L}^2$; that is, $\beta_t(T)=W_t T W_t^*$, $T\in\mathfrak{L}$, by the definition. Let $\boldsymbol{E}_n$ be the projection on $\boldsymbol{L}^2$ defined by the formula

$$(\boldsymbol{E}_n f)(k)=\begin{cases} f(n), & k=n, \\ 0, & k\neq n. \end{cases}$$

Definition 2.1. Let $\mathfrak{M}$ be a closed subspace of $\boldsymbol{L}^2$. We shall say that $\mathfrak{M}$ is: left-invarant, if $\mathfrak{L}_+\mathfrak{M}\subset\mathfrak{M}$; left-reducing, if $\mathfrak{L}\mathfrak{M}\subset\mathfrak{M}$; left-pure, if $\mathfrak{M}$ contains no non-trivial left-reducing subspace containing $\mathfrak{M}$ is all of $\boldsymbol{L}^2$. The right-hand versions of these concepts are defined similarly, and a closed subspace which is both left- and right-invariant will be called two-sided invariant.

We write Z for $M\cap M'$ and identify it $L^\infty(X, \mu)$ for some locally compact Hausdorff space X with a σ-finite measure μ ($\mu(X)=\infty$) such that

$$\int_X f d\mu = \phi(f), \qquad f \in L^\infty(X, \mu).$$

Since α is ergodic on Z and $\phi\circ\alpha=\phi$, there exists an invertible measure-preserving ergodic transformation τ on X such that $\alpha(f)(t)=f(\tau^{-1}t)$, $f\in L^\infty(X, \mu)$, $t\in X$.

At first, we consider a direct integral of M with respect to Z according to [**1**]. By [**1**, Part II, Chapter 6, Theorems 1 and 2], there exists a μ-measurable field $t\to H(t)$ of non-zero complex Hilbert spaces over X, a μ-measurable field $t\to M(t)$ of factors in the $H(t)$'s and an isomorphism of H onto $\int^\oplus H(t)\,d\mu(t)$ which transforms M into $\int^\oplus M(t)d\mu(t)$. Therefore, we identify H, M and Z with $\int^\oplus H(t)d\mu(t)$, $\int^\oplus M(t)d\mu(t)$ and the space of diagonal operators, respectively. By [**1**, Part II, Chapter 5, Corollary of Theorem 2], there exists a μ-measurable field $t\to\phi_t$ of faithful, normal finite traces on $M(t)_+$'s such that $\phi=\int^\oplus\phi_t\,d\mu(t)$. Let $L^2(M, \phi_t)$ be the non-commutative L^2-space associated with $M(t)$ and ϕ_t. Then the field $t\to L^2(M(t), \phi_t)$ of complex Hilbert spaces over X is μ-measurable and $L^2(M, \phi)=\int^\oplus L^2(M(t), \phi_t)d\mu(t)$. Further, by [**1**, Part II, Chapter 4, Definition 1], the field $t\to M(t)$ of achieved Hilbert algebras over X in $\int^\oplus L^2(M(t), \phi_t)d\mu(t)$ is μ-measurable. Let $\ell_{x(t)}$ (resp. $r_{x(t)}$) be the left (resp. right) multiplication on $L^2(M(t), \phi_t)$ and put $\ell(M(t))=\{\ell_{x(t)}: x(t)\in M(t)\}$ (resp. $r(M(t))=\{r_{x(t)}: x(t)\in M(t)\}$). Then the field $t\to \ell(M(t))$ (resp. $t\to r(M(t))$) of factors over X is μ-measurable and $\ell(M)=\int^\oplus\ell(M(t))d\mu(t)$ (resp. $r(M)=\int^\oplus r(M(t))d\mu(t)$). Next we define the Hilbert space $\boldsymbol{L}^2{}_t$ by

$$\boldsymbol{L}^2{}_t=\{f_t\colon \boldsymbol{Z}\to L^2(M(t), \phi_t) \mid \sum_{n\in\boldsymbol{Z}}\|f_t(n)\|_2^2<\infty\}$$

and define the operators $L_{x(t)}$ on $\boldsymbol{L}^2{}_t$ by $(L_{x(t)}f_t)(n)=\ell_{x(t)}f_t(n)$. Then the field $t\to\boldsymbol{L}^2{}_t$ of complex Hilbert spaces over X is μ-measurable and $\boldsymbol{L}^2=\int^\oplus\boldsymbol{L}^2{}_t\,d\mu(t)$ and the field $t\to L(M(t))$ of factors over X is μ-measurable and $L(M)=\int^\oplus L(M(t)d\mu(t)$. Therefore, by [**1**, Part II, Chapter 3, Theorem 4], the field $t\to L(M(t))'$ of semi-finite factors over X is μ-measurable and $L(M)'=\int^\oplus L(M(t))'\,d\mu(t)$. By the definition of L_x (resp. $L_{x(t)}$), we may identify $L(M)$ (resp. $L(M(t))$) with the von Neumann algebra tensor product $\boldsymbol{C}_{\ell^2(\boldsymbol{Z})}\otimes\ell(M)$ (resp. $\boldsymbol{C}_{\ell^2(\boldsymbol{Z})}\otimes\ell(M(t))$), where $\boldsymbol{C}_{\ell^2(\boldsymbol{Z})}$ denotes the algebras of scalar multiples of the identity acting on $\ell^2(\boldsymbol{Z})$. From this, we can identify the commutant of $L(M)$: $L(M)'=(\boldsymbol{C}_{\ell^2(\boldsymbol{Z})}\otimes\ell(M))'=\boldsymbol{C}_{\ell^2(\boldsymbol{Z})}'\otimes\ell(M)'=B(\ell^2(\boldsymbol{Z}))\otimes r(M)$, where $B(\ell^2(\boldsymbol{Z}))$ is the full algebra of operators on $\ell^2(\boldsymbol{Z})$. Analogously, we can identify the commutant of $L(M(t))$: $L(M(t))'=B(\ell^2(\boldsymbol{Z}))\otimes r(M(t))$. Then we have $L(M)'=\int^\oplus B(\ell^2(\boldsymbol{Z}))\otimes r(M(t))d\mu(t)$. Put $\tilde{\phi}(r_x)=\phi(x)$ (resp. $\tilde{\phi}_t(r_{x(t)})=\phi_t(x(t))$). Let $Tr\otimes\tilde{\phi}$ (resp. $Tr\otimes\tilde{\phi}_t$) be the tensor product of Tr and $\tilde{\phi}$ (resp. $\tilde{\phi}_t$) on $B(\ell^2(\boldsymbol{Z}))\otimes r(M)$ (resp. $B(\ell^2(\boldsymbol{Z}))\otimes r(M(t))$), where Tr is the canonical trace on $B(\ell^2(\boldsymbol{Z}))$. Then $t\to Tr\otimes\tilde{\phi}_t$ is a μ-measurable field of faithful normal semi-finite traces over X and $Tr\otimes\tilde{\phi}=\int^\oplus Tr\otimes\tilde{\phi}_t d\mu(t)$. By [**9**, Lemma 2.3], $\boldsymbol{E}_0L(M)'\boldsymbol{E}_0$ is unitarily isomorphic to $r(M)$ and so, in particular, $\boldsymbol{E}_0$ is a finite projection in $L(M)'$.

Let $\boldsymbol{E}_n(t)$ be the projection on $\boldsymbol{L}^2{}_t$ defined by the formula

$$(\boldsymbol{E}_n(t)f_t)(k)=\begin{cases} f_t(n), & k=n,\\ 0, & k\neq n.\end{cases}$$

Then $t \to \boldsymbol{E}_0(t)$ is a μ-measurable field of projections over X and $\boldsymbol{E}_0 = \int^{\oplus} \boldsymbol{E}_0(t) d\mu(t)$. By [1, Part II, Chapter 5, Theorem 2], $\boldsymbol{E}_0(t)$ is a finite projection for almost everywhere t in X. Since M is finite, $M(t)$ is a finite factor almost everywhere and so ϕ_t is a faithful normal finite trace on $M(t)$. Hence we have $(Tr \otimes \widetilde{\phi_t})(\boldsymbol{E}_0(t)) = \phi_t(1) < \infty$ *a.e.* Therefore we put

$$\Psi_t = \frac{1}{(Tr \otimes \widetilde{\phi_t})(\boldsymbol{E}_0(t))} Tr \otimes \widetilde{\phi_t}.$$

Then $t \to \Psi_t$ is a μ-measurable field of faithful normal semifinite traces on $M(t)$ such that $\Psi_t(\boldsymbol{E}_0(t)) = 1$ and we put $\Psi = \int^{\oplus} \Psi_t d\mu(t)$.

Next we will define an $L(Z)$-trace following ([1, Chapter III, § 4]). Since the algebra $L(Z)$ is $*$-isomorphic to the algebra $L^{\infty}(X, \mu)$, we define $\mathfrak{Z}$ to be the set of nonnegative measurable functions, finite or not, on X. For every $T = \int^{\oplus} T(t) d\mu(t) \in L(M)'$, let $\phi(T)$ be the function $t \to \Psi_t(T(t))$ which is an element in $\mathfrak{Z}$. By [1, Part III, Chapter 4, Exercise 4], ϕ is a faithful normal semifinite $L(Z)$-trace on $L(M)'_+$ such that $\phi(\boldsymbol{E}_0) = \mathrm{I}$. The $L(Z)$-trace Φ induces a map ρ from $(\boldsymbol{E}_0 L(M)' \boldsymbol{E}_0)_+$ into $(L(Z)\boldsymbol{E}_0)_+$, by $\rho(T) = \boldsymbol{E}_0 \Phi(T)$, $T \in (\boldsymbol{E}_0 L(M)' \boldsymbol{E}_0)_+$. Then ρ is a faithful normal finite center valued trace on $(\boldsymbol{E}_0 L(M)' \boldsymbol{E}_0)_+$.

Lemma 2.2. *For each* $c \in (L(Z)\boldsymbol{E}_0)_+$, $\rho(c) = c$.

Proof. Let $c \in (L(Z)\boldsymbol{E}_0)_+$. Then there exists an element $c_1 \in L(Z)$ such that $c = c_1 \boldsymbol{E}_0$. Hence we have

$$\rho(c) = \boldsymbol{E}_0 \Phi(c) = \boldsymbol{E}_0 \Phi(c_1 \boldsymbol{E}_0) = c_1 \boldsymbol{E}_0 \Phi(\boldsymbol{E}_0) = c_1 \boldsymbol{E}_0 I = c.$$

This completes the proof.

Hence we define a multiplicity function of a left-invariant subspace of $\boldsymbol{L}^2$ as in [9]. Let $\mathfrak{M}$ be a left-pure, left-invariant subspace with the wandering subspace $\mathfrak{F} = \mathfrak{M} \ominus L_\delta \mathfrak{M}$. We denote the projection of $\boldsymbol{L}^2$ onto $\mathfrak{F}$ by $P(\mathfrak{M})$. By [6. Proposition 3.1], we know that the projection $P(\mathfrak{M})$ lies in $L(M)'$. By the preceding discussions, we may write $P(\mathfrak{M}) = \int^{\oplus} P(t) d\mu(t)$, where $P(t)$ is a projection in $B(\ell^2(\boldsymbol{Z})) \otimes r(M(t))$ for almost all t. The multiplicity function of $\mathfrak{M}$ is the function defined by the equation $m(t) = \Psi_t(P(t))$. Since the field $t \to P(t)$ of projections is μ-measurable, m is a non-negative measurable function over X. By the definition of Φ, it is clear that $\Phi(P(\mathfrak{M}))(t) = \Psi_t(P(t))$. Therefore we have the following theorem as in [4, Theorem 3.4] and [9, Theorem 3.1].

Theorem 2.3. For $i = 1, 2$, *let* $\mathfrak{M}_i$ *be a left-pure, left-invariant subspace of* $\boldsymbol{L}^2$ *with a multiplicity function* m_i. *Let* $P(\mathfrak{M}_i)$ *be the projection of* $\boldsymbol{L}^2$ *onto* $\mathfrak{F}_i = \mathfrak{M}_i \ominus L_\delta \mathfrak{M}_i$. *Then the following assertions are equivalent:*

(1) *there exists a partial isometry* V *in* $\mathfrak{R}$ *such that* $P_{\mathfrak{M}_1} = V P_{\mathfrak{M}_2} V^*$, *where* $P_{\mathfrak{M}_i}$ *is the projection of* $\boldsymbol{L}^2$ *onto* $\mathfrak{M}_i$;

(2) $m_1(t) \leq m_2(t)$, *a.e.*;

(3) $\Phi(P(\mathfrak{M}_1)) \leq \Phi(P(\mathfrak{M}_2))$; *and*

(4) $P(\mathfrak{M}_1) \precsim P(\mathfrak{M}_2)$ *in* $L(M)'$.

Furthermore, if the condition (1) *is satisfied, then* $\mathfrak{M}_1 = V\mathfrak{M}_2$.

PROOF. (1) → (2) and (4) → (1) are clear from [**9**, Theorem 3. 1]. (2) → (3) is clear from [**1**, Part III, Chapter 4, Exercise 4]. (2) → (4). Since $m_1(t) \leq m_2(t) < \infty$, by [**1**, Part III, Chapter 2, Proposition 13], $P_1(t) \precsim P_2(t)$. Suppose that $m_1(t) < m_2(t) = \infty$. Since $B(\ell^2(\boldsymbol{Z})) \otimes r(M(t))$ is a factor, $P_1(t) \precsim P_2(t)$. Finally, if $m_1(t) = m_2(t) = \infty$, then $P_1(t)$ and $P_2(t)$ are infinite projections. By [**1**, Part III, Chapter 8, Corollary 5], $P_1(t) \sim P_2(t)$. Thus $P_1(t) \precsim P_2(t)$ *a.e.* $t \in X$. Therefore, by [**1**, Part III, Chapter 1, Exercise 15], $P(\mathfrak{M}_1) \precsim P(\mathfrak{M}_2)$. This completes the proof.

3. Invariant subspace structure

Keep the notations and the assumptions in § 2. Our aim in this section is to construct a left-pure left-full left-invariant subspace of $\boldsymbol{L}^2$ such that the multiplicity function $m(t) = \infty$ for almost everywhere t in X. To do this, we need some lemmas.

LEMMA 3. 1 (cf. [**18**, Lemma 3. 1]). *Let* $\{\mathfrak{M}_i\}_{i \in I}$ *be a finite or countable collection of left-pure, left-invariant subspace of* $\boldsymbol{L}^2$ *such that* $\mathfrak{M}_i$ *is orthogonal to* $\mathfrak{M}_j$, *for* $i \neq j$. *Then* $\mathfrak{M} = \sum_{i \in I} \oplus \mathfrak{M}_i$ *is a left-pure, left-invariant subspace with the multiplicity function* $\sum_{i \in I} m_i(t)$, *where* m_i *is the multiplicity function of* $\mathfrak{M}_i$.

Let χ_F be a characteristic function of a measurable subset F in X. We define a projection P_F in $L(M)'$ by

$$(P_F f)(n) = \begin{cases} \ell_{\chi_F} f(0), & n = 0, \\ 0, & n \neq 0, \end{cases} \qquad f \in \boldsymbol{L}^2.$$

Thus it is clear that $P_F = L_{\chi_F} \boldsymbol{E}_0 \in L(M)'$. By Lemma 2. 2,

$$\Phi(P_F) = \Phi(L_{\chi_F} \boldsymbol{E}_0) = L_{\chi_F} \Phi(\boldsymbol{E}_0) = L_{\chi_F} I = L_{\chi_F}.$$

Since $P_F \leq \boldsymbol{E}_0$, $\{L_\delta{}^n P_F L_\delta{}^{*n}\}_{n \in \boldsymbol{Z}}$ is mutually orthogonal. Thus, we define a closed subspace $\mathfrak{M}(P_F) = \sum_{n \in \boldsymbol{Z}} \oplus (L_\delta{}^n P_F L_\delta{}^{*n}\} \boldsymbol{L}^2$. As in [**18**, Lemma 3. 2] and [**9**, Lemma 5. 1], we have

LEMMA 3. 2. (i) *$\mathfrak{M}(P_F)$ is a left-pure, left-invariant subspace of $\boldsymbol{L}^2$ with the multiplicity function* χ_F.

(ii) *If* $\mu(F) < \infty$, *then* $\mathfrak{M}(P_F)$ *is the closed linear span of* $\{\mathfrak{L}_+ e_0\}$, *where* $e_0(n) = 0$ *if* $n \neq 0$ *and* $e_0(0) = \chi_F$.

PROOF. (i) It is clear that $\mathfrak{M}(F)$ is a left-pure, left-invariant subspace of $\boldsymbol{L}^2$. Since $\Phi(P_F) = L_{\chi_F}$, the multiplicity function of $\mathfrak{M}(P_F)$ is χ_F.

(ii) Since $e_0(n) = \delta_{n,0}\, \chi_F$, we have

$$(\sum_{n=0}^{\infty} L_\delta{}^n P_F L_\delta{}^{*n} e_0)(k) = \sum_{n=0}^{\infty} u^n (P_F L_\delta{}^{*n} e_0)(k-n)$$
$$= u^k \mathfrak{L}_{\chi_F} (L_\delta{}^{*k} e_0)(0) = u^k \mathfrak{L}_{\chi_F} u^{*k} e_0(k) = \mathfrak{L}_{\alpha^k(\chi_F)} \delta_{k,0} \chi_F$$
$$= \chi_{\tau^k(F)} \delta_{k,0} \chi_F = \delta_{k,0} \chi_F = e_0(k).$$

Thus, $e_0 \in \mathfrak{M}(F)$ and so $[\mathfrak{L}_+ e_0]_2 \subset \mathfrak{M}(P_F)$. Conversely, for every $n \geq 0$, let $f \in L_\delta{}^n P_F L_\delta{}^{*n} \boldsymbol{L}^2$. Then we have, for all $k \in \boldsymbol{Z}$,

$$f(k) = (L_\delta{}^n P_F L_\delta{}^{*n} f)(k) = u^n (P_F L_\delta{}^{*n} f)(k-n)$$
$$= u^n \mathfrak{L}_{\chi_F} \delta_{k,n} (L_\delta{}^{*n} f)(k-n) = \delta_{k,n} \mathfrak{L}_{\chi_{\tau^n(F)}} f(k).$$

Since $\mathfrak{N} = \{x \in M : \phi(x^* x) < \infty\}$ is dense in $L^2(M, \phi)$, there exists a sequence $\{x_i\}_{i \in \boldsymbol{Z}}$ in $\mathfrak{N}$ such that $\|x_i - f(n)\|_2 \to 0$. Then we have

$$(L_{x_i} L_\delta{}^n e_0)(k) = x_i (L_\delta{}^n e_0)(k) = x_i u^n e_0(k-n)$$
$$= \delta_{k,n} x_i u^n \chi_F = \delta_{k,n} x_i \alpha^n(\chi_F) = \delta_{k,n} x_i \chi_{\tau^n(F)}$$
$$\to \delta_{k,n} \chi_{\tau^n(F)} f(n) = f(k).$$

This implies that $\| L_{x_i} L_\delta{}^n e_0 - f \|_2 \to 0$. Thus, $L_\delta{}^n P_F L_\delta{}^{*n} \boldsymbol{L}^2 \subset [\mathfrak{L}_+ e_0]_2$ where $[\mathfrak{L}_+ e_0]_2$ is the closure of $\mathfrak{L}_+ e_0$ in $\boldsymbol{L}^2$, and so $\mathfrak{M}(P_F) \subset [\mathfrak{L}_+ e_0]_2$. This completes the proof.

Let E and F be measurable subsets of X such that there are measurable subsets $\{E_n\}_{n=0}^{\infty}$ and $\{F_n\}_{n=0}^{\infty}$ with the following properties:

(1) $E_n \subset E$ and $F_n \subset F$, $n \geq 0$;

(2) $E_n \cap E_m = F_n \cap F_m = \phi$, $n \neq m$;

(3) $\mu(E \setminus \bigcup_{n=0}^{\infty} E_n) = \mu(F \setminus \bigcup_{n=0}^{\infty} F_n) = 0$; and

(4) $F_n = \tau^n(F_n)$, $n \geq 0$.

Then we have the following lemma.

LEMMA 3.4 ([18, Lemma 3.4]). *$U = \sum_{k=0}^{\infty} L_{\chi_{F_k}} L_\delta{}^k$ is a partial isometry in $\mathfrak{L}_+$ with initial projection L_{χ_E} and final projection L_{χ_F}.*

LEMMA 3.5. *Keep the notations as above. Suppose that $\mu(E) = \mu(F) < \infty$. Then there exists a left-pure, left-invariant subspace $\mathfrak{M}$ of $\mathfrak{M}(P_E)$ such that $\Phi(P(\mathfrak{M})) = \chi_F$ and $\sum_{n \in \boldsymbol{Z}} L_\delta{}^n P(\mathfrak{M}) L_\delta{}^{*n} = R_{\chi_E}$.*

PROOF. We define a projection P in $L(M)'$ by

$$(Pf)(k) = \begin{cases} \chi_{F_k} f(k), & k \geq 0, \\ 0, \quad k < 0, & f \in \boldsymbol{L}^2. \end{cases}$$

That is, $P=\sum_{k=0}^{\infty} L_{\chi_{F_k}} \boldsymbol{E}_k$. Then it is clear that $(L_\delta^m PL_\delta^{*m})(L_\delta^n PL_\delta^{*n})=0$, for $n, m\in\boldsymbol{Z}$, $n\neq m$. This implies that P is a wandering projection in $L(M)'$. Therefore, we define a closed subspace $\mathfrak{M}$ by $\mathfrak{M}=(\sum_{n=0}^{\infty} L_\delta^n PL_\delta^{*n})\boldsymbol{L}^2$. Then it is clear that $\mathfrak{M}$ is left-pure and left-invariant. Further, $\Phi(P)=\sum_{n=0}^{\infty}\Phi(L_{\chi_{F_k}}\boldsymbol{E}_k)=\sum_{k=0}^{\infty} L_{\chi_{F_k}}\Phi(R_\delta^k\boldsymbol{E}_0 R_\delta^{*k})=\sum_{k=0}^{\infty} L_{\chi_{F_k}}\Phi(\boldsymbol{E}_0)=\sum_{k=0}^{\infty} L_{\chi_{F_k}}=L_{\chi_F}$. Thus the multiplicity function of $\mathfrak{M}$ is χ_F.

On the other hand, since $P_E=L_{\chi_F}\boldsymbol{E}_0=\sum_{k=0}^{\infty} L_{\chi_{E_k}}\boldsymbol{E}_0$, we have, for $k\geq 0$,

$$L_\delta^k L_{\chi_{E_k}}\boldsymbol{E}_0 L_\delta^{-k}=L_\delta^k L_{\chi_{E_k}} L_\delta^{-k} L_\delta^k \boldsymbol{E}_0 L_\delta^{-k}=L_{\alpha^k(\chi_{E_k})}\boldsymbol{E}_k$$
$$=L_{\chi_{\tau^k(E_k)}}\boldsymbol{E}_k=L_{\chi_{F_k}}\boldsymbol{E}_k,$$

and so

$$\sum_{n=0}^{\infty} L_\delta^n L_{\chi_{E_k}}\boldsymbol{E}_0 L_\delta^{-n}\geq\sum_{n=k}^{\infty} L_\delta^n L_{\chi_{E_k}}\boldsymbol{E}_0 L_\delta^{-n}$$
$$=\sum_{n=k}^{\infty} L_\delta^{n-k} L_\delta^k L_{\chi_{E_k}}\boldsymbol{E}_0 L_\delta^{-k} L_\delta^{-n+k}=\sum_{n=k}^{\infty} L_\delta^{n-k} L_{\chi_{F_k}}\boldsymbol{E}_k L_\delta^{-n+k}$$
$$=\sum_{n=0}^{\infty} L_\delta^n L_{\chi_{F_k}}\boldsymbol{E}_k L_\delta^{-n}.$$

Thus, $\sum_{n=0}^{\infty} L_\delta^n P_E L_\delta^{-n}\geq\sum_{n=0}^{\infty} L_\delta^n PL_\delta^{-n}$ and so $\mathfrak{M}(P_F)\supset\mathfrak{M}(P)$. Since $\sum_{n=-\infty}^{\infty} L_\delta^n L_{\chi_{E_k}}\boldsymbol{E}_0 L_\delta^{-n}=\sum_{n=-\infty}^{\infty} L_\delta^n L_{\chi_{E_k}}\boldsymbol{E}_k L_\delta^{-n}$, we have $\sum_{n=-\infty}^{\infty} L_\delta^n PL_\delta^{-n}=\sum_{n=-\infty}^{\infty} L_\delta^n P_E L_\delta^{-n}=R_{\chi_E}$. This completes the proof.

THEOREM 3.6. *Let m be a measurable function on X such that $m(t)=\infty$ for almost all $t\in X$. Then there eixsts a left-pure, left-full, left-invariant subspace $\mathfrak{M}_\infty$ of $\boldsymbol{L}^2$ such that the multiplicity function of $\mathfrak{M}_\infty$ is m.*

PROOF. Since (X, μ) is σ-finite, there exists a family $\{E_n\}_{n=1}^{\infty}$ of measurable subsets of X such that $X=\bigcup_{n=1}^{\infty} E_n$, $E_1\subset E_2\subset\cdots\subset E_n\subset\cdots$ and $\mu(E_n)<\infty$, $n\geq 1$. As in the proof of [**2**, Theorem 3.5], we can define the measurable subsets $\{F_n\}_{n=1}^{\infty}$, $\{E_n^{(k)}\}_{k=1}^{\infty}$ and $\{F_n^{(k)}\}_{k=1}^{\infty}$ with the following properties: for $n\geq 1$,

(1) $F_n=\sum_{k=0}^{\infty} F_n^{(k)}$ and $E_n=\sum_{k=0}^{\infty} E_n^{(k)}$;

(2) $E_n^{(k)}=\tau^k(F_n^{(k)})$, $k\geq 0$; and

(3) $F_n\cap F_m=\phi$, for $n\neq m$.

By Lemma 3.5, for all $n\geq 1$, there exists a left-pure, left-invariant subspace $\mathfrak{M}_n$ of $\mathfrak{M}(P_{F_n})$ such that $\Phi(P(\mathfrak{M}_n))=\chi_{E_n}$. Put $F_0=X\diagdown\bigcup_{n=1}^{\infty} F_n$. Since $\{F_n\}_{n=1}^{\infty}$ is mutually disjoint,

$\{\mathfrak{M}(P_{F_n})\}_{n=1}^{\infty}$is mutually orthogonal. Put $P=P_{F_0}+\sum_{n=1}^{\infty}P(\mathfrak{M}_n)$. Then P is a wandering projection and

$$\Phi(P)=\Phi(P_{F_0})+\sum_{n=1}^{\infty}\Phi\,(P(\mathfrak{M}_n))=\chi_{F_0}+\sum_{n=1}^{\infty}\chi_{E_n}=\chi_{F_0}+\infty \mathrm{I}\ =\infty\mathrm{I}.$$

Thus we define a left-pure, left-invariant subspace $\mathfrak{M}$ by $(\sum_{k=0}^{\infty}L_\delta{}^k PL_\delta{}^{-k})\boldsymbol{L}^2$. Further, since $\sum_{k=-\infty}^{\infty}L_\delta{}^kP(\mathfrak{M}_n)L_\delta{}^{-k}=\sum_{k=-\infty}^{\infty}L_\delta{}^kP_{F_n}L_\delta{}^{-k}$ by Lemma 3. 5,

$$\sum_{k=-\infty}^{\infty}L_\delta{}^kPL_\delta{}^{-k}=\sum_{k=-\infty}^{\infty}L_\delta{}^kP_{F_0}L_\delta{}^{-k}+\sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty}L_\delta{}^kP(\mathfrak{M}_n)L_\delta{}^{-k}$$

$$=\sum_{k=-\infty}^{\infty}L_\delta{}^kP_{F_0}L_\delta{}^{-k}+\sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty}L_\delta{}^kP_{F_n}\,L_\delta{}^{-k}$$

$$=\sum_{k=-\infty}^{\infty}L_\delta{}^kP_{F_0+\sum_{n=1}^{\infty}F_n}L_\delta{}^{-k}=\sum_{k=-\infty}^{\infty}L_\delta{}^k\boldsymbol{E}_0L_\delta{}^{-k}=\sum_{k=-\infty}^{\infty}\boldsymbol{E}_k=I.$$

This implies that $\mathfrak{M}$ is left-full. This completes the proof.

By Theorem 3. 6, we can construct a left-pure, left-full, left-invariant subspace of $\boldsymbol{L}^2$ such that $m(t)=\infty$ for almost all $t\in X$. We denote this space by $\mathfrak{M}_\infty$. Then we have the following.

THEOREM 3.7. *Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of $\boldsymbol{L}^2$. Then there exists a partial isometry V in $\mathfrak{R}$ such that $P_{\mathfrak{M}}=VP_{\mathfrak{M}_\infty}V^*$, so that $\mathfrak{M}=V\mathfrak{M}_\infty$.*

PROOF. Since $\Phi(P(\mathfrak{M}))\leq\infty \mathrm{I}$, by Theorem 3. 1, we have this theorem.

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Tomomi Kominato
Matsudai Highschool
Matsudai, Higashikubiki,
Niigata, 942–15
Japan

Kichi-Suke Saito
Department of Mathematics
Faculty of Science
Niigata University
Niigata, 950–21
Japan