Parallel submanifolds of Cayley plane

By
Kazumi TSUKADA

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1. Introduction

A submanifold M of a Riemannian manifold $\widetilde{M}$ is called to be parallel if the second fundamental form of M is parallel. Several authors have completely classified parallel submanifolds when the ambient spaces are the Euclidean space and the symmetric spaces of rank one except Cayley plane and its non-compact dual. Parallel submanifolds of the Euclidean space and the sphere have been classified by D. Ferus [2], [3], [4] and those of the real hyperbolic space by M. Takeuchi [12]. Parallel Kaehler submanifolds of the complex projective space and the complex hyperbolic space have been classified by H. Nakagawa and R. Takagi [11] and by M. Kon [6] respectively. H. Naitoh in [8], [9], and [10] has classified totally real parallel submanifolds of the complex space form and consequently has completely classified parallel submanifolds of the complex space form. Parallel submanifolds of the quaternion projective space and its non-compact dual have been classified by the author [13]. In this paper we will study parallel submanifolds of Cayley plane $P_2(\mathbf{Cay})$.

We need the classification of totally geodesic submanifolds of $P_2(\mathbf{Cay})$ to classify parallel submanifolds of $P_2(\mathbf{Cay})$. On this, the following result is obtained:

THEOREM (J. A. WOLF [14]). *Let N be a connected complete totally geodesic submanifold of Cayley plane $P_2(\boldsymbol{Cay})$ with dim $N \geqq 2$. Then N is an r-dimensional sphere S^r $(2 \leqq r \leqq 8)$, a real projective plane $P_2(\boldsymbol{R})$, a complex projective plane $P_2(\boldsymbol{C})$, or a quaternion projective plane $P_2(\boldsymbol{H})$. Moreover, if two connected complete totally geodesic submanifolds are homeomorphic, then they are equivalent under an element of $I_0(P_2(\boldsymbol{Cay}))$, where $I_0(P_2(\boldsymbol{Cay}))$ denotes the identity component of the full group of isometries of $P_2(\boldsymbol{Cay})$.*

Especially maximal totally geodesic submanifolds of $P_2(\mathbf{Cay})$ are $P_2(\boldsymbol{H})$ and S^8.

In this paper we will show the following:

THEOREM. *Let f be an immersion with parallel second fundamental form of a connected manifold M (dim $M \geqq 2$) into Cayley plane $P_2(\boldsymbol{Cay})$. Then there exists a totally geodesic submanifold $P_2(\boldsymbol{H})$ or S^8 of $P_2(\boldsymbol{Cay})$ which contains the image $f(M)$ of M by f.*

By this Theorem a parallel submanifold M of $P_2(\mathbf{Cay})$ is reduced to either of the following cases:

(a) $M \underset{f_1}{\subset} P_2(\boldsymbol{H}) \underset{f_2}{\subset} P_2(\mathbf{Cay})$,

or

(b) $M \underset{f_1}{\subset} S^8 \underset{f_2}{\subset} P_2(\mathbf{Cay})$,

where f_1 is parallel and f_2 is totally geodesic. Meanwhile parallel submanifolds of $P_2(\boldsymbol{H})$ and S^8 have already been classified. Therefore we have completely classified parallel submanifolds of $P_2(\mathbf{Cay})$.

2. Preliminaries

Let $\tilde{M}$ be an m-dimensional Riemannian manifold with the Riemannian connection $\tilde{\nabla}$ and M be an n-dimensional Riemannian manifold with the Riemannian connection ∇. We denote by $\tilde{R}$ the curvature tensor of $\tilde{\nabla}$. Les f be an isometric immersion of M into $\tilde{M}$. The metrics on the tangent bundles $T\tilde{M}$, TM are denoted by $<\ ,>$. The metric and the connection on the pull back $f^*T\tilde{M}$ induced from $<\ ,>$ and $\tilde{\nabla}$ are also denoted by $<\ ,>$ and $\tilde{\nabla}$. We have an orthogonal decomposition:

$$f^*T\tilde{M}=TM+NM,$$

where NM denotes the normal bundle of f. We denote by $\nabla^{\perp}$ the normal connection on NM induced from $\tilde{\nabla}$. Then we have Gauss-Weingarten formulas:

$$\tilde{\nabla}_X Y=\nabla_X Y+\sigma(X, Y),$$
$$\tilde{\nabla}_X \xi=-A_\xi X+\nabla_X^{\perp}\xi,$$

for vector fields X, Y on M and a normal vector field ξ. Here the tensor fields σ and A_ξ are called the second fundamental form and the shape operator respectively, which are related by $<A_\xi X, Y>=<\sigma(X, Y), \xi>$. We define a tensor $\bar{\nabla}\sigma$ by

$$\bar{\nabla}\sigma(X, Y, Z)=\nabla_X^{\perp}\sigma(Y, Z)-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z),$$

for vector fields X, Y, Z on M. The isometric immersion f is said to be *totally geodesic* if $\sigma=0$ on M, and f is said to be *parallel* if $\bar{\nabla}\sigma=0$ on M. For a point $p\in M$, put

$$N_p^1 M=\{\sigma(X, Y)\in N_pM, X, Y\in T_pM\}_{\boldsymbol{R}},$$

which is called the *first normal space*. Put $O_p^1M=T_pM+N_p^1M$, which is called the *first osculating space*. If f is parallel and M is connected, the dimensions of N_p^1M and O_p^1M are constant on M. Therefore $N^1M=\bigcup_{p\in M}N_p^1M$ and $O^1M=\bigcup_{p\in M}O_p^1M$ are subbundles of $f^*T\tilde{M}$. Moreover we have

LEMMA 2.1 (H. NAITOH [7]). *If f is parallel and $\tilde{M}$ is locally Riemannian symmetric, then the following holds:*

(a) $\tilde{R}(X, Y)Z\in T_pM$

(b) $\widetilde{R}(X, Y)\xi \in N^1_p M$

(c) $\sigma(V, \widetilde{R}(X, Y)Z)-\widetilde{R}(\sigma(V, X), Y)Z-\widetilde{R}(X, \sigma(V, Y))Z-\widetilde{R}(X, Y)\sigma(V, Z)=0$

(d) $-A_{\widetilde{R}(X, Y)\xi}(V)-\widetilde{R}(\sigma(V, X), Y)\xi-\widetilde{R}(X, \sigma(V, Y))\xi+\widetilde{R}(X, Y)A_\xi V=0,$

for $X, Y, Z, V \in T_pM$ *and* $\xi \in N^1_p M$.

If a subspace W of the tangent space $T_p\widetilde{M}$ at $p \in \widetilde{M}$ satisfies $\widetilde{R}(X, Y)Z \in W$ for $X, Y, Z \in W$, then W is called a *curvature invariant subspace*. It is well-known that for a curvature invariant subspace W at p of a Riemannian symmetric space $\widetilde{M}$, there exists a unique complete totally geodesic submanifold N of $\widetilde{M}$ such that $p \in N$, $T_pN=W$ (S. Helgason [5]). We prepare the following key lemma to prove Theorem.

LEMMA 2.2 (H. NAITOH [9]). *Let f be a parallel immersion of a connected Riemannian manifold M into a Riemannian symmetric space $\widetilde{M}$. If $O^1_p M$ is a curvature invariant subspace of $T_p\widetilde{M}$ for some point $p \in M$, then there exists a unique complete totally geodesic submanifold N of $\widetilde{M}$ such that $f(M)$ is contained in N and $T_pN=O^1_p M$.*

In fact H. Naitoh proved this when $\widetilde{M}$ is the complex space form (see Theorem 2.4 in [9]). Following his proof, we see that the statement holds whenever $\widetilde{M}$ is a Riemannian symmetric space.

3. The Cayley algebra and the curvature tensor of P_2(Cay)

The set of Cayley numbers, which is denoted by **Cay**, is an 8-dimensional vector space over the field $\boldsymbol{R}$ of real numbers with basis elements $e_0=1, e_1, \ldots, e_7$. For these basis elements a multiplication is defined as follows:

$e_ie_0=e_0e_i=e_i$, and e_ie_j $(i, j \geqq 1)$ is given by the following table.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

We extend the multiplication onto **Cay** canonically. Then **Cay** is a non-associative division algebra, which is called the *Cayley algebra*. To $a=\alpha_0 e_0+\sum_{i=1}^{7} \alpha_i e_i$, we associate the conjugate Cayley number $\bar{a}=\alpha_0 e_0-\sum_{i=1}^{7} \alpha_i e_i$. We define an inner product $\langle a, b\rangle$ by

$<a, b> = \sum_{i=0}^{7} \alpha_i \beta_i$ for $a = \sum_{i=0}^{7} \alpha_i e_i$ and $b = \sum_{i=0}^{7} \beta_i e_i$ and the norm $\|a\|$ by $\|a\| = \sqrt{<a, a>}$.

Then similarly to the quaternion algebra $\boldsymbol{H}$, we have $\overline{ab} = \bar{b}\,\bar{a}$, $\|ab\| = \|a\|\,\|b\|$, and $a\bar{b} + b\bar{a} = \bar{a}b + \bar{b}a = 2 <a, b> e_0$.

We remark that the field $\boldsymbol{R}$ of real numbers, the field $\boldsymbol{C}$ of complex numbers, and the algebra $\boldsymbol{H}$ of quaternions are canonically regarded as the subalgebras of the Cayley algebra **Cay**. In fact, the mappings $\alpha \rightarrow \alpha e_0$, $\alpha + \beta i \rightarrow \alpha e_0 + \beta e_1$, and $\alpha + \beta i + \gamma j + \delta k \rightarrow \alpha e_0 + \beta e_1 + \gamma e_2 + \delta e_3$ for $\alpha, \beta, \gamma, \delta \in \boldsymbol{R}$ are injective homomorphisms of $\boldsymbol{R}$, $\boldsymbol{C}$, and $\boldsymbol{H}$ into **Cay** respectively. In particular we identify the Cayley algebra **Cay** with pairs $\boldsymbol{H}+\boldsymbol{H}$ of quaternions as follows: To $a = \sum_{i=0}^{7} \alpha_i e_i$, we attach $[\alpha, \beta] \in \boldsymbol{H}+\boldsymbol{H}$, where $\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, $\beta = \alpha_4 - \alpha_5 i - \alpha_6 j - \alpha_7 k$. For this correspondence, the followings hold:

$$[\alpha, \beta]\,[\gamma, \delta] = [\alpha\gamma - \delta\bar{\beta},\ \bar{\alpha}\delta + \gamma\beta]$$

$$\overline{[\alpha, \beta]} = [\bar{\alpha}, -\beta]$$

$$<[\alpha, \beta], [\gamma, \delta]> = <\alpha, \gamma> + <\beta, \delta>,$$

where $\bar{\alpha}$ is the conjugate number of α in $\boldsymbol{H}$ and $<\alpha, \beta>$ is the inner product on $\boldsymbol{H}$.

Though **Cay** is not associative, the following formulas hold (cf. I. Yokota [15] p. 208): For $a, b, u, v \in$ **Cay**,

(3.1) $$<au, v> = <u, \bar{a}v>, \quad <ua, v> = <u, v\bar{a}>,$$

(3.2) $$a(\bar{a}u) = (a\bar{a})u, \quad a(u\bar{a}) = (au)\bar{a}, \quad u(a\bar{a}) = (u\bar{a})a,$$

$$a(au) = (aa)u, \quad a(ua) = (au)a, \quad u(aa) = (ua)a,$$

(3.3) $$\bar{b}(au) + \bar{a}(bu) = 2 <a, b> u = (ua)\bar{b} + (ub)\bar{a},$$

(3.4) for an orthonormal basis $e_0, a_1, \ldots, a_7$,

$$a_i(a_j u) = -a_j(a_i u) \ (i \neq j), \qquad a_i(a_i u) = -u \text{ and}$$

especially $a_i a_j = -a_j a_i$, $a_i^2 = -e_0$.

Let $\widetilde{M}$ be either Cayley plane or its non-compact dual. Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}$ is given as follows.

LENNA 3.1 (R. B. BROWN and A. GRAY [1]). *The tangent space $T_p\widetilde{M}$ at p of $\widetilde{M}$ is identified with* ***Cay+Cay****, viewed as pairs of Cayley numbers. Under this identification, the metric tensor $\tilde{g}$ at p is given by $\tilde{g}((a, b), (c, d)) = <a, c> + <b, d>$ and the curvature tensor $\widetilde{R}$ at p is given by*

(3.5) $$\begin{aligned}&\widetilde{R}((a, b), (c, d))(e, f)\\ &= \frac{k}{4}(-4<a, e>c + 4<c, e>a + (ed)\bar{b} - (eb)\bar{d} + (ad - cb)\bar{f},\\ &\bar{a}(cf) - \bar{c}(af) - 4<b, f>d + 4<d, f>b - \bar{e}(ad - cb)),\end{aligned}$$

where k is positive or negative according as $\widetilde{M}$ is Cayley plane or its non-compact dual.

We devote the rest of this section to describing curvature invariant subspaces of $\widetilde{M}$

By Lemma 3. 1, we identify the tangent space $T_p\widetilde{M}$ with **Cay**+**Cay**. We put subspaces $\mathfrak{m}_{S^r}$ and $\mathfrak{m}_{\boldsymbol{K}}$ ($\boldsymbol{K}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$) of $T_p\widetilde{M}$ as follows:

$$\mathfrak{m}_{S^r}=\{(a, 0); a\in W^r\},$$

where W^r denotes an r-dimensional subspace of **Cay** ($2\leqq r\leqq 8$) and

$$\mathfrak{m}_{\boldsymbol{K}}=\{(\alpha, \beta); \alpha, \beta\in \boldsymbol{K}\},$$

where $\boldsymbol{K}=\boldsymbol{R}, \boldsymbol{C}$, or $\boldsymbol{H}$ is regarded as the subalgebra of **Cay**. Then by (3. 5), $\mathfrak{m}_{S^r}$ and $\mathfrak{m}_{\boldsymbol{K}}$ ($\boldsymbol{H}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$) are curvature invariant subspaces. Moreover we see that the complete totally geodesic submanifolds N of $\widetilde{M}$ such that $T_pN=\mathfrak{m}_{S^r}$, $\mathfrak{m}_{\boldsymbol{R}}$, $\mathfrak{m}_{\boldsymbol{C}}$, and $\mathfrak{m}_{\boldsymbol{H}}$ are an r-dimensional sphere S^r, a real projective plane $P_2(\mathbf{R})$, a complex projective plane $P_2(\mathbf{C})$, and a quaternion projective plane $P_2(\mathbf{H})$, respectively if $\widetilde{M}$ is Cayley plane. So we call the curvature invariant subspaces $\mathfrak{m}_{S^r}$, $\mathfrak{m}_{\boldsymbol{R}}$, $\mathfrak{m}_{\boldsymbol{C}}$, and $\mathfrak{m}_{\boldsymbol{H}}$ in $T_p\widetilde{M}$ *S^r-type, $P_2(\boldsymbol{R})$-type, $P_2(\boldsymbol{C})$-type,* and *$P_2(\boldsymbol{H})$-type* respectively. Wolf's Theorem stated in section 1 implies that any curvature invariant subspace of $T_p\widetilde{M}$ is equivalent to one of $\mathfrak{m}_{S^r}$ and $\mathfrak{m}_{\boldsymbol{K}}$($\boldsymbol{K}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$) under an element of the isotropy subgroup of $I_0(\widetilde{M})$ at p.

4. Proof of Theorem

Let $\widetilde{M}$ be either Cayley plane or its non-compact dual and f be an isometric immersion with parallel second fundamental form of a connected Riemannian manifold M (dim $M\geqq 2$) into $\widetilde{M}$. If the following holds, by Lemma 2. 2 we obtain Theorem.

PROPOSITION 4. 1. *For a point $p\in M$, the tangent space T_pM and the first osculating space O^1_pM are both curvature invariant subspaces of $T_p\widetilde{M}$. Moreover one of the following cases occurs:*

(1) *T_pM is S^r-type ($3\leqq r\leqq 8$) and O^1_pM is S^n-type ($r\leqq n\leqq 8$),*

(2) *T_pM is S^2-type and O^1_pM is S^n-type ($2\leqq n\leqq 5$) or $P_2(\boldsymbol{C})$-type,*

(3) *T_pM is $P_2(\boldsymbol{H})$-type and O^1_pM is equal to T_pM,*

(4) *T_pM is $P_2(\boldsymbol{C})$-type and O^1_pM is equal to T_pM or is $P_2(\boldsymbol{H})$-type,*

(5) *T_pM is $P_2(\boldsymbol{R})$-type and O^1_pM is equal to T_pM or is $P_2(\boldsymbol{C})$-type.*

Proof of Proposition 4. 1. As usual we identify $T_p\widetilde{M}$ with **Cay**+**Cay**. By Lemma 2. 1 (a), T_pM is a curvature invariant subspace of $T_p\widetilde{M}$. Therefore it is sufficient to consider the following five cases:

Case 1: $T_pM=\mathfrak{m}_{S^r}$ ($3\leqq r\leqq 8$),

Case 2: $T_pM=\mathfrak{m}_{S^2}$,

Case 3: $T_pM=\mathfrak{m}_{\boldsymbol{H}}$, Case 4: $T_pM=\mathfrak{m}_C$,

Case 5: $T_pM=\mathfrak{m}_R$.

We determine the first osculating space O_p^1M for each case.

Case 1. $O_p^1M=\mathfrak{m}_{S^n}$ ($r\leqq n\leqq 8$).

Proof. In this case, we decompose the normal space N_pM as follows:

$$N_pM=\{(c, 0);\ c\in(W^r)^\perp\}+\{(0, d);\ d\in\mathbf{Cay}\}.$$

We denote by σ' and σ'' the components of the second fundamental form σ according to this decomposition. That is,

$$\sigma((a, 0), (b, 0))=(\sigma'((a, 0), (b, 0)), \sigma''((a, 0), (b, 0))),$$

where $a, b\in W^r$, $\sigma'((a, 0), (b, 0))\in(W^r)^\perp$, and $\sigma''((a, 0), (b, 0))\in\mathbf{Cay}$.

We shall show that σ'' vanishes. In fact applying Lemma 2.1 (c), we have

$$\begin{aligned}&(\sigma'((a, 0), \widetilde{R}((b, 0), (c, 0))(d, 0)), \sigma''((a, 0), \widetilde{R}((b, 0), (c, 0))(d, 0)))\\&-\widetilde{R}((\sigma'((a, 0), (b, 0)), \sigma''((a, 0), (b, 0))), (c, 0))(d, 0)\\&-\widetilde{R}((b, 0), (\sigma'((a, 0), (c, 0)), \sigma''((a, 0), (c, 0))))(d, 0)\\&-\widetilde{R}((b, 0), (c, 0))(\sigma'((a, 0), (d, 0)), \sigma''((a, 0), (d, 0)))=0,\end{aligned}$$

for $a, b, c, d\in W^r$.

By (3.5), we have

$$\begin{aligned}&-4\langle b, d\rangle\sigma''((a, 0), (c, 0))+4\langle c, d\rangle\sigma''((a, 0), (b, 0))\\&-\bar{d}(c\sigma''((a, 0), (b, 0)))+\bar{d}(b\sigma''((a, 0), (c, 0)))\\&-\bar{b}(c\sigma''((a, 0), (d, 0)))+\bar{c}(b\sigma''((a, 0), (d, 0)))=0\end{aligned}\tag{4.1}$$

Putting $c=a$ and $d=b$ in (4.1) for an orthonormal system $\{a, b\}$ of W^r, we get

$$\begin{aligned}&-4\sigma''((a, 0), (a, 0))-2\bar{b}(a\sigma''((a, 0), (b, 0)))\\&\quad+\bar{b}(b\sigma''((a, 0), (a, 0)))+\bar{a}(b\sigma''((a, 0), (b, 0)))=0\end{aligned}$$

and using (3.2),

$$\begin{aligned}&-3\sigma''((a, 0), (a, 0))-2\bar{b}(a\sigma''((a, 0), (b, 0)))\\&\quad+\bar{a}(b\sigma''((a, 0), (b, 0)))=0.\end{aligned}\tag{4.2}$$

Similarly we have

$$\begin{aligned}&-3\sigma''((b, 0), (b, 0))-2\bar{a}(b\sigma''((a, 0), (b, 0)))\\&\quad+\bar{b}(a\sigma''((a, 0), (b, 0)))=0.\end{aligned}\tag{4.3}$$

Adding (4.2) and (4.3), we obtain

$$
\begin{aligned}
0= & -3\{\sigma''((a, 0), (a, 0))+\sigma''((b, 0), (b, 0))\} \\
& -\{\bar{b}(a\sigma''((a, 0)(b, 0)))+\bar{a}(b\sigma''((a, 0), (b, 0)))\} \\
= & -3\{\sigma''((a, 0), (a, 0))+\sigma''((b, 0), (b, 0))\} \\
& -2\langle a, b\rangle\sigma''((a, 0), (b, 0)) \\
= & -3\{\sigma''((a, 0), (a, 0))+\sigma''((b, 0), (b, 0))\}.
\end{aligned}
$$

Hence we get

$$\sigma''((a, 0), (a, 0))+\sigma''((b, 0), (b, 0))=0. \tag{4.4}$$

For an arbitrary unit element $a\in W^r$, we take $b, c\in W^r$ such that $\{a, b, c\}$ is an orthonormal system of W^r. By (4.4), we get $\sigma''((a, 0), (a, 0))=-\sigma''((b, 0), (b, 0))=\sigma''((c, 0), (c, 0))=-\sigma''((a, 0), (a, 0))$ and hence $\sigma''((a, 0), (a, 0))=0$. Since a is arbitrary, we have $\sigma''=0$. Therefore the first normal space $N_p^1 M$ is contained in $\{(c, 0);\ c\in (W^r)^{\perp}\}$ and hence there exists an n-dimensional subspace W^n of **Cay** such that $W^r\subseteq W^n$ and $O_p^1 M=\{(a, 0);\ a\in W^n\}$.

Case 2. $O_p^1M=\mathfrak{m}_{S^n}$ $(2\leqq n\leqq 5)$ or O_p^1M is equivalent to $\mathfrak{m}_C$.

Proof. For any 2-dimensional subspace W^2 of **Cay**, $\{(a, 0);\ a\in W^2\}$ is equivalent to $\{(\alpha, 0);\ \alpha\in \boldsymbol{C}\}$ under an element of the isotropy subgroup of $I_0(\widetilde{M})$ at p. Therefore we may assume that $T_pM=\{(\alpha, 0);\ \alpha\in\boldsymbol{C}\}$. Let $\mathfrak{gl}(T_p\widetilde{M})$ be the Lie algebra of all linear endomorphisms of $T_p\widetilde{M}$. We denote by $\mathfrak{R}$ the subspace of $\mathfrak{gl}(T_p\widetilde{M})$ linearly spanned by $\widetilde{R}(X, Y)$, $X, Y\in T_pM$. Since T_pM is a curvature invariant subspace, $\mathfrak{R}$ is a Lie subalgebra of $\mathfrak{gl}(T_p\widetilde{M})$. Moreover T_pM and N_pM are invariant subspaces by the action of $\mathfrak{R}$. An irreducibly invariant subspace of N_pM by the action of $\mathfrak{R}$ is given by $\{(\alpha a, 0);\ \alpha\in\boldsymbol{R}\}$ or $\{(0, \lambda f);\ \lambda\in\boldsymbol{C}\}$, where a and f are unit elements of **Cay** and $\langle a, \boldsymbol{C}\rangle=\{0\}$. In fact using (3.5), we can easily see that these spaces are irreducible. Suppose that V is an irreducibly invariant subspace of N_pM by the action of $\mathfrak{R}$ and V has an element (a, f) such that $\langle a, \boldsymbol{C}\rangle=\{0\}$ and $f\neq 0$. Since $\widetilde{R}((e_1, 0), (e_0, 0))(a, f)=\frac{k}{4}(0, -2e_1f)$ and $\widetilde{R}((e_1, 0), (e_0, 0))(0, e_1f)=\frac{k}{4}(0, 2f)$, the subspace $\{(0, \lambda f);\ \lambda\in\boldsymbol{C}\}$ is contained in V. Therefore V has to coincide with $\{(0, \lambda f);\ \lambda\in\boldsymbol{C}\}$.

By Lemm 2.1 (b), the first normal space N_p^1M is an invariant subspace of N_pM by the action of $\mathfrak{R}$. Since $\dim N_p^1M\leqq 3$, the following three cases may occur:

(i) $N_p^1M=\{(a, 0);\ a\in W^t\}$, where W^t $(0\leqq t\leqq 3)$ is a t-dimensional subspace of **Cay** such that $\langle W^t, \boldsymbol{C}\rangle=\{0\}$,

(ii) $N_p^1M=\{(0, \lambda f);\ \lambda\in\boldsymbol{C}\}$, where f is a unit element of **Cay**,

(iii) $N_p^1M=\{(0, \lambda f);\ \lambda\in\boldsymbol{C}\}+\{(\alpha d, 0);\ \alpha\in\boldsymbol{R}\}$, where f and d are unit elements of **Cay** and $\langle d, \boldsymbol{C}\rangle=\{0\}$.

(i) In this case, we have $O_p^1M=\mathfrak{m}_{S^{2+t}}$ clearly.

(ii) In this case, we can easily check that $O_p^1 M$ is a curvature invariant subspace and is equivalent to $\mathfrak{m}_C$.

(iii) This case does not occur.

We denote by σ_0 the $(d, 0)$-component of the second fundamental form σ. Applying Lemma 2.1 (d), we have

$$\widetilde{R}(\sigma((e_0, 0), (e_0, 0)), (e_1, 0))(0, f)$$
$$+\widetilde{R}((e_0, 0), \sigma((e_0, 0), (e_1, 0)))(0, f) \in T_pM.$$

Since $\widetilde{R}((\alpha, 0), (0, \lambda f))(0, f) \in T_pM$ for $\alpha, \lambda \in \boldsymbol{C}$,

$$\sigma_0((e_0, 0), (e_0, 0))\widetilde{R}((d, 0), (e_1, 0))(0, f)$$
$$+\sigma_0((e_0, 0), (e_1, 0))\widetilde{R}((e_0, 0), (d, 0))(0, f) \in T_pM.$$

By (3.4) and (3.5) we have

$$\sigma_0((e_0, 0), (e_0, 0))(0, \bar{d}(e_1 f) - \bar{e}_1(df))$$
$$+\sigma_0((e_0, 0), (e_1, 0))(0, df - \bar{d}f)$$
$$=(0, 2\sigma_0((e_0, 0), (e_0, 0))e_1(df) + 2\sigma_0((e_0, 0), (e_1, 0))df) \in T_pM.$$

Therefore we have $\sigma_0((e_0, 0), (e_0, 0)) = \sigma_0((e_0, 0), (e_1, 0)) = 0$.
By Lemma 2.1 (d), it follows that

$$\widetilde{R}(\sigma((e_1, 0), (e_0, 0)), (e_1, 0))(0, f)$$
$$+\widetilde{R}((e_0, 0), \sigma((e_1, 0), (e_1, 0)))(0, f) \in T_pM.$$

Computing similarly, we have $\sigma_0((e_1, 0), (e_1, 0)) = 0$. Consequently σ_0 vanishes. This is a contradiction.

Case 3. $O_p^1 M = T_pM$. That is, the second fundamental form σ vanishes.

Proof. We identify **Cay** with $\boldsymbol{H}+\boldsymbol{H}$ and we simply write α for $[\alpha, 0]$ if there is no danger of confusion. For later use we prepare some formulas:

$$(4.5) \quad \widetilde{R}((\alpha, \beta), (\gamma, \delta))(\varepsilon, \lambda)$$
$$=\frac{k}{4}(-4\langle\alpha, \varepsilon\rangle\gamma + 4\langle\gamma, \varepsilon\rangle\alpha + (\varepsilon\delta)\bar{\beta} - (\varepsilon\beta)\bar{\delta} + (\alpha\delta - \gamma\beta)\bar{\lambda},$$
$$\bar{\alpha}(\gamma\lambda) - \bar{\gamma}(\alpha\lambda) - 4\langle\beta, \lambda\rangle\delta + 4\langle\delta, \lambda\rangle\beta - \bar{\varepsilon}(\alpha\delta - \gamma\beta)),$$

$$(4.6) \quad \widetilde{R}((\alpha, \beta), (\gamma, \delta))([0, \varepsilon], [0, \lambda])$$
$$=\frac{k}{4}([0, (\bar{\beta}\delta - \bar{\delta}\beta)\varepsilon + (\beta\bar{\gamma} - \delta\bar{\alpha})\lambda], [0, (\alpha\bar{\gamma} - \gamma\bar{\alpha})\lambda + (\alpha\delta - \gamma\beta)\varepsilon]),$$

$$(4.7) \quad \widetilde{R}((\alpha, \beta), ([0, \gamma], [0, \delta]))(\varepsilon, \lambda)$$
$$=\frac{k}{4}([0, -4\langle\alpha, \varepsilon\rangle\gamma + 2\bar{\beta}\bar{\varepsilon}\delta + \bar{\lambda}(\bar{\alpha}\delta - \beta\gamma)],$$
$$[0, -4\langle\beta, \lambda\rangle\delta + 2\alpha\lambda\gamma - \varepsilon(\bar{\alpha}\delta - \beta\gamma)]),$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda \in \boldsymbol{H}$.

In Case 3 the normal space N_pM is given by

$$N_pM=\{([0,\gamma],[0,\delta]);\gamma,\delta\in\boldsymbol{H}\}.$$

We define σ' and σ'' by

$$\sigma((\alpha,\alpha'),(\beta,\beta'))=([0,\sigma'((\alpha,\alpha'),(\beta,\beta'))],[0,\sigma''((\alpha,\alpha'),(\beta,\beta'))]),$$

where $\alpha,\alpha',\beta,\beta'\in\boldsymbol{H}$ $\sigma'((\alpha,\alpha'),(\beta,\beta'))$, $\sigma''((\alpha,\alpha'),(\beta,\beta'))\in\boldsymbol{H}$. Applying Lemma 2.1 (c), we have

$$\begin{aligned}
&\sigma((\alpha,\alpha'),\widetilde{R}((\beta,\beta'),(\gamma,\gamma'))(\delta,\delta'))-\widetilde{R}(\sigma((\alpha,\alpha'),(\beta,\beta')),(\gamma,\gamma'))(\delta,\delta')\\
&-\widetilde{R}((\beta,\beta'),\sigma((\alpha,\alpha'),(\gamma,\gamma')))(\delta,\delta')-\widetilde{R}((\beta,\beta'),(\gamma,\gamma'))\sigma((\alpha,\alpha'),(\delta,\delta'))\\
&=0
\end{aligned}$$

Using (4.5), (4.6), and (4.7), we get

$$\begin{aligned}
&\sigma((\alpha,\alpha'),(-4<\beta,\delta>\gamma+4<\gamma,\delta>\beta+\delta\gamma'\bar{\beta}'-\delta\beta'\bar{\gamma}'+(\beta\gamma'-\gamma\beta')\bar{\delta}',\\
&\bar{\beta}\gamma\delta'-\bar{\gamma}\beta\delta'-4<\beta',\delta'>\gamma'+4<\gamma',\delta'>\beta'-\bar{\delta}(\beta\gamma'-\gamma\beta')))\\
&+([0,-4<\gamma,\delta>\sigma'((\alpha,\alpha'),(\beta,\beta'))+2\bar{\gamma}'\bar{\delta}\sigma''((\alpha,\alpha'),(\beta,\beta'))\\
&+\bar{\delta}'(\bar{\gamma}\sigma''((\alpha,\alpha'),(\beta,\beta'))-\gamma'\sigma'((\alpha,\alpha'),(\beta,\beta')))],\\
(4.8)\qquad&[0,-4<\gamma',\delta'>\sigma''((\alpha,\alpha'),(\beta,\beta'))+2\gamma\delta'\sigma'((\alpha,\alpha'),(\beta,\beta'))\\
&-\delta(\bar{\gamma}\sigma''((\alpha,\alpha'),(\beta,\beta'))-\gamma'\sigma'((\alpha,\alpha'),(\beta,\beta')))])\\
&-([0,-4<\beta,\delta>\sigma'((\alpha,\alpha'),(\gamma,\gamma'))+2\bar{\beta}'\bar{\delta}\sigma''((\alpha,\alpha'),(\gamma,\gamma'))\\
&+\bar{\delta}'(\bar{\beta}\sigma''((\alpha,\alpha'),(\gamma,\gamma'))-\beta'\sigma'((\alpha,\alpha'),(\gamma,\gamma')))],\\
&[0,-4<\beta',\delta'>\sigma''((\alpha,\alpha'),(\gamma,\gamma'))+2\beta\delta'\sigma'((\alpha,\alpha'),(\gamma,\gamma'))\\
&-\delta(\bar{\beta}\sigma''((\alpha,\alpha'),(\gamma,\gamma'))-\beta'\sigma'((\alpha,\alpha'),(\gamma,\gamma')))])\\
&-([0,(\bar{\beta}'\gamma'-\bar{\gamma}'\beta')\sigma((\alpha,\alpha'),(\delta,\delta'))+(\bar{\beta}'\bar{\gamma}-\bar{\gamma}'\bar{\beta})\sigma''((\alpha,\alpha'),(\delta,\delta'))],\\
&[0,(\beta\bar{\gamma}-\gamma\bar{\beta})\sigma''((\alpha,\alpha'),(\delta,\delta'))+(\beta\gamma'-\gamma\beta')\sigma'((\alpha,\alpha'),(\delta,\delta'))])\\
&=0.
\end{aligned}$$

Putting $\beta=\alpha$, $\delta=\gamma$, $\alpha'=\beta'=\gamma'=\delta'=0$ in (4.8) for an orthonormal system $\{\alpha,\gamma\}$ in $\boldsymbol{H}$, we have

$$\begin{aligned}
0=&([0,\sigma'((\alpha,0),(4\alpha,0))],[0,\sigma''((\alpha,0),(4\alpha,0))])\\
&+([0,-4\sigma'((\alpha,0),(\alpha,0))],[0,-\gamma(\bar{\gamma}\sigma''((\alpha,0),(\alpha,0)))])\\
&-(0,[0,\gamma(\bar{\alpha}\sigma''((\alpha,0),(\gamma,0)))])-(0,[0,(\alpha\bar{\gamma}-\gamma\bar{\alpha})\sigma''((\alpha,0),(\gamma,0))])\\
=&(0,[0,3\sigma''((\alpha,0),(\alpha,0))-\alpha\bar{\gamma}\sigma''((\alpha,0),(\gamma,0))]).
\end{aligned}$$

Therefore $\quad 3\sigma''((\alpha,0),(\alpha,0))-\alpha\bar{\gamma}\sigma''((\alpha,0),(\gamma,0))=0.$

Similarly we have $3\sigma''((\gamma, 0), (\gamma, 0))-\gamma\bar{\alpha}\sigma''((\alpha, 0), (\gamma, 0))=0$. Adding two equations, we have

$$0=3\sigma''((\alpha, 0), (\alpha, 0))+3\sigma''((\gamma, 0), (\gamma, 0))-2<\alpha, \gamma>\sigma''((\alpha, 0), (\gamma, 0))$$
$$=3\sigma''((\alpha, 0), (\alpha, 0))+3\sigma''((\gamma, 0), (\gamma, 0)).$$

For an arbitrary unit element $\alpha \in \boldsymbol{H}$, we take $\beta, \gamma \in \boldsymbol{H}$ such that $\{\alpha, \beta, \gamma\}$ is an orthonormal system in $\boldsymbol{H}$. Then we have

$$\sigma''((\alpha, 0), (\alpha, 0))=-\sigma''((\beta, 0), (\beta, 0))=\sigma''((\gamma, 0), (\gamma, 0))=-\sigma''((\alpha, 0), (\alpha, 0))$$

and hence $\sigma''((\alpha, 0), (\alpha, 0))=0.$
Since σ'' is symmetric, we have

$$(4.9) \qquad \sigma''((\alpha, 0), (\beta, 0))=0 \qquad \text{for } \alpha, \beta \in \boldsymbol{H}.$$

Putting $\beta'=\alpha'$, $\delta'=\gamma'$, $\alpha=\beta=\gamma=\delta=0$ in (4.8) for an orthonormal system $\{\alpha', \gamma'\}$ in $\boldsymbol{H}$, we have

$$3\sigma'((0, \alpha'), (0, \alpha'))-3\bar{\alpha}'\gamma'\sigma'((0, \alpha'), (0, \gamma'))=0.$$

By the similar computation, we obtain

$$(4.10) \qquad \sigma'((0, \alpha'), (0, \beta'))=0 \qquad \text{for } \alpha', \beta' \in \boldsymbol{H}.$$

Putting $\alpha=\gamma=\beta'=\delta'=0$, $\gamma'=\alpha'$, $\delta=\beta$ in (4.8), we have

$$0=-([0, <\beta, \beta>\sigma'((0, \alpha'), (0, \alpha'))], [0, <\beta, \beta>\sigma''((0, \alpha'), (0, \alpha'))])$$
$$+([0, 2\bar{\alpha}'\bar{\beta}\sigma''((0, \alpha'), (\beta, 0))], [0, \beta\alpha'\sigma'((0, \alpha'), (\beta, 0))])$$
$$-([0, -4<\beta, \beta>\sigma'((0, \alpha')), (0, \alpha'))], [0, -<\beta, \beta>\sigma''((0, \alpha'), (0, \alpha'))])$$
$$-([0, -\bar{\alpha}'\bar{\beta}\sigma''((0, \alpha'), (\beta, 0))], [0, \beta\alpha'\sigma'((0, \alpha'), (\beta, 0))])$$
$$=([0, 3\bar{\alpha}'\bar{\beta}\sigma''((0, \alpha'), (\beta, 0))], 0).$$

Hence we have

$$(4.11) \qquad \sigma''((0, \alpha'), (\beta, 0))=0 \qquad \text{for } \alpha', \beta \in \boldsymbol{H}.$$

Putting $\alpha'=\gamma'=\beta=\delta=0$, $\gamma=\alpha$, $\delta'=\beta'$ in (4.8), we have

$$(4.12) \qquad \sigma'((\alpha, 0), (0, \beta'))=0 \qquad \text{for } \alpha, \beta' \in \boldsymbol{H}.$$

By (4.11) and (4.12), we have

$$(4.13) \qquad \sigma((\alpha, 0), (0, \beta))=0 \qquad \text{for } \alpha, \beta \in \boldsymbol{H}.$$

Next putting $\alpha'=\gamma'=\beta=\delta=0$ and $\delta'=1$ in (4.8), we have

$$([0, \sigma'((\alpha, 0), (-\gamma\beta', 0))], 0)-([0, -\beta'\sigma'((\alpha, 0), (\gamma, 0))], 0)=0$$

and hence

$$\sigma'((\alpha, 0), (\gamma\beta', 0))=\beta'\sigma'((\alpha, 0), (\gamma, 0)).$$

Particularly it follows that $\sigma'((\alpha, 0), (\gamma ij, 0))=j\sigma'((\alpha, 0), (\gamma i, 0))=ji\sigma'((\alpha, 0), (\gamma, 0))$ and $\sigma'((\alpha, 0), (\gamma ij, 0))=ij\sigma'((\alpha, 0), (\gamma, 0))$.

Consequently we have

$$(4.14)\qquad \sigma'((\alpha, 0), (\gamma, 0))=0 \qquad \text{for } \alpha, \gamma \in \boldsymbol{H}.$$

Calculating similarly we get

$$(4.15)\qquad \sigma''((0, \alpha'), (0, \gamma'))=0 \qquad \text{for } \alpha', \gamma' \in \boldsymbol{H}.$$

By (4.9) and (4.14), we have $\sigma((\alpha, 0), (\gamma, 0))=0$ and by (4.10) and (4.15), $\sigma((0, \alpha'), (0, \gamma'))=0$.

Consequently the second fundamental form σ vanishes.

Case 4. $O_p^1M=T_pM$ or O_p^1M is equivalent to $\mathfrak{m}_{\boldsymbol{H}}$.

Case 5. $O_p^1M=T_pM$ or O_p^1M is equivalent to $\mathfrak{m}_{\boldsymbol{C}}$.

Similarly to Case 2, we denote by $\mathfrak{R}$ the subspace of $\mathfrak{gl}(T_p\widetilde{M})$ linearly spanned by $\widetilde{R}(X, Y)$, $X, Y \in T_pM$. Then $\mathfrak{R}$ is a Lie subalgebra of $\mathfrak{gl}(T_p\widetilde{M})$. We can prove Case 4 and Case 5 by the same argument as Case 2.

Proof of Case 4. If V is an irreducibly invariant subspace of N_pM by the action of $\mathfrak{R}$, then $\dim V=4$ and V is given by $V=\{(\alpha c, \beta c); \alpha, \beta \in \boldsymbol{C}\}$, where c is a unit element of **Cay** and $\langle c, \boldsymbol{C}\rangle=\{0\}$. Moreover T_pM+V is a curvature invariant subspace which is equivalent to $\mathfrak{m}_{\boldsymbol{H}}$. In fact, the subspace of **Cay** spanned by e_0, e_1, c, e_1c is a subalgebra of **Cay** which is isomorphic to $\boldsymbol{H}$. Since N_p^1M is an invariant subspace of N_pM by the action of $\mathfrak{R}$ and since $\dim N_p^1M\leqq 10$, it follows that $\dim N_p^1M=0$ or 4 or 8. If $\dim N_p^1M=0$, we have $O_p^1M=T_pM$. If $\dim N_p^1M=4$, by the above fact, O_p^1M is a curvature invariant subspace which is equivalent to $\mathfrak{m}_{\boldsymbol{H}}$. If $\dim N_p^1M=8$, applying Lemma 2.1 (d) we can show that this case does not occur. Its proof is quite similar to that of Case 2–(iii).

Proof of Case 5. If V is an irreducibly invariant subspace of N_pM by the action of $\mathfrak{R}$, then $\dim V=2$ and V is linearly spanned by (e, f) and $(-f, e)$ such that the real part of e = the real part of $f=0$ and $\|e\|^2+\|f\|^2=1$. Here if e and f are linearly dependent in **Cay**, then T_pM+V is a curvature invariant subspace of $T_p\widetilde{M}$ which is equivalent to $\mathfrak{m}_{\boldsymbol{C}}$. Actually in this case V is given by $V=\{(\alpha e, \beta e); \alpha, \beta \in \boldsymbol{R}\}$, where e is a unit element of **Cay** such that the real part of $e=0$. Moreover the subspace of **Cay** spanned by e_0 and e is a subalgebra of **Cay** which is isomorphic to $\boldsymbol{C}$.

Since N_p^1M is invariant in N_pM by the action of $\mathfrak{R}$ and since $\dim N_p^1M\leqq 3$, it follows that $\dim N_p^1M=0$ or 2. If $\dim N_p^1M=0$, we have $O_p^1M=T_pM$. If $\dim N_p^1M=2$, then N_p^1M is linearly spanned by (e, f) and $(-f, e)$ such that the real part of e = the real part of $f=0$ and $\|e\|^2+\|f\|^2=1$. We shall show that e and f are linearly dependent. If it is shown, by the above argument, O_p^1M is a curvature invariant subspace which is equivalent to $\mathfrak{m}_C$. We assume that e and f are linearly independent. We denote by σ_1 and σ_2 the (e, f)-component and $(-f, e)$-component of the second fundamental form σ respectively, that is,

$$\sigma((\alpha, \beta), (\gamma, \delta))=\sigma_1((\alpha, \beta), (\gamma, \delta))(e, f)+\sigma_2((\alpha, \beta), (\gamma, \delta))(-f, e)$$

for $\alpha, \beta, \gamma, \delta \in \boldsymbol{R}$, $\sigma_1((\alpha, \beta), (\gamma, \delta))$, $\sigma_2((\alpha. \beta), (\gamma, \delta)) \in \boldsymbol{R}$.
Applying Lemma 2. 1 (c), we have

$$\begin{aligned}&\sigma((e_0, 0), \widetilde{R}((0, e_0), (e_0, 0))(0, e_0))-\widetilde{R}(\sigma((e_0, 0), (0, e_0)), (e_0, 0))(0, e_0)\\&-\widetilde{R}((0, e_0), \sigma((e_0, 0), (e_0, 0)))(0, e_0)-\widetilde{R}((0, e_0), (e_0, 0))\sigma((e_0, 0), (0, e_0))\\&=0.\end{aligned}$$

By (3. 5), we get

$$\begin{aligned}&-\sigma_1((e_0, 0),(e_0, 0))(e, f)-\sigma_2((e_0, 0), (e_0, 0))(-f, e)\\&+\sigma_1((e_0, 0), (0, e_0))(f, 2e)-\sigma_2((e_0, 0), (0, e_0))(-e, 2f)\\&+\sigma_1((e_0, 0), (e_0, 0))(e, 4f)-\sigma_2((e_0, 0), (e_0, 0))(f, -4e)\\&+\sigma_1((e_0, 0), (0, e_0))(-f, e)-\sigma_2((e_0, 0), (0, e_0))(e, f)=0\end{aligned}$$

and hence

$$\begin{aligned}&3\{\sigma_2((e_0, 0), (e_0, 0))+\sigma_1((e_0, 0), (0, e_0))\}e\\&+3\{\sigma_1((e_0, 0), (e_0, 0))-\sigma_2((e_0, 0), (0, e_0))\}f=0.\end{aligned}$$

Since e *and* f are linearly independent, we have

$$\text{(4. 16)}\quad \begin{cases}\sigma_1((e_0, 0), (e_0, 0))=\sigma_2((e_0, 0), (0, e_0))\\ \sigma_2((e_0, 0), (e_0, 0))=-\sigma_1((e_0, 0), (0, e_0)).\end{cases}$$

Similarly we get

$$\begin{aligned}&\sigma((0, e_0), \widetilde{R}((e_0, 0), (0, e_0))(e_0, 0))-\widetilde{R}(\sigma((0, e_0), (e_0, 0)), (0, e_0))(e_0, 0)\\&-\widetilde{R}((e_0, 0), \sigma((0, e_0), (0, e_0)))(e_0, 0)-\widetilde{R}((e_0, 0), (0, e_0))\sigma((0, e_0), e_0, 0))\\&=0\end{aligned}$$

and by the same computation as above we obtain

$$\text{(4. 17)}\quad \begin{aligned}&\sigma_1((0, e_0), (0, e_0))=-\sigma_2((e_0, 0), (0, e_0))\\&\sigma_2((0, e_0), (0, e_0))=\sigma_1((e_0, 0), (0, e_0)).\end{aligned}$$

By (4. 16) and (4. 17), we may select e and f in **Cay** such that

$$\sigma((e_0, 0), (e_0, 0)) = -\sigma((0, e_0), (0, e_0)) = \lambda(e, f) \text{ and}$$

$$\sigma((e_0, 0), (0, e_0)) = \lambda(-f, e),$$

where λ is a non-zero real number.
Applying Lemma 2. 1 (d), we have

$$\widetilde{R}(\sigma((e_0, 0), (e_0, 0)), (0, e_0))(e, f) + \widetilde{R}((e_0, 0), \sigma((e_0, 0), (0, e_0)))(e, f) \in T_pM$$

and hence

$$\widetilde{R}((e, f), (0, e_0))(e, f) + \widetilde{R}((e_0, 0), (-f, e))(e, f) \in T_pM.$$

By (3. 5), we get

$$(-3ef, -4\langle f, f\rangle e_0 - \langle e, e\rangle e_0) + (-4\langle f, e\rangle e_0 - ef, 2\langle f, f\rangle e_0 - \langle e, e\rangle e_0) \in T_pM.$$

Consequently ef is a real number and hence e and f are linearly dependent. This is a contradiction.

Since the above proof is valid for the non-compact dual of Cayley plane, the following holds.

COROLLARY 4. 2. *Let $\widetilde{M}$ be the non-compact dual of Cayley plane whose curvature tensor is given by* (3. 5) *and f be an immersion with parallel second fundamental form of a connected manifold M ($\dim M \geqq 2$) into $\widetilde{M}$. Then there exists an* 8*-dimensional totally geodesic submanifold N of $\widetilde{M}$ in which the image $f(M)$ of M by f is contained. Here N is the non-compact dual of $P_2(\boldsymbol{H})$ or the* 8*-dimensional real hyperbolic space with constant sectional curvature k.*

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Kazumi Tsukada
Department of Mathematics
Niigata University
Ikarashi 2-8050
Niigata 950-21, Japan