A stochastic approximation method approximating the roots of time varying regression functions*

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1. Introduction

Let $M(x)$ be a regression function on the real line R and θ be the unique root of the equation $M(x)=0$. In practical problems there exist some cases where the use of a process which converges to θ from below (in some sense) is advantageous to us. For instance, θ may be an optimal level in operating a system where the costs caused by operating at a level above θ are considerably greater than those caused by operating at a level below θ. Eichhorn and Zacks [3] considered this situation in the problem of finding an optimal dosage of drugs which have secondary harmful effects in addition to their therapeutic effects. Anbar [1] has proposed the modified Robbins-Monro (R-M) procedure X_n and has proved that X_n converges almost surely (a. s.) to θ as n tends to infinity and that, with probability one, X_n exceeds θ only finitely many times. Isogai [6] has obtained the same results as above for the case where there exist errors in setting the x-levels.

However some situation may occur where the regression function $M(x)$ varies with the time. This situation has been treated by several authors (for example, by Dupač [2] and Watanabe [9]). In this paper we also consider this situation. Let $M_n(x)$ be a regression function on R at time n and θ_n be the unique root of the equation $M_n(x)=0$. The modified R-M procedure X_n defined by Anbar [1] will be used. The aim of this paper is to derive the rate of convergence of $X_n-\theta_n$ to zero by using the method due to Heyde [4], and to show that, with probability one, $X_n-\theta_n$ exceeds zero only finitely many times.

In Section 2 we shall give notations and prove several auxiliary results which are needed for Section 3. In Section 3 the main results will be proved.

2. Preliminaries and auxiliary results

In this section we shall give notations which are used throughout this paper and

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prove several auxiliary results which are needed for the next section.

The modified R-M procedure by ANBAR [1] is defined as follows:

X_1 is any random variable with $E[X_1^2] < \infty$,

(2.1) $$X_{n+1} = X_n - a_n(y_n + b_n) \qquad \text{for } n=1, 2, \dots,$$

where $\{a_n\}$ is a sequence of positive numbers, $\{b_n\}$ is a sequence of real numbers and $\{y_n\}$ is a sequence of random variables. In this paper we assume the following assumptions be valid.

Assumption. (i) $\{Y_n(x)\}$ is a sequence of random variables which depend on parameter $x \in R$, and for each n and x the expectation of $Y_n(x)$ exists, and the regression function $M_n(x)$ is Borel measurable on R and has the unique root θ_n of the equation $M_n(x)=0$, where

(2.2) $$M_n(x) = E[Y_n(x)].$$

(ii) y_n in (2.1) is a random variable which has conditional distribution given $X_1, \dots, X_n$ equal to that of $Y_n(X_n)$ given X_n.

By Assumption (ii) we get

(2.3) $$E[y_n | X_1, \dots, X_n] = M_n(X_n) \qquad a.s.,$$

where $E[\cdot|\cdot]$ denotes the conditional expectation operator. We can rewrite (2.1) as

(2.4) $$X_{n+1} = X_n - a_n M_n(X_n) - a_n b_n + a_n v_n \qquad \text{for } n=1, 2, \dots,$$

where

$$v_n = M_n(X_n) - y_n \qquad \text{for } n=1, 2, \dots .$$

Clearly

(2.5) $$E[v_n | X_1, \dots, X_n] = 0 \qquad a.s.,$$

so that $\{v_n\}$ is a martingale difference. In this paper $C_1, C_2, \dots$ denote appropriate positive constants.

LEMMA 2.1. *Let $\{\alpha_n\}$ be a sequence of positive numbers with $\lim_{n\to\infty} \alpha_n = \alpha > \frac{1}{2}$.* Then

$$\sum_{m=n_0}^{n} (m\gamma_m)^{-2} \sim (2\alpha-1)^{-1}(n\gamma_n^2)^{-1} \qquad as\ n \to \infty,$$

where n_0 is a positive integer such that

$$1 - \alpha_n n^{-1} > 0 \qquad for\ all\ n \geqq n_0,$$

$$\gamma_n = \prod_{j=n_0}^{n} (1 - \alpha_j j^{-1}) \qquad for\ all\ n \geqq n_0$$

and "$a_n \sim b_n$ as $n \to \infty$" means $\lim_{n\to\infty} a_n/b_n = 1$.

PROOF. It suffices to show that

$$(2.1.1)\qquad (2\alpha-1)n\gamma_n^2 \sum_{m=n_0}^{n} (m\gamma_m)^{-2} \to 1 \qquad \text{as } n \to \infty.$$

For any $\varepsilon > 0$ with $2(\alpha-\varepsilon) > 1$ there exists a positive integer $n_1 \geqq n_0$ such that for all $j \geqq n_1$

$$\alpha+\varepsilon > \alpha_j > \alpha-\varepsilon \quad \text{and} \quad 1-(\alpha+\varepsilon)j^{-1} > 0,$$

which implies

$$(2.1.2)\qquad \prod_{j=m+1}^{n} (1-(\alpha+\varepsilon)j^{-1}) < \prod_{j=m+1}^{n} (1-\alpha_j j^{-1}) < \prod_{j=m+1}^{n} (1-(\alpha-\varepsilon)j^{-1})$$

for all $n > m \geqq n_1$. Let any $\eta (0 < \eta < 1)$ be fixed. By using (2. 3) of Sacks [8] there exists a positive integer $n_2 \geqq n_1$ such that for all $n > m \geqq n_2$

$$(2.1.3)\qquad \begin{aligned} &\prod_{j=m+1}^{n} (1-(\alpha+\varepsilon)j^{-1}) > (1-\eta)n^{-(\alpha+\varepsilon)}m^{\alpha+\varepsilon} \\ &\prod_{j=m+1}^{n} (1-(\alpha-\varepsilon)j^{-1}) < (1+\eta)n^{-(\alpha-\varepsilon)}m^{\alpha-\varepsilon}, \end{aligned}$$

which, together with (2. 1. 2), yields

$$(2.1.4)\qquad \begin{aligned} &\gamma_n^2 \sum_{m=n_0}^{n} (m\gamma_m)^{-2} \\ &\geqq \sum_{m=n_2}^{n} m^{-2}(\gamma_n\gamma_m^{-1})^2 > (1-\eta)^2 n^{-2(\alpha+\varepsilon)} \sum_{m=n_2}^{n} m^{2(\alpha+\varepsilon)-2} \end{aligned}$$

for all $n \geqq n_2$. According to Lemma 4 of Sacks [8] there exists a positive integer $n_3 \geqq n_2$ such that for all $n \geqq n_3$

$$(2.1.5)\qquad \begin{aligned} &\sum_{m=n_2}^{n} m^{2(\alpha+\varepsilon)-2} > (1-\eta)(2(\alpha+\varepsilon)-1)^{-1}n^{2(\alpha+\varepsilon)-1} \\ &\sum_{m=n_2}^{n} m^{2(\alpha-\varepsilon)-2} < (1+\eta)(2(\alpha-\varepsilon)-1)^{-1}n^{2(\alpha-\varepsilon)-1}, \end{aligned}$$

which, together with (2. 1. 4), implies

$$(2.1.6)\qquad \liminf_{n\to\infty} (2\alpha-1)n\gamma_n^2 \sum_{m=n_0}^{n} (m\gamma_m)^{-2} \geqq 1.$$

Since by (2. 1. 2) and (2. 1. 3) $n\gamma_n^2 \leqq C_1 n^{1-2(\alpha-\varepsilon)}$ for all $n \geqq n_3$ and some constant $C_1 > 0$, using $2(\alpha-\varepsilon) > 1$ we get

$$(2.1.7)\qquad n\gamma_n^2 \to 0 \qquad \text{as } n \to \infty.$$

It follows from (2. 1. 2), (2. 1. 3) and (2. 1. 5) that

$$\begin{aligned} &\gamma_n^2 \sum_{m=n_0}^{n} (m\gamma_m)^{-2} \\ &\leqq \gamma_n^2 \sum_{m=n_0}^{n_3-1} (m\gamma_m)^{-2} + (1+\eta)^3(2(\alpha-\varepsilon)-1)^{-1}n^{-1} \qquad \text{for all } n \geqq n_3, \end{aligned}$$

which, together with (2. 1. 7), yields

$$(2.1.8)\qquad \limsup_{n\to\infty} (2\alpha-1)n\gamma_n^2 \sum_{m=n_0}^{n} (m\gamma_m)^{-2} \leqq 1.$$

Thus by (2. 1. 6) and (2. 1. 8) we obtain (2. 1. 1). This completes the proof.

The following proposition gives the a.s. and mean square convergences of $X_n-\theta_n$ to zero.

PROPOSITION 2. 2. *Suppose that the following conditions are fulfilled:*

There exist three sequences of positive numbers $\{\varepsilon_n\}$, $\{\rho_n\}$ *and* $\{A_n\}$, *and a sequence of real numbers* $\{b_n\}$ *such that*

(2. 2. 1) $(x-\theta_n)M_n(x)>0$ *if* $|x-\theta_n|\geqq\varepsilon_n$ *for all* $n\geqq 1$,

(2. 2. 2) $\inf\{|M_n(x)|: \varepsilon_n<|x-\theta_n|<\varepsilon_n^{-1}\}\geqq\rho_n$ *for all* $n\geqq 1$,

(2. 2. 3) $|M_n(x)|\leqq A_n(1+|x-\theta_n|)$ *for all* $x\in R$ *and all* $n\geqq 1$,

(2. 2. 4) $\sum_{n=1}^{\infty} a_n^2\ E[(M_n(X_n)-Y_n(X_n))^2]<\infty$,

(2. 2. 5) $\lim_{n\to\infty}\varepsilon_n=0$, $0<\varepsilon_n\leqq 1$ *for all* $n\geqq 1$, $\lim_{n\to\infty} a_nA_n=0$,

$$\sum_{n=1}^{\infty} a_n\rho_n=\infty,\quad \sum_{n=1}^{\infty} a_n|b_n|<\infty$$

and

(2. 2. 6) $\sum_{n=1}^{\infty}|\theta_n-\theta_{n+1}|<\infty$.

Then we have

$$\lim_{n\to\infty}|X_n-\theta_n|=0 \quad a.s.$$

and

$$\lim_{n\to\infty} E[(X_n-\theta_n)^2]=0.$$

PROOF. Let $Z_n=X_n-\theta_n$ for each $n\geqq 1$. By (2. 4), (2. 2. 3), (2. 2. 4) and $E[X_1^2]<\infty$ we get

(2. 2. 7) $E[X_n^2]<\infty$ for all $n\geqq 1$,

which implies

(2. 2. 8) $E[Z_n^2]<\infty$ for all $n\geqq 1$.

It follows from (2. 4) that

(2. 2. 9) $Z_{n+1}=T_n(Z_1,\dots,Z_n)+U_n$ for each $n\geqq 1$,

where

(2. 2. 10) $T_n(r_1\dots,r_n)=T_n(r_n)=r_n+(\theta_n-\theta_{n+1})-a_nM_n(r_n+\theta_n)-a_nb_n$

for $(r_1,\dots,r_n)\in R^n$ and $n\geqq 1$, and $U_n=a_nv_n$ for each $n\geqq 1$.

By (2. 5) and (2. 2. 4) we have

(2. 2. 11) $E[U_n|Z_1,\dots,Z_n]=0$ a.s. for all $n\geqq 1$

and

(2. 2. 12) $\sum_{n=1}^{\infty} E[U_n^2]<\infty$.

According to (2. 2. 5) there exists a positive integer N such that

(2.2.13) $\quad 1-a_nA_n\geqq 0 \qquad$ for all $n\geqq N$.

We shall estimate $T_n(r_1,\dots,r_n)$ given by (2.2.10). Let $n(\geqq N)$ be fixed.

Case 1 where $|r_n|\leqq\varepsilon_n$.

By (2.2.3) we get

$$|T_n(r_n)|\leqq|r_n|+|\theta_n-\theta_{n+1}|+a_nA_n(1+|r_n|)+a_n|b_n|,$$

which, together with $|r_n|\leqq\varepsilon_n\leqq 1$, implies

(2.2.14) $\quad |T_n(r_n)|\leqq\varepsilon_n+|\theta_n-\theta_{n+1}|+2a_nA_n+a_n|b_n|.$

Case 2 where $\varepsilon_n<|r_n|<\varepsilon_n^{-1}$.

Assume that $r_n>0$. Since by (2.2.1) $M_n(r_n+\theta_n)=|M_n(r_n+\theta_n)|$, it follows from (2.2.2) that

$$\begin{aligned}T_n(r_n)&\leqq|r_n|+|\theta_n-\theta_{n+1}|-a_n|M_n(r_n+\theta_n)|+a_n|b_n|\\&\leqq|r_n|+|\theta_n-\theta_{n+1}|-a_n\rho_n+a_n|b_n|.\end{aligned}$$

On the other hand, by (2.2.3) and (2.2.13) we get

$$\begin{aligned}T_n(r_n)&\geqq|r_n|-|\theta_n-\theta_{n+1}|-a_n|M_n(r_n+\theta_n)|-a_n|b_n|\\&\geqq|r_n|-|\theta_n-\theta_{n+1}|-a_nA_n(1+|r_n|)-a_n|b_n|\\&\geqq-\{|\theta_n-\theta_{n+1}|+a_nA_n+a_n|b_n|\}.\end{aligned}$$

Thus the above relations imply

(2.2.15) $\quad |T_n(r_n)|\leqq\max\{|\theta_n-\theta_{n+1}|+a_nA_n+a_n|b_n|,$

$$|r_n|-a_n\rho_n+|\theta_n-\theta_{n+1}|+a_n|b_n|\}.$$

When $r_n<0$, in the same way as above we get (2.2.15).

Case 3 where $|r_n|\geqq\varepsilon_n^{-1}$.

In the same way as case 2 we have

(2.2.16) $\quad |T_n(r_n)|\leqq\max\{|\theta_n-\theta_{n+1}|+a_nA_n+a_n|b_n|,$

$$|r_n|+|\theta_n-\theta_{n+1}|+a_n|b_n|\}.$$

Define $\gamma_n(r_1,\dots,r_n)$ as follows: for $(r_1,\dots,r_n)\in R^n$

$$\gamma_n(r_1,\dots,r_n)=\gamma_n(r_n)=\begin{cases}a_n\rho_n & \text{if } |r_n|<\varepsilon_n^{-1}\\ 0 & \text{otherwise.}\end{cases}$$

By (2.2.14), (2.2.15) and (2.2.16) we obtain

(2.2.17) $\quad |T_n(r_1,\dots,r_n)|\leqq\max\{\varepsilon_n+|\theta_n-\theta_{n+1}|+2a_nA_n+a_n|b_n|,$

$$|r_n|-\gamma_n(r_1,\dots,r_n)+|\theta_n-\theta_{n+1}|+a_n|b_n|\}$$

for all $(r_1, \dots, r_n) \in R^n$ and all $n \geqq N$. It follows from (2. 2. 5) and (2. 2. 6) that

$$\text{(2. 2. 18)} \quad \lim_{n\to\infty} \{\varepsilon_n + |\theta_n - \theta_{n+1}| + 2a_n A_n + a_n |b_n|\} = 0$$

and

$$\text{(2. 2. 19)} \quad \sum_{n=1}^{\infty} \{|\theta_n - \theta_{n+1}| + a_n |b_n|\} < \infty.$$

Since $\lim_{n\to\infty} \varepsilon_n^{-1} = \infty$, for any sequence of real numbers $\{r_m\}$ with $\sup_{m\geqq 1} |r_m| < \infty$ there exists a positive integer $N_1 (\geqq N)$ such that $\sup_{m\geqq 1} |r_m| < \varepsilon_n^{-1}$ for all $n \geqq N_1$. Hence by the definition of γ_n we get $\gamma_n(r_1, \dots, r_n) = a_n \rho_n$ for all $n \geqq N_1$, which, together with (2. 2. 5), yields

$$\text{(2. 2. 20)} \quad \sum_{n=1}^{\infty} \gamma_n(r_1, \dots, r_n) = \infty.$$

We note that Lemma 1 of WATANABE [9] remains valid by replacing any constant X_1 in [9] by any random variable X_1 with $E[X_1^2] < \infty$. Thus by (2. 2. 8), (2. 2. 9), (2. 2. 11), (2. 2. 12), (2. 2. 17) to (2. 2. 20) and Lemma 1 of WATANABE [9] we obtain the result. This completes the proof.

The following lemma gives the rate of convergence of $E[(X_n - \theta_n)^2]$ to zero, which will be useful for proving theorems in the next section.

LEMMA 2. 3. *Assume the following conditions:*

$$\text{(2. 3. 1)} \quad (x - \theta_n) M_n(x) \geqq 0 \qquad \textit{for all } x \in R \textit{ and all } n \geqq 1,$$

there exist two sequences of positive constants $\{K_{n1}\}$ *and* $\{K_{n2}\}$ *such that*

$$\text{(2. 3. 2)} \quad K_{n1} |x - \theta_n| \leqq |M_n(x)| \leqq K_{n2} |x - \theta_n| \qquad \textit{for all } x \in R \textit{ and all } n \geqq 1,$$

$$\text{(2. 3. 3)} \quad \lim_{n\to\infty} K_{n2}^2 / n = 0$$

and

$$\text{(2. 3. 4)} \quad K_1 = \inf_{n\geqq 1} K_{n1} > 0,$$

there exists a sequence of nonnegative valued Borel measurable function $L_n(x)$ *on* R *such that*

$$\text{(2. 3. 5)} \quad E[(M_n(x) - Y_n(x))^2] \leqq L_n(x) \qquad \textit{for all } x \in R \textit{ and all } n \geqq 1$$

and

$$\text{(2. 3. 6)} \quad l_n = o(\log_2 n),$$

where $l_n = E[L_n(X_n)] < \infty$ and $\log_2 n = \log(\log n)$,

$$\text{(2. 3. 7)} \quad |\theta_n - \theta_{n+1}| = o(n^{-3/2} (\log_2 n)^{1/2})$$

and $a_n = A n^{-1}$ *where* A *is an arbitrary positive number with*

$$\text{(2. 3. 8)} \quad 2AK_1 > 1.$$

Let $\{b_n\}$ *be any sequence of real numbers satisfying*

(2. 3. 9) $\quad b_n^2 \leqq C_1(\log_2 n)/n \qquad$ *for all* $n \geqq 3$

$b_1 = b_2 = 0,$

where C_1 *is some positive constant. Then, Proposition* 2. 2 *holds and there exists a positive constant* C *such that*

(2. 3. 10) $\quad E[(X_n - \theta_n)^2] \leqq C(\log_2 n)/n \qquad$ *for all* $n \geqq 3.$

PROOF. Since $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} K_{n1}/n = \infty$ by (2. 3. 4), there exists a sequence of positive numbers $\{\pi_n\}$ such that

$$\pi_n \leqq 1 \qquad \text{for all } n \geqq 1, \qquad \lim_{n\to\infty} \pi_n = 0, \qquad \sum_{n=1}^{\infty} (K_{n1}\pi_n)/n = \infty$$

and $\sum_{n=1}^{\infty} a_n \pi_n = \infty$. In Proposition 2. 2 let $\varepsilon_n = \pi_n$, $\rho_n = K_{n1}\pi_n$, $A_n = K_{n2}$. Then by the conditions of this lemma all the conditions of Proposition 2. 2 are fulfilled. Thus Proposition 2. 2 holds. Next, we shall show (2. 3. 10). Let $Z_n = X_n - \theta_n$ for each $n \geqq 1$. By (2. 4) and (2. 5) we get

(2. 3. 11) $$E[Z_{n+1}^2] = E[Z_n^2] + (\theta_n - \theta_{n+1})^2 + a_n^2 E[M_n^2(X_n)] + a_n^2 E[v_n^2] + a_n^2 b_n^2$$
$$+ 2(\theta_n - \theta_{n+1}) E[Z_n] + 2a_n^2 b_n E[M_n(X_n)] - 2a_n E[Z_n M_n(X_n)]$$
$$- 2a_n b_n E[Z_n] - 2a_n(\theta_n - \theta_{n+1}) E[M_n(X_n)] - 2a_n b_n (\theta_n - \theta_{n+1})$$

for all $n \geqq 1$. Define $Q_n(x)$ as follows: for each $n \geqq 1$

$$Q_n(x) = \begin{cases} M_n(x)/(x - \theta_n) & \text{if } x \neq \theta_n \\ \alpha_n & \text{if } x = \theta_n, \end{cases}$$

where α_n is an arbitrary constant with $K_{n1} \leqq \alpha_n \leqq K_{n2}$. Hence

(2. 3. 12) $\quad M_n(x) = Q_n(x)(x - \theta_n) \qquad$ for all $x \in R$ and all $n \geqq 1$.

Here, by (2. 3. 1) and (2. 3. 2) we get

(2. 3. 13) $\quad K_{n1} \leqq Q_n(x) \leqq K_{n2} \qquad$ for all $x \in R$ and all $n \geqq 1$.

It follows from (2. 3. 12) and (2. 3. 13) that

(2. 3. 14) $\quad E[M_n^2(X_n)] \leqq K_{n2}^2 E[Z_n^2] \qquad$ for all $n \geqq 1$

and

(2. 3. 15) $\quad |E[M_n(X_n)]| \leqq K_{n2} E[|Z_n|] \qquad$ for all $n \geqq 1$.

By (2. 3. 15) and the inequality

(2. 3. 16) $\quad 2|ab| \leqq ka^2 + k^{-1}b^2 \qquad$ for any $a, b \in R$ and any $k > 0$,

we have

(2. 3. 17) $\quad 2a_n^2 |b_n E[M_n(X_n)]| \leqq K_{n2} a_n^2 E[Z_n^2] + K_{n2} a_n^2 b_n^2 \qquad$ for all $n \geqq 1$.

By (2. 3. 5) we get

(2. 3. 18) $E[v_n^2] \leqq l_n$ for all $n \geqq 1$.

Since by (2. 3. 12) and (2. 3. 13) $E[Z_n M_n(X_n)] \geqq K_{n1} E[Z_n^2]$, we have

(2. 3. 19) $-2a_n E[Z_n M_n(X_n)] \leqq -2K_{n1} a_n E[Z_n^2]$ for all $n \geqq 1$.

From (2. 3. 8) there exists a positive number ε such that $2AK_1(1-\varepsilon) > 1$. It follows from (2. 3. 12), (2. 3. 13) and (2. 3. 16) that for all $n \geqq 1$

(2. 3. 20) $2a_n |b_n E[Z_n]| \leqq 2^{-1} K_1 \varepsilon a_n E[Z_n^2] + 2(K_1\varepsilon)^{-1} a_n b_n^2$,

(2. 3. 21) $2a_n |\theta_n - \theta_{n+1}| \, |E[M_n(X_n)]|$

$\leqq 2^{-1} K_1 \varepsilon a_n E[Z_n^2] + 2K_{n2}^2 (K_1\varepsilon)^{-1} a_n (\theta_n - \theta_{n+1})^2$,

(2. 3. 22) $2a_n |b_n| \, |\theta_n - \theta_{n+1}| \leqq a_n b_n^2 + a_n(\theta_n - \theta_{n+1})^2$

and

(2. 3. 23) $2|\theta_n - \theta_{n+1}| \, |E[Z_n]| \leqq 2^{-1} K_1 \varepsilon a_n E[Z_n^2] + 2(K_1\varepsilon)^{-1} a_n^{-1} (\theta_n - \theta_{n+1})^2$.

Hence by (2. 3. 11), (2. 3. 14), (2. 3. 17) to (2. 3. 23) and the assumption that $K_{n1} \geqq K_1$ for all $n \geqq 1$, we have

(2. 3. 24) $E[Z_{n+1}^2] \leqq (1 - 2AK_1 n^{-1} \delta_n) E[Z_n^2]$

$+ \{1 + 2K_{n2}^2 (K_1\varepsilon)^{-1} a_n + a_n + 2(K_1\varepsilon)^{-1} a_n^{-1}\} (\theta_n - \theta_{n+1})^2$

$+ a_n^2 l_n + (1 + K_{n2}) a_n^2 b_n^2 + (1 + 2(K_1\varepsilon)^{-1}) a_n b_n^2$,

where $\delta_n = 1 - (3\varepsilon/4) - K_{n2}^2 (2K_1)^{-1} a_n - K_{n2} (2K_1)^{-1} a_n$. Since by (2. 3. 3) and (2. 3. 4) $\lim_{n\to\infty} K_{n2}/n = 0$, from (2. 3. 3) there exist a positive integer $n_0 \geqq 3$ and some constant $C_2 > 0$ such that

(2. 3. 25) $(K_{n2}^2 + K_{n2})(2K_1)^{-1} a_n < \varepsilon/4$ and $a_n \leqq 1$ for all $n \geqq n_0$

and

(2. 3. 26) $1 + 2K_{n2}^2 (K_1\varepsilon)^{-1} a_n + a_n + 2(K_1\varepsilon)^{-1} a_n^{-1} \leqq C_2 a_n^{-1}$ for all $n \geqq 1$.

By (2. 3. 9), (2. 3. 24), (2. 3. 25) and (2. 3. 26) we get

(2. 3. 27) $E[Z_{n+1}^2] \leqq (1 - \xi n^{-1}) E[Z_n^2] + C_3 n (\theta_n - \theta_{n+1})^2 + C_3 n^{-2} l_n$

$+ C_3 n^{-2} \log_2 n$ *for all* $n \geqq n_0$,

where $\xi = 2AK_1(1-\varepsilon) (> 1)$. Let $d_n = E[Z_n^2]$ and

$$\widetilde{\beta}_{mn} = \begin{cases} \prod_{j=m+1}^{n} (1 - \xi j^{-1}) & \text{if } n > m \\ 1 & \text{if } n = m. \end{cases}$$

It is clear that there exist a positive integer $n_1 (> n_0)$ and a positive constant C_4 such that

(2. 3. 28) $0 < \widetilde{\beta}_{mn} \leqq C_4 n^{-\xi} m^{\xi}$ for all $n \geqq m \geqq n_1 - 1$.

Since by (2. 2. 8) and (2. 3. 27)

$$d_{n+1} \leqq \widetilde{\beta}_{n_1-1,\,n} d_{n_1} + C_3 \sum_{m=n_1}^{n} \widetilde{\beta}_{mn} m(\theta_m - \theta_{m+1})^2$$

$$+ C_3 \sum_{m=n_1}^{n} \widetilde{\beta}_{mn}\, m^{-2} l_m + C_3 \sum_{m=n_1}^{n} \widetilde{\beta}_{mn}\, m^{-2} \log_2 m \qquad \text{for all } n \geqq n_1$$

and $d_{n_1} < \infty$, using (2. 3. 28) we have

$$\text{(2. 3. 29)} \quad d_{n+1} \leqq C_5 \{ n^{-\xi} + n^{-\xi} \sum_{m=n_1}^{n} m^{\xi+1} (\theta_m - \theta_{m+1})^2$$

$$+ n^{-\xi} \sum_{m=n_1}^{n} m^{\xi-2} l_m + n^{-\xi} \sum_{m=n_1}^{n} m^{\xi-2} \log_2 m \} \qquad \text{for all } n \geqq n_1.$$

According to Lemma 4 of Sacks [8], (2. 3. 6), (2. 3. 7) and the fact that $\xi > 1$ we get

$$n^{1-\xi} (\log_2 n)^{-1} \sum_{m=n_1}^{n} m^{\xi-2} \log_2 m \leqq C_6,$$

$$n^{1-\xi} (\log_2 n)^{-1} \sum_{m=n_1}^{n} m^{\xi-2} l_m \leqq C_6,$$

$$n^{1-\xi} (\log_2 n)^{-1} \sum_{m=n_1}^{n} m^{\xi+1} (\theta_m - \theta_{m+1})^2 \leqq C_6$$

and

$$n^{1-\xi} (\log_2 n)^{-1} \leqq C_6$$

for all $n \geqq n_1$, which, together with (2. 3. 29), imply

$$d_{n+1} / (n^{-1} \log_2 n) \leqq C_7 \qquad \text{for all } n \geqq n_1.$$

Thus there exists a positive constant C such that

$$d_n \leqq C n^{-1} \log_2 n \qquad \text{for all } n \geqq 3.$$

This completes the proof.

3. Results

In this section the results of the previous section are used to prove the main results. Assume the following conditions:

$$\text{(3. 1)} \qquad |\theta_n - \theta_{n+1}| = o(n^{-3/2} (\log_2 n)^{1/2}),$$

$$\text{(3. 2)} \qquad M_n(x) = \alpha_1(n)(x - \theta_n) + \alpha_2(n)(x - \theta_n)^2 + o((x - \theta_n)^2)$$

for each $n \geqq 1$, where $\alpha_1(n) > 0$ and $\alpha_2(n)$ are constants depending only on n with

$$\text{(3. 3)} \qquad \lim_{n \to \infty} \alpha_1(n) = \alpha_1 > 0$$

and

$$\text{(3. 4)} \qquad \sum_{n=3}^{\infty} (n^{-3} \log_2 n)^{1/2} |\alpha_2(n)| < \infty,$$

and $o(x^2)$ means "$o(x^2)/x^2 \to 0$ as $|x| \to 0$", there exists a sequence of positive numbers $\{\eta_n\}$ such that

(3.5) $\quad \bar{R} = \sup_{n\geqq 1} \sup_{x\in R} E[|V_n(x)|^{2+\eta_n}] < \infty$

and

(3.6) $\quad 0 < \underline{\eta} \leqq \bar{\eta} < \infty,$

where $V_n(x) = \mathrm{M}_n(x) - Y_n(x)$, $\underline{\eta} = \inf_{n\geqq 1} \eta_n$ and $\bar{\eta} = \sup_{n\geqq 1} \eta_n$,

there exists a finite positive constant σ^2 such that for any sequence of real numbers $\{x_n\}$ with $x_n - \theta_n \to 0$ as $n \to \infty$

(3.7) $\quad E[V_n^2(x_n)] \to \sigma^2 \quad$ as $n \to \infty$.

The following result concerning the rate of $x_n - \theta_n$ to zero is analogous to that of HEYDE [4].

THEOREM 3.1. *Let* $\{b_n\}$ *be any sequence of real numbers satisfying*

(3.1.1) $\quad b_n = o((n^{-1}\log_2 n)^{1/2})$

$$b_1 = b_2 = 0.$$

Assume all the conditions of Lemma 2.3 *except for* (2.3.7) *and* (2.3.9), *and the conditions* (3.1) *to* (3.7). *Then we have*

$$X_n = \theta_n + A\sigma(2A\alpha_1(n)-1)^{-1/2}\zeta_n(2n^{-1}\log_2 n)^{1/2} \quad a.s. \quad \text{for all } n \geqq 3,$$

where the sequence of random variables $\{\zeta_n, n \geqq 3\}$ *has its set of a.s. limit points confined to* $[-1, 1]$ *with*

$$\limsup_{n\to\infty} \zeta_n = 1 \quad a.s., \quad and \quad \liminf_{n\to\infty} \zeta_n = -1 \quad a.s.$$

PROOF. We shall proceed the proof along that of Theorem 2.1 in HEYDE [4]. We note that under the conditions of this theorem all the conditions of Lemma 2.3 are fulfilled. It follows from (2.3.2), (2.3.8), (3.2) and (3.3) that

(3.1.2) $\quad \alpha_1(n) \geqq K_1 \quad$ for all $n \geqq 1$

and

(3.1.3) $\quad 2A\alpha_1 \geqq 2AK_1 > 1.$

Let $\delta_n = \alpha_2(n)(X_n - \theta_n)^2 + o((X_n - \theta_n)^2)$ and $Z_n = X_n - \theta_n$. By (2.4) and (3.2) we get

(3.1.4) $$Z_{n+1} = \beta_{2n}Z_3 + \sum_{m=3}^{n} \beta_{mn}(\theta_m - \theta_{m+1}) - A\sum_{m=3}^{n} \beta_{mn} m^{-1}\delta_m + A\sum_{m=3}^{n} \beta_{mn} m^{-1} v_m - AD_n \quad \text{for all } n \geqq 3,$$

where

$$\beta_{mn} = \begin{cases} \prod_{j=m+1}^{n} (1 - A\alpha_1(j) j^{-1}) & \text{if } n > m \\ 1 & \text{if } n = m \end{cases}$$

and

(3.1.5) $$D_n=\sum_{m=3}^{n}\beta_{mn}m^{-1}b_m.$$

From (3.3) and (3.1.2) there exists a positive integer n_0 such that

$$0<1-A\alpha_1(j)j^{-1}\leqq 1-AK_1j^{-1} \qquad \text{for all } j\geqq n_0,$$

by which we get

(3.1.6) $$|\beta_{mn}|\leqq C_2m^{AK_1}n^{-AK_1} \qquad \text{for all } n\geqq m\geqq 1$$

and

(3.1.7) $$\beta_{mn}=\gamma_n\gamma_m^{-1} \qquad \text{for all } n\geqq m\geqq n_0,$$

where

$$\gamma_n=\prod_{j=n_0}^{n}(1-A\alpha_1(j)j^{-1})(>0) \qquad \text{for all } n\geqq n_0.$$

Since by (2.2.8) $|Z_3|<\infty$ a.s., using (3.1.3) and (3.1.6) we have

(3.1.8) $$\beta_{2n}Z_3=o((n^{-1}\log_2 n)^{1/2}) \qquad \text{a.s.}$$

It follows from (3.1.6) that

(3.1.9) $$\begin{aligned}&n^{1/2}(\log_2 n)^{-1/2}\sum_{m=3}^{n}|\beta_{mn}|\,|\theta_m-\theta_{m+1}|\\ &\leqq C_2\,n^{1/2-AK_1}(\log_2 n)^{-1/2}\sum_{m=3}^{n}m^{AK_1}|\theta_m-\theta_{m+1}|.\end{aligned}$$

According to Lemma 4 of Sacks [8] and (3.1.3) we get

$$\sum_{m=3}^{n}m^{AK_1-3/2}(\log_2 m)^{1/2}\sim\left(AK_1-\frac{1}{2}\right)^{-1}n^{AK_1-1/2}(\log_2 n)^{1/2} \qquad \text{as } n\to\infty,$$

which, together with (3.1) and the Toeplitz lemma (Loève [7], page 238), implies that the right hand side (R. H. S.) of (3.1.9) converges to zero as n tends to infinity.
Hence

(3.1.10) $$\sum_{m=3}^{n}\beta_{mn}(\theta_m-\theta_{m+1})=o((n^{-1}\log_2 n)^{1/2}).$$

In the same manner as (3.1.10), by using (3.1.1) we get

(3.1.11) $$D_n=o((n^{-1}\log_2 n)^{1/2}).$$

By (3.1.6) we have

(3.1.12) $$\begin{aligned}&n^{1/2}(\log_2 n)^{-1/2}\sum_{m=3}^{n}|\beta_{mn}|m^{-1}|\alpha_2(m)|Z_m^2\\ &\leqq C_3n^{1/2-AK_1}(\log_2 n)^{-1/2}\sum_{m=3}^{n}m^{AK_1-1/2}(\log_2 m)^{1/2}\times(m\log_2 m)^{-1/2}|\alpha_2(m)|Z_m^2.\end{aligned}$$

Since by (2.3.10) and (3.4)

$$\sum_{n=3}^{\infty}(n\log_2 n)^{-1/2}|\alpha_2(n)|E[Z_n^2]<\infty,$$

it holds that

$$\sum_{n=3}^{\infty}(n\log_2 n)^{-1/2}|\alpha_2(n)|Z_n^2<\infty \qquad \text{a.s.},$$

which, together with (3. 1. 3), (3. 1. 12) and the Kronecker lemma (Loève [7], page 238), implies that the R. H. S. of (3. 1. 12) converges to zero as n tends to infinity. Hence

$$(3.1.13) \qquad \sum_{m=3}^{n}\beta_{mn}m^{-1}\alpha_2(m)Z_m^2=o((n^{-1}\log_2 n)^{1/2}) \qquad \text{a.s.}$$

Since by Lemma 2. 3 $Z_m\to 0$ a.s. as $m\to\infty$, there exists a positive constant C_4, possibly depending on ω, such that

$$|o(Z_m^2)|\leqq C_4 Z_m^2 \qquad \text{a.s.} \qquad \text{for all } m\geqq 1,$$

which, together with (3. 1. 6), yields

$$n^{1/2}(\log_2 n)^{-1/2}\sum_{m=3}^{n}|\beta_{mn}m^{-1}o(Z_m^2)|$$

$$\leqq C_5 n^{1/2-AK_1}(\log_2 n)^{-1/2}\sum_{m=3}^{n}m^{AK_1-1/2}(\log_2 m)^{1/2}\times(m\log_2 m)^{-1/2}Z_m^2.$$

It follows from (2. 3. 10) that

$$\sum_{n=3}^{\infty}(n\log_2 n)^{-1/2}E[Z_n^2]<\infty,$$

which implies

$$\sum_{n=3}^{\infty}(n\log_2 n)^{-1/2}Z_n^2<\infty \qquad \text{a.s.}$$

Thus, in the same manner as (3. 1. 13) we get

$$\sum_{m=3}^{n}\beta_{mn}m^{-1}o(Z_m^2)=o((n^{-1}\log_2 n)^{1/2}) \qquad \text{a.s.},$$

which, together with (3. 1. 13), yields

$$(3.1.14) \qquad \sum_{m=3}^{n}\beta_{mn}m^{-1}\delta_m=o((n^{-1}\log_2 n)^{1/2}) \qquad \text{a.s.}$$

In view of (3. 1. 4), (3. 1. 8), (3. 1. 10), (3. 1. 11) and (3. 1. 14), it suffices, in order to prove the theorem, to show the following:

$$(3.1.15) \qquad \limsup_{n\to\infty}\zeta'_n=1 \qquad \text{a.s.},$$

$$(3.1.16) \qquad \liminf_{n\to\infty}\zeta'_n=-1 \qquad \text{a.s.}$$

and $\{\zeta'_n,\ n\geqq 3\}$ has its set of a.s. limit points confined to $[-1, 1]$, where

$$\zeta'_n=\sigma^{-1}(2A\,\alpha_1(n)-1)^{1/2}(2n^{-1}\log_2 n)^{-1/2}\sum_{m=3}^{n}\beta_{mn}m^{-1}v_m \qquad \text{for } n\geqq 3.$$

Let $U_n=n^{-1}\gamma_n^{-1}v_n$ for all $n\geqq n_0$. Since $\{v_n\}$ is a martingale difference, $\{U_n\}$ is also a martingale difference.

Let

$$(3.1.17)\quad s_n^2 = E\Big[\Big(\sum_{m=n_0}^{n} U_m\Big)^2\Big] = \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2} E[v_m^2] \qquad \text{for all } n \geqq n_0.$$

Since $X_m - \theta_m \to 0$ a.s. as $m \to \infty$, it follows from (3. 7) that

$$(3.1.18)\quad \lim_{m\to\infty} E[v_m^2 \mid \mathscr{F}_{m-1}] = \lim_{m\to\infty} E[V_m^2(X_m) \mid X_m] = \sigma^2 \qquad \text{a.s.},$$

where $\mathscr{F}_m$ is the σ-field generated by $X_1, \dots, X_{m+1}$. On the other hand, by (3. 5) and the Hölder inequality we get

$$E[v_m^2 \mid \mathscr{F}_{m-1}] \leqq \bar{R}^{2/(2+\eta_m)} \leqq \max\{1, \bar{R}\} < \infty \qquad \text{for all } m \geqq 1,$$

which implies

$$\sup_{m\geqq 1} E[v_m^2 \mid \mathscr{F}_{m-1}] \leqq \max\{1, \bar{R}\}.$$

Hence by the bounded convergence theorem and (3. 1. 18) we have

$$(3.1.19)\quad \lim_{n\to\infty} E[v_n^2] = \sigma^2.$$

From (3. 1. 3) and Lemma 2. 1 we get

$$(3.1.20)\quad \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2} \sim (2A\alpha_1 - 1)^{-1}(n\gamma_n^2)^{-1} \qquad \text{as } n\to\infty,$$

which, together with (3. 1. 3), (3. 1. 6) and (3. 1. 7), implies

$$(3.1.21)\quad \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2} \to \infty \qquad \text{as } n\to\infty.$$

It follows from (3. 1. 17), (3. 1. 19), (3. 1. 20), (3. 1. 21) and the Toeplitz lemma that

$$(3.1.22)\quad s_n^2 \sim \sigma^2 \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2} \sim \sigma^2(2A\alpha_1 - 1)^{-1}(n\gamma_n^2)^{-1} \qquad \text{as } n\to\infty.$$

Set

$$\zeta_n'' = (2s_n^2 \log_2 s_n^2)^{-1/2} \sum_{m=n_0}^{n} U_m \qquad \text{for } n \geqq n_0.$$

By Theorem 1 of HEYDE and SCOTT [5] it will hold that

$$(3.1.23)\quad \limsup_{n\to\infty} \zeta_n'' = 1 \qquad \text{a.s.},$$

$$(3.1.24)\quad \liminf_{n\to\infty} \zeta_n'' = -1 \qquad \text{a.s.}$$

and $\{\zeta_n'', n \geqq n_0\}$ has its set of a.s. limit points confined to $[-1, 1]$ if the following conditions are fulfilled:

$$(3.1.25)\quad \sum_{n=n_0}^{\infty} s_n^{-4} E[U_n^4 I(|U_n| < \delta s_n)] < \infty \qquad \text{for some } \delta > 0,$$

$$(3.1.26)\quad \sum_{n=n_0}^{\infty} s_n^{-1} E[|U_n| I(|U_n| \geqq \varepsilon s_n)] < \infty \qquad \text{for all } \varepsilon > 0$$

and

$$(3.1.27)\quad s_n^{-2} \sum_{m=n_0}^{n} U_m^2 \to 1 \qquad \text{a.s.} \quad \text{as } n\to\infty,$$

where $I(A)$ denotes the indicator function of the set A. First we shall show (3. 1. 25).

By (3. 1. 22) it suffices to show that

$$(3.1.28)\qquad \sum_{n=n_0}^{\infty} n^{-2}E[v_n^4 I(|v_n|<\delta n^{1/2})]<\infty \qquad \text{for some } \delta>0.$$

We may assume that $0<\eta_n<2$ for all $n\geqq 1$ in (3. 5). Let $\delta(0<\delta<1)$ be fixed. From (3. 5) and (3. 6) it is clear that

$$\sum_{n=n_0}^{\infty} n^{-2}E[v_n^4 I(|v_n|<\delta n^{1/2})]$$

$$\leqq \sum_{n=n_0}^{\infty} \delta^{2-\eta_n} n^{-(1+\eta_n/2)} E[|v_n|^{2+\eta_n}]$$

$$\leqq \bar{R} \sum_{n=n_0}^{\infty} n^{-(1+\underline{\eta}/2)}<\infty,$$

which concludes (3. 1. 28). Next we shall show (3. 1. 26). By (3. 1. 22) it suffices to show that

$$(3.1.29)\qquad \sum_{n=n_0}^{n} n^{-1/2}E[|v_n|I(|v_n|\geqq \varepsilon n^{1/2})]<\infty \qquad \text{for all } \varepsilon>0.$$

From (3. 5) and (3. 6) it is clear that

$$\sum_{n=n_0}^{n} n^{-1/2}E[|v_n|I(|v_n|\geqq \varepsilon n^{1/2})]$$

$$\leqq \max\{1, \varepsilon^{-(1+\bar{\eta})}\}\bar{R}\sum_{n=n_0}^{\infty} n^{-(1+\underline{\eta}/2)}<\infty,$$

which concludes (3. 1. 29). Finally we shall show (3. 1. 27). It is clear that

$$(3.1.30)\qquad s_n^{-2}\sum_{m=n_0}^{n} U_m^2 = s_n^{-2}\sum_{m=n_0}^{n}(U_m^2 - E[U_m^2 | \mathscr{F}_{m-1}])$$

$$+s_n^{-2}\sum_{m=n_0}^{n} E[U_m^2 | \mathscr{F}_{m-1}].$$

By (3. 1. 18), (3. 1. 21), (3. 1. 22) and the Toeplitz lemma we have

$$(3.1.31)\qquad s_n^{-2}\sum_{m=n_0}^{n} E[U_m^2 | \mathscr{F}_{m-1}]\to 1 \quad \text{a.s.} \quad \text{as } n\to\infty.$$

Set $v_m' = v_m I(|v_m|\leqq m^{1/2})$ for each $m\geqq n_0$. By making use of the Kolmogorov strong law of large numbers of martingale we shall obtain

$$(3.1.32)\qquad s_n^{-2}\sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2}((v_m')^2 - E[(v_m')^2 | \mathscr{F}_{m-1}])\to 0 \quad \text{a.s.} \quad \text{as } n\to\infty$$

if it holds that

$$(3.1.33)\qquad \sum_{n=n_0}^{\infty} s_n^{-4} n^{-4}\gamma_n^{-4}\,\mathrm{Var}\,[(v_n')^2]<\infty,$$

where Var $[X]$ denotes variance of X. In order to show (3. 1. 33) it suffices, from (3. 1. 22), to show

$$\sum_{n=n_0}^{\infty} n^{-2}E[(v_n')^4]<\infty,$$

which is satisfied by the fact that

$$\sum_{n=n_0}^{\infty} n^{-2}E[(v'_n)^4] \leqq \bar{R} \sum_{n=n_0}^{\infty} n^{-(1+\underline{\gamma}/2)} < \infty.$$

Hence the relation (3. 1. 32) holds. By the Markov inequality we have

$$\sum_{n=n_0}^{\infty} P(v_n \neq v'_n) \leqq \sum_{n=n_0}^{\infty} n^{-(1+\gamma_n/2)} E[|v_n|^{2+\gamma_n}]$$

$$\leqq \bar{R} \sum_{n=n_0}^{\infty} n^{-(1+\underline{\gamma}/2)} < \infty,$$

which implies $P(v_n \neq v'_n \text{ i.o.})=0$. Hence by (3. 1. 32) we get

$$(3.1.34) \quad s_n^{-2} \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2}(v_m^2 - E[(v'_m)^2 | \mathscr{F}_{m-1}]) \to 0 \qquad \text{a.s.} \qquad \text{as } n \to \infty.$$

As $v_m^2 = (v'_m)^2 + v_m^2 I(|v_m| > m^{1/2})$, we get

$$(3.1.35) \quad s_n^{-2} \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2}(v_m^2 - E[v_m^2 | \mathscr{F}_{m-1}])$$

$$= s_n^{-2} \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2}(v_m^2 - E[(v'_m)^2 | \mathscr{F}_{m-1}])$$

$$- s_n^{-2} \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2} E[v_m^2 I(|V_m| > m^{1/2}) | \mathscr{F}_{m-1}].$$

Since

$$\sum_{n=n_0}^{\infty} n^{-1}E[v_n^2 I(|v_n| > n^{1/2})] \leqq \bar{R} \sum_{n=n_0}^{\infty} n^{-(1+\underline{\gamma}/2)} < \infty,$$

it follows from (3. 1. 22) that

$$\sum_{n=n_0}^{\infty} s_n^{-2} n^{-2}\gamma_n^{-2} E[v_n^2 I(|v_n| > n^{1/2})] < \infty,$$

which implies

$$\sum_{n=n_0}^{\infty} s_n^{-2} n^{-2}\gamma_n^{-2} E[v_n^2 I(|v_n| > n^{1/2}) | \mathscr{F}_{n-1}] < \infty \qquad \text{a.s.}$$

Thus, by the Kronecker lemma we obtain

$$s_n^{-2} \sum_{m=n_0}^{n} m^{-2}\gamma_m^{-2} E[v_m^2 I(|v_m| > m^{1/2}) | \mathscr{F}_{m-1}] \to 0 \qquad \text{a.s.} \qquad \text{as } n \to \infty,$$

which, together with (3. 1. 34) and (3. 1. 35), yields

$$(3.1.36) \quad s_n^{-2} \sum_{m=n_0}^{n} (U_m^2 - E[U_m^2 | \mathscr{F}_{m-1}]) \to 0 \qquad \text{a.s.} \qquad \text{as } n \to \infty.$$

Combining (3. 1. 30), (3. 1. 31) and (3. 1. 36) we have (3. 1. 27). It follows from (3. 3) and the Toeplitz lemma that

$$\sum_{j=n_0}^{n} \alpha_1(j) j^{-1} \sim \alpha_1 \log n \qquad \text{as } n \to \infty.$$

After some calculations, using the above result, (3. 1. 22) and the Taylor theorem, we obtain

$$(3.1.37) \quad \log_2 s_n^2 \sim \log_2 n \qquad \text{as } n \to \infty.$$

By (3. 5) we get $\sup_{n \geqq 1} E[v_n^2] < \infty$, which implies that $|v_n| < \infty$ a.s. for all $n \geqq 1$. Hence using (3. 3), (3. 1. 3) and (3. 1. 6) we have

(3. 1. 38) $\sigma^{-1}(2A\alpha_1(n)-1)^{1/2}(2n^{-1}\log_2 n)^{-1/2}\sum_{m=3}^{n_0-1} m^{-1}\beta_{mn} v_m=0$ a.s.

Combining (3. 3), (3. 1. 7), (3. 1. 22), (3. 1. 23), (3. 1. 24), (3. 1. 37), (3. 1. 38) and the fact that $\{\zeta_n'', n\geqq n_0\}$ has its set of a.s. limit points confined to $[-1, 1]$, we obtain (3. 1. 15), (3. 1. 16) and the fact that $\{\zeta_n', n\geqq 3\}$ has its set of a.s. limit points confined to $[-1, 1]$. This completes the proof.

Let

(3. 8) $D_n\geqq D(2n^{-1}\log_2 n)^{1/2}+o((n^{-1}\log_2 n)^{1/2})$,

where b_n's are nonnegative numbers satisfying (3. 1. 1), D_n is defined as (3. 1. 5) and D is any positive number such that

(3. 9) $D>\sigma(2A\alpha_1-1)^{-1/2}$ for σ and α_1 in (3. 3) and (3. 7).

The following theorem can be proved by making use of the relations in the proof of Theorem 3. 1.

THEOREM 3. 2. *Assume the conditions* (3. 1) *to* (3. 9). *Then, under all the conditions of Lemma* 2.3 *except for* (2. 3. 7) *and* (2. 3. 9), *we have*

(3. 2. 1) $X_n-\theta_n\to 0$ *a.s.* *as* $n\to\infty$

and, with probability one, $X_n-\theta_n>0$ *only finitely many time.*

PROOF. By Lemma 2. 3 we obtain (3. 2. 1). Let $Z_n=X_n-\theta_n$. It follows from (3. 1. 4), (3. 1. 8), (3. 1. 10) and (3. 1. 14) that

$$Z_{n+1}=A\sum_{m=3}^{n}\beta_{mn} m^{-1} v_m-AD_n+o((n^{-1}\log_2 n)^{1/2}) \quad \text{a.s.} \qquad \text{for all } n\geqq 3,$$

which, together with (3. 3), (3. 8), (3. 9) and (3. 1. 15), implies

$$\begin{aligned} &P\{Z_n>0 \text{ i.o.}\}\\ &\leqq P\{\limsup_{n\to\infty}(\sigma^{-1}(2A\alpha_1(n)-1)^{1/2})(2n^{-1}\log_2 n)^{-1/2}\sum_{m=3}^{n}\beta_{mn} m^{-1} v_m\\ &\qquad \geqq \limsup_{n\to\infty}(\sigma^{-1}(2A\alpha_1(n)-1)^{1/2})(2n^{-1}\log_2 n)^{-1/2}D_n\}\\ &=P\{1\geqq\sigma^{-1}(2A\alpha_1-1)^{1/2}D\}=0. \end{aligned}$$

Thus $P\{Z_n>0 \text{ i.o.}\}=0$, which concludes that, with probability one, $X_n-\theta_n>0$ occur only finitely many times. This completes the proof.

We shall give an example of $\{b_n\}$ satisfying (3. 8) and (3. 9).

EXAMPLE.

Let

$b_1=b_2=0$

$$b_n = D(2n^{-1} \log_2 n)^{1/2} \qquad \text{for all } n \geqq 3,$$

where D is any positive number such that $D > 3\sigma(2A\,\alpha_1 - 1)^{1/2}/2$ for σ and α in (3. 3) and (3. 7).

We shall, now, consider two cases where ANBAR [1] and DUPAČ [2] treated. First we shall consider the case of ANBAR [1] in which it holds that

$$M_n(x) = M(x) \qquad \text{for all } x \in R \text{ and all } n \geqq 1$$

$$\theta_n = \theta \qquad \text{for all } n \geqq 1 \text{ with } \theta \text{ being the unique root of } M(x)=0.$$

Theorem 3. 2 gives

COROLLARY 3. 3 *Theorem* 2 *of* ANBAR [1] *holds.*

Next, we shall consider the case of DUPAČ [2] in which $Y(x)$ is a random variable for each $x \in R$ with $Y(x)$ being a Borel measurable function on R, the expectation of $Y(x)$, $M(x) = E[Y(x)]$, exists and is a Borel measurable function on R, and

$$M_n(x) = M(x - \theta_n + \theta_1) \qquad \text{for all } n \geqq 1$$

where θ_1 is the unique root of $M(x) = 0$ and $\{\theta_n\}$ is a sequence of real numbers, so θ_n is the unique root of the equation $M_n(x) = 0$.

COROLLARY 3. 4. *Suppose the following conditions:*

(3. 4. 1) $\quad (x - \theta_1) M(x) \geqq 0 \qquad$ *for all* $x \in R$,

(3. 4. 2) $\quad K_1 |x - \theta_1| \leqq |M(x)| \leqq K_2 |x - \theta_1| \qquad$ *for all* $x \in R$ *and some positive constants* K_1 *and* K_2,

(3. 4. 3) $\quad M(x) = \alpha_1(x - \theta_1) + \alpha_2(x - \theta_1)^2 + o((x - \theta_1)^2)$

with $\alpha_1 > 0$ *and* α_2 *being real numbers,*

there exist finite positive constants η *and* σ^2 *such that*

(3. 4. 4) $\quad \sup_{x \in R} E[|V(x)|^{2+\eta}] < \infty$

and

(3. 4. 5) $\quad E[V^2(x)] \to \sigma^2 \quad$ *as* $x \to \theta_1$ *with* $V(x) = M(x) - Y(x)$,

(3. 4. 6) $\quad |\theta_n - \theta_{n+1}| = o(n^{-3/2} (\log_2 n)^{1/2})$,

(3. 4. 7) $\quad a_n = A\,n^{-1}$

where A *is an arbitrary real number with* $2AK_1 > 1$. *Assume that* $\{b_n\}$ *is a sequence of nonnegative numbers satisfying* (3. 1. 1) *and*

$$D_n \geqq D(2n^{-1} \log_2 n)^{1/2} + o((n^{-1} \log_2 n)^{1/2})$$

where D_n *is defined as* (3. 1. 5) *and* D *is any positive number with* $D > \sigma(2AK_1 - 1)^{-1/2}$. *Then Theorem* 3. 2 *holds.*

PROOF. In Theorem 3.1 set $\alpha_1(n)=\alpha_1$, $\alpha_2(n)=\alpha_2$, $\eta_n=\eta$, $V_n(x)=V(x-\theta_n+\theta_1)$ for all $n\geqq 1$. Then all the conditions (3.1) to (3.8) are satisfied. From (3.4.2) and (3.4.3) we get

$$D>\sigma(2AK_1-1)^{-1/2}\geqq\sigma(2A\alpha_1-1)^{-1/2},$$

by which (3.9) is fulfilled. The relation (3.4.1) implies the relation (2.3.1). In Lemma 2.3 set $K_{n1}=K_1$ and $K_{n2}=K_2$ for all $n\geqq 1$. By the Hölder inequality we get

$$E[(M_n(x)-Y_n(x))^2]\leqq\{E[|V(x-\theta_n+\theta_1)|^{2+\eta}]\}^{2/(2+\eta)}.$$

Set $L_n(x)=\{E[|V(x-\theta_n+\theta_1)|^{2+\eta}]\}^{2/(2+\eta)}$ in (2.3.5). Since $l_n\leqq\{\sup_{x\in R}E[|V(x)|^{2+\eta}]\}^{2/(2+\eta)}$, we have (2.3.6) from (3.4.4). Another conditions of Lemm 2.3 are all satisfied. Thus, since all the conditions of Theorem 3.2 are fulfilled, Theorem 3.2 holds. This completes the proof.

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