A note on a geometric version of the Siegel formula for quadratic forms of signature (2, 2k)

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Introduction

In [6], as a special case of Theorem 1, the author proved the following:

Let Q be an even integral quadratic form of signature $(2, q)$ with level N, let D be the symmetric domain attached to Q, and let Γ_Q be the group of proper units for Q. Then there are elliptic cusp forms of weight $(2+q)/2$ with respect to a certain congruence subgroup of level N, whose Fourier coefficients are period integrals of holomorphic q-forms on the modular variety $\Gamma_Q\backslash D$.

In this note, we prove a variant of the Siegel formula, which fits in with the above result:

There is an Eisenstein series of level N of weight $(2+q)/2$, whose Fourier coefficients are period integrals of a certain $\left(\frac{q}{2}, \frac{q}{2}\right)$ type differential form on $\Gamma_Q\backslash D$.

Though our result is substantially a paraphrase of the result of Siegel [10], the author believes that there is some meaning to write it in this form, if one is interested in the Hodge components of the dual cocycles of certain cycles $\Gamma_\ell\backslash X_\ell$ of $\Gamma_Q\backslash D$ obtained by embedding (cf. [5]).

Notation

$\boldsymbol{Z}$ is the ring of integers, and $\boldsymbol{Q}$ (resp. $\boldsymbol{R}$, resp. $\boldsymbol{C}$) is the field of the rational (resp. real, resp. complex) numbers. Throughout in this paper, n denotes the number of the variables of a quadratic form Q, $SO(Q)$ the special orthogonal group for Q over the real number field and G the connected component of the identity of it. We write by D the symmetric space G/K, where K is a maximal compact subgroup of G. If Q is of signature $(2+, q-)$ $(2+q=n)$, then D has a G-invariant complex structure.

§ 1 Geometric Preliminaries

1. 1 *G-invariant forms on D*

Let G be $SO_0(2, q)$ and K a maximal compact subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$, the Lie algebras of G and K, respectively. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{p}$, and let k_0 be an element of the center of K such that $J=ad(k_0)|\mathfrak{p}$ defines the complex structure on $\mathfrak{p}$. Let $\mathfrak{p}^{\pm}$ be the eigenspaces with respect to J in $\mathfrak{p}_{\boldsymbol{C}}=\mathfrak{p}\otimes_{\boldsymbol{R}}\boldsymbol{C}$ with eigenvalues $\pm\sqrt{-1}$, respectively.

Let us consider the bigraded relative cohomology $H^{p,q}(\mathfrak{g}, \mathfrak{k}: \boldsymbol{C})$ of $\mathfrak{g}$ mod $\mathfrak{k}$, with coefficients in a trivial $\mathfrak{g}$-molule $\boldsymbol{C}$ (cf. [2], Chap. I § 1, and Chap. II § 2).

Then the space of G-invariant (p, q) type differential forms on the Hermitian symmetric domain $D=G/K$, is isomorphic to $H^{p,q}(\mathfrak{g}, \mathfrak{k}; \boldsymbol{C})$ (cf. [2], Chap. I § 1.6). Considering similarly for the compact dual $\mathfrak{g}^c$ of $\mathfrak{g}$ defined by

$$\mathfrak{g}^c=\mathfrak{k}\oplus\sqrt{-1}\,\mathfrak{p},$$

and noting a natural isomorphism

$$H^{p,q}(\mathfrak{g}, \mathfrak{k}; \boldsymbol{C})\xrightarrow{\sim}H^{p,q}(\mathfrak{g}^c, \mathfrak{k}; \boldsymbol{C}),$$

we can show that the space of G-invariant (p, q) type differential forms on D, is isomorphic to the space of G^c-invariant (p, q) type differential forms on the compact Hermitian symmetric space $V=G^c/K$, where G^c is the compact dual of G. Moreover, by the theorem of de Rham, the latter space is isomorphic to the (p, q) type cohomology group $H^{p,q}(V)$ of the projective variety V.

Since V is a complex quadric in our case, a result of Cartan [3] shows the following proposition.

PROPOSITION (1. 1).

There exists (1, 1) *type G-invariant real form ω on D, which is unique up to constant multiple. Moreover if we put $q=2k$, there exist two linearly independent G-invariant real (k, k) type forms ξ and η on D, such that*

$$\xi=\overset{k}{\wedge}\omega,\ \eta\wedge\eta=(-1)^k\,\xi\wedge\xi,\ \textit{and}\ \xi\wedge\eta=0.$$

1. 2 *Totally real submanifolds X_ℓ*

Let Q be a real quadratic form on $L_{\boldsymbol{R}}=\boldsymbol{R}^n$, of signature $(2+,\ q-)$ with $q=n-2$, and let G be the connected component of the identity in the real orthogonal group $SO(Q)$.

Let ω, ξ, and η be the differential forms on $D=G/K$ obtained by Proposition (1. 1) of the previous section. Let ℓ be an element of $L_{\boldsymbol{R}}$ with $Q(\ell)>0$. Put $G_\ell=\{g\in G \mid g(\ell)=\ell\}$. Then, by the theorem of Witt, G_ℓ is isomorphic to $SO_0(1, q)$ for any ℓ with $Q(\ell)>0$. Choose a point x of D, and define X_ℓ as G_ℓ-orbit of x: $X_\ell=G_\ell(x)$. Let K_x be the isotropy subgroup of G at x. Then

$$X_\ell \simeq G_\ell/G_\ell\cap K_x\cong SO_0(1, q)/SO(q).$$

Hence X_ℓ is also a symmetric space of $\boldsymbol{BD}$ type of real dimension q.

LEMMA (1. 2).

Restrict ω, ξ, and η to X_ℓ. Then we have

$$\omega|_{X_\ell}=0,\ \textit{accordingly}\quad \xi|_{X_\ell}=0,$$

and $\eta|_{X_\ell}$ is an everywhere non-vanishing G_ℓ-invariant q-form on X_ℓ. Especially $\eta|_{X_\ell}$

defines a G_ℓ-invariant measure on X_ℓ.

PROOF. Evidently, the G-invariance of ω, ξ, and η implies the G_ℓ-invariance of $\omega|X_\ell$, $\xi|X_\ell$, and $\eta|X_\ell$, respectively. Therefore it suffices to show the vanishing or non-vanishing of $\omega|X_\ell$, $\xi|X_\ell$, and $\eta|X_\ell$ at one point x of D.

Fix a maximal compact subgroup K of G. Let $\mathfrak{p}_\ell$ and $\mathfrak{k}_\ell$ be the Lie algebras of G_ℓ and $G_\ell \cap K$, respectively. It is easy to check that there exist Cartan decompositions:

$$\mathfrak{g}_\ell = \mathfrak{k}_\ell \oplus \mathfrak{p}_\ell, \text{ and } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

such that $\mathfrak{k}_\ell \subset \mathfrak{k}$ and $\mathfrak{g}_\ell \subset \mathfrak{p}$. And we can easily verify that

$$\mathfrak{p}_\ell \cup J\mathfrak{p}_\ell = \{0\}, \text{ and } \mathfrak{p}_\ell \oplus J\mathfrak{p}_\ell = \mathfrak{p},$$

for the complex structure J (cf. [4], pp. 238–239, and 527). Therefore X_ℓ is a totally real submanifold. Namely X_ℓ is defined in D, locally by the equations:

$$Im\ x_1 = 0, \ldots\ldots, Im\ z_q = 0,$$

where $(z_1, \ldots\ldots, z_q)$ is a holomorphic local coordinate at a point $x \in X_\ell$ of D.

Take a basis of $\mathfrak{p}_C^+$ and ist conjugate basis of $\mathfrak{p}_C^-$, such that with respect to these bases the action of the factor $SO(q)$ of $K \simeq SO(2) \times SO(q)$ is given by the standard one, and the action of the $SO(2)$ factor of K is given by multiplication by exp $(i\theta)$ $(\theta \in \boldsymbol{R})$.

Let $\{dz_1, \ldots\ldots, dz_q\}$ be a basis of the holomorphic cotangent space of D at x, corresponding to the dual basis of the above basis of $\mathfrak{p}_C^+$ by means of the natural identification of $\mathfrak{p}_C^+$ with the holomorphic tangent space at x. Then ω is represented locally at x by

$$\omega = (\text{a real number}) \times \sqrt{-1}\{dz_1 \wedge d\bar{z}_1 + \ldots + dz_q \wedge d\bar{z}_q\}.$$

Along X_ℓ, we have $d\bar{z}_1 = dz_1, \ldots, d\bar{z}_q = dz_q$, because X_ℓ is totally real. Hence $\omega|X_\ell = 0$ at x, and accordingly over all X_ℓ. Consequently, $\xi|X_\ell = \overset{k}{\wedge}\omega|X_\ell = \overset{k}{\wedge}(\omega|X_\ell) = 0$. It is easy to show that a local presentation of η is given by

$$\eta = (\text{a real number}) \times \sum_{\substack{I \cup J = \{1, \ldots, q\}, \\ I \cap J = \phi}} \omega_{IJ}\, dz_I \wedge d\bar{z}_J,$$

where the summation runs over all partitions of the set $\{1, \ldots, q\}$ into two ordered subsets I and J with cardinality $\frac{q}{2} = k$, and dz_I (resp. $d\bar{z}_J$) is defined by

$$dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \ldots \wedge dz_{i_k} \text{ (resp. } d\bar{z}_J = d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \ldots \wedge d\bar{z}_{j_k})$$

for $I = (i_1, i_2, \ldots i_k)$(resp. $J = (j_1, j_2, \ldots, j_k)$). Moreover $\omega_{IJ} = \text{sgn}\begin{pmatrix} 1, \ldots, k, k+1, \ldots, q \\ i_1, \ldots, i_k, j_1, \ldots, j_k \end{pmatrix}$ is the signature of permutation. (See [3].) We can choose a local coordinate $\{x_1, x_2, \ldots, x_q\}$ of X_ℓ such that $dz_i = d\bar{z}_i = dx_i$ $(i = 1, \ldots, q)$. Hence

$$\eta = (\text{an non-zero real number}) \times dx_1 \wedge dx_2 \ldots \wedge dx_q \neq 0,$$

which completes the proof of the lemma.

REMARK (1.3). *Put* $\Gamma_\ell = \Gamma_Q \cap G_\ell$.

$\eta|X_\ell$ is G_ℓ-invariant, and *a fortiori* Γ_ℓ-invariant. Thus an integral

$$\int_{\Gamma_\ell \backslash X_\ell} \eta$$

is defined. On the other hand, the groups G_ℓ and $G_\ell \cap K$ are unimodular, because G_ℓ is semisimple and $G_\ell \cap K$ is compact. Therefore there exists a G_ℓ-invariant measure on X_ℓ, which is unique up to constant multiple. Accordingly, it is a constant multiple of the measure which is obtained by means of the differential form $\eta|X_\ell$. Thus

$$\int_{\Gamma_\ell \backslash X_\ell} \eta$$

is a constant multiple of the volume of $\Gamma_\ell \backslash X_\ell$ with respect to a G_ℓ-invariant measure, which is known to be finite (cf. Siegel [13], for example).

§ 2 The main result

Let Q be an even integral quadratic form on $L = \boldsymbol{Z}^n$, with signature $(2+, 2k-)$ ($n = 2+2k$, and $k \geq 1$). And when $k=1$, we assume that Q is not a kernel form. Let h be an element of L^*, the dual latlice of L. For any rational number $r = a/b$ with $(a, b) = 1$ and $b > 0$, we put

$$\gamma_h(r) = i^{(1-k)} |\det Q|^{-1/2} b^{-n/2} \sum_{\ell \in bL/L} \exp[\pi i Q(\ell + h)].$$

Let $M_w(\Gamma(N))$ be the space of elliptic cusp forms of weight w with respect to the principal congruence subgroup $\Gamma(N)$, and let $M_w(\Gamma_0(N), (\frac{\Delta_Q}{}))$ be the space of elliptic cusp forms of weight w with respect to $\Gamma_0(N)$ with the multiplicator $(\frac{\Delta_Q}{})$. Then we have the following theorem

THEOREM (*Siegel formula*).

For any point z of the complex upper half plane H, we have

$$E^0{}_{k+1}(z; h) = \delta_h + c(\eta) \sum_{\substack{\ell \in L^* \bmod \Gamma_Q, \\ \ell \equiv h \bmod L,\, Q(\ell) > 0}} \left\{ \int_{\Gamma_\ell \backslash Z_\ell} \eta \right\} \exp[\pi i Q(\ell) z]$$

$$= \delta_h + \sum_r \gamma_h(r)(z-r)^{-n/2}.$$

And $E^0{}_{k+1}(z; h)$ *defined by this equality belongs to* $M_{k+1}(\Gamma(N))$, *where N is the level of Q. Especially when* $h \in L$, *the Eisenstein series* $E^0{}_{k+1}(z; 0)$ *belongs to* $M_{k+1}(\Gamma_0(N), (\frac{\Delta_Q}{}))$. *Here the summation is taken over all rational numbers r. (When* $n=4$, *the sun of r's with fixed denominator is considered first, and next we move denominators.) The number* $c(\eta)$ *is a constant depending only on* η, $\delta_h = 1$ *or* 0, *according as* $h \in L$ *or* $h \notin L$.

2. 1. *Weil Representation.*

First we re call basic facts on Weil representation (cf. Weil [14])

Let Q be a quadratic form defined over $\boldsymbol{R}^n$, of signature $(p+, q-)$. We assume that q is even, and for simplicity that p is also even. Then n is even.

Since n is even, in view of the results of Saito [7] on a trivialization of projective Weil representation of $SL_2(\boldsymbol{R})$, we can define a representation w_Q of $SL_2(\boldsymbol{R})$ on the Schwartz-Bruhat space $S(\boldsymbol{R}^n)$ over $\boldsymbol{R}^n$ as follows.

For any f of $S(\boldsymbol{R}^n)$, we put

$$[w_Q\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) f](x)=a^{n/2} f(ax) \qquad (a\in\boldsymbol{R}^{\times},\ x\in\boldsymbol{R}^n),$$

$$[w_Q\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) f](x)=\exp[\pi i Q(x)b] f(x) \qquad (b\in\boldsymbol{R}),$$

$$[w_Q\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) f](x)=i^{(p-q)/2}|\det Q|^{n/2}\int_{\boldsymbol{R}^n}\exp[2\pi i(x, y)_Q] f(y)dy,$$

where dy is the usual Lebesgue measure on $\boldsymbol{R}^n$, and $(x, y)_Q$ is a symmetric bilinear form associated to Q defined by

$$(x, y)_Q=\frac{1}{2}Q(x+y)-\frac{1}{2}Q(x)-\frac{1}{2}Q(y)$$

for any $x, y\in\boldsymbol{R}^n$.

Moreover, if $c\neq 0$, we have

$$[w_Q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f](x)=(-i)^{(p-q)/2}(sgn\ c)^{(p-q)/2}|c|^{-n/2}|\det Q|^{1/2}$$
$$\int_{\boldsymbol{R}^n}\exp[\pi i\{ac^{-1}Q(\mathrm{x})-2c^{-1}(x, y)_Q+c^{-1}dQ(y)\}] f(y)dy$$

2. 2. *Poisson summation formula*

We fix a quadratic form Q on $L=\boldsymbol{Z}^n$ with values in even integers $2\boldsymbol{Z}$, of signature $(p+, q-)$. Put $L=\boldsymbol{Z}^n$ and

$$L^*=\{h\in\boldsymbol{Q}^n\,|\,(h, l)_Q\in\boldsymbol{Z} \text{ for all } l\in\boldsymbol{Z}^n\}.$$

Then L is a subgroup of finite index in L^*.

Let h be an element of L^*. Then we put

$$\theta((f, h)=\sum_{\ell\in L} f(\ell+h)$$

for any function $f\in S(\boldsymbol{R}^n)$. Especially when $h\in L$, we write

$$\theta(f)=\sum_{\ell\in L} f(\ell).$$

PROPOSITION (2. 1).

(i) *For any* $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z})$, *we have*

$$\theta(w_Q(\gamma)f, h)=\sum_{K\in L^*/L} c(h, k)_\gamma\, \theta(f, k),$$

where $c(h, k)_\gamma$ *is a constant given by*

$$c(h, k)_\gamma=\begin{cases}\delta_{k,\, ah}a^{n/2}\exp[\pi iabQ(h)], \text{ if } c=0,\\ (-i)^{(p-q)/2}(\operatorname{sgn} c)^{(p-q)/2}|\det Q|^{-1/2}|c|^{-n/2}\\ \qquad\sum_{r\in L/cL}\exp[\pi i\{ac^{-1}Q(h+r)-2c^{-1}(h+r, k)_Q+c^{-1}dQ(k)\}], \text{ if } c\neq 0.\end{cases}$$

Here $\delta_{*,\,*}$ *is the Kronecker delta function defined on* L^*/L.

(ii) *Let* N *be the level of* Q. *Namely,* N *is the least positive integer such that* NQ^{-1} *is also a quadratic form on* L^* *with values in even integers (especially we have* $NL^*\subset L$).

Then for $\gamma=\begin{pmatrix} a & c\\ b & d\end{pmatrix}\in SL_2(\mathbf{Z})$ *with* $c\equiv 0 \bmod N$,

$$c(h, k)_\gamma=\begin{cases}\delta_{k,\, ah}\left(\frac{\Delta_Q}{d}\right)\exp[\pi iabQ(h)], \text{ if } d>0,\\ \delta_{k,\, ah}(-1)^s\left(\frac{\Delta_Q}{|d|}\right)\exp[\pi iabQ(h)], \text{ if } d<0.\end{cases}$$

Here $s=(p-q)/2$, $\Delta_Q=(-1)^{n/2}|\det Q|=(-1)^{n/2}\det Q$, *and* $\left(\frac{\Delta_Q}{*}\right)$ *is the Jacobi symbol.*

Proof. The part (*i*) is identical with the part (*i*) of Proposition 1.6 of Shintani [9]. (Note that our $c(h, k)_\gamma$ is different from $c(h, k)_\gamma$ of [9].) So we omit the proof.

Now we show the part (ii). Following the argument of [9], we can check that $c(h, k)_\gamma=0$ if $k\not\equiv ah \bmod L$. Let us calculate $c(h, k)_\gamma$. Put $k=ah$, and assume $c\neq 0$. Then, in view of $ad-bc=1$, we have

$$ac^{-1}Q(h+r)-2c^{-1}(h+r, k)_Q+c^{-1}dQ(k)=ac^{-1}Q(r)+abQ(h).$$

Therefore, the formula of $c(h, k)_\gamma$ of the part (*i*) reads

$$c(h, ah)_\gamma=(-i)^s(\operatorname{sign} c)^s|\det Q|^{-1/2}|c|^{-n/2}\sum_{r\in L/cL}\exp[\pi iabQ(h)]\sum_{r\in L/cL}\exp[\pi i\{\left\{\frac{a}{c}\right\}Q(r)\}].$$

By the reciprocity law of Gaussian sum (Satz 2 of Siegel [12]),

$$(-i)^s(\operatorname{sgn} c)^s|\det Q|^{-1/2}|c|^{-n/2}\sum_{r\in L/cL}\exp\left[\pi i\left\{\frac{a}{c}Q(r)\right\}\right]$$

$$=|\det Q|^{-1}(\operatorname{sgn} a)^s|a|^{-n/2}\sum_{r\in L/aQL}\exp\left[\pi i\left\{-\frac{c}{a}Q^{-1}(r)\right\}\right].$$

Here Q of L/aQL is a matrix representing the bilinear form $(\ ,\ )_Q$.
Since $(a, N)=(a, \det Q)=1$, we can write $r=Qr_1+ar_2$ with $r_1\in L/aL$ and $r_2\in L/QL$. Therefore the last sum of the above formula is reduced to

$$\sum_{r_1\in L/aL}\exp\left[\pi i\left(-\frac{c}{a}\right)Q(r_1)\right]\sum_{r_2\in L/QL}\exp[\pi i(-ca)Q^{-1}(r_2)],$$

which is equal to

$$|\det Q| \sum_{r_1 \in L/aL} \exp\left[\pi i\left(-\frac{c}{a}\right)Q(r_1)\right].$$

because $cQ^{-1}(r_2)$ is even integer by the assumption $N|c$.

Thus

$$c(h, ah)_r = (sgn\ a)^s |a|^{-n/2} \sum_{r_1 \in L/aL} \exp\left[\pi i\left(-\frac{c}{a}\ Q\right)(r_1)\ \right].$$

A standard calculation of the last Gaussian sum shows the part (ii) of the proposition.

2. 3. *Intertwining property of certain test functions*

Let Q be the quadratic form considered in the previous section. Let R be a minimal majorant of Hermite of Q. We denote by H (resp. H^-) the complex upper (resp. lower) half plane. For any points $z \in H$ and $\zeta \in H^-$, we define a function $f(x;\ z,\ \zeta;\ Q,\ R)$ of $S(\boldsymbol{R}^n)$ by

$$f(x;\ z,\ \zeta;\ Q,\ R) = \exp\left[\pi i\left\{Q(x)\frac{z+\zeta}{2} + R(x)\frac{z-\zeta}{2}\right\}\right].$$

where $x \in \boldsymbol{R}^n$. Then f has the following intertwining property.

Proposition (2. 2).

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{R})$,

$$[w_Q(g)f](x;\ z,\ \zeta;\ Q,\ R) = (cz+d)^{-p/2}(c\zeta+d)^{-q/2} f(x;\ \frac{az+b}{cz+d},\ \frac{a\delta+b}{c\delta+d};\ Q,\ R).$$

Proof. First, note that it suffices to check this for the generators of $SL_2(\boldsymbol{R})$. For $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, the equality of our proposition follows immediately from the definition of w_Q. Let us discuss the case $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Both sides are holomorphic in z and ζ. Therefore it suffices to show our equality when $z = iy$ and $\zeta = -iv$ $(y,\ v > 0)$. Moreover, by applying a linear transformation on the variables x, we may assume that

$$Q(x) = x_1^2 + x_2^2 + \dots\ x_p^2 - x_{p+1}^2 - \dots - x_n^2,$$

and

$$R(x) = x_1^2 + x_2^2 + \dots + x_p^2 + x_{p+1}^2 + \dots + x_n^2.$$

Then

$$f(x;\ z,\ \zeta;\ Q\ R) = \exp[-\pi \{ \sum_{i=l}^{p} x_i^2 \} y - \pi \{ \sum_{i=p+1}^{n} x_i^2 \} v].$$

In this case, the formula in question is well-known.

2. 4. *Theta transformation formulae*

Let Q be a quadratic form given in Section 2. 2, and let R be a minimal majorant of

Q. For $z\in H$, and $h\in L^*/L$, we put

$$\theta(z; Q, R; h)=\sum_{\substack{\ell\in L^*\\ \ell\equiv h \bmod L}} f(\ell; z, \bar{z}; Q, R)$$

$$=\sum_{\substack{\ell\in L^*\\ \ell\equiv h \bmod L}} \exp\left[\pi i\left\{Q(\ell)\frac{z+\bar{z}}{2}+R(\ell)\frac{z-\bar{z}}{2}\right\}\right].$$

Since $f(x; z, \bar{z}; Q, R)$ belongs to $S(\boldsymbol{R}^n)$ as a function in x, $\theta(z; Q, R; h)$ converges absolutely on any compact subset of H.

Combining Proposition (2. 1) with Proposition (2. 2), we have the following proposition.

PROPOSITION (2. 3). (i) *For any* $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$, *we have*

$$\theta\left(\frac{az+b}{cz+d}; Q, R; h\right)=(cz+d)^{p/2}(c\bar{z}+d)^{q/2}\sum_{k\in L^*/L} c(h, k)_\gamma\, \theta(z; Q, R; k),$$

where $c(h, k)_\gamma$ *is a constant depending only on* h, k, *and* γ, *which is defined in Proposition* (2. 1).

(ii) *Especially when* $h=0$, *for any* $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$ *with* $c\equiv 0 \bmod N$, *we have*

$$\theta\left(\frac{az+b}{cz+d}; Q, R; 0\right)=(cz+d)^{p/2}(c\bar{z}+d)^{q/2}\varepsilon_Q(d)\theta(z; Q, R; 0),$$

where

$$\varepsilon_Q(d)=\begin{cases}\left(\dfrac{\Delta_Q}{d}\right), & \text{if } d>0,\\[2ex] (-1)^s\left(\dfrac{\Delta_Q}{|d|}\right), & \text{if } d<0.\end{cases}$$

2. 5. *The Siegel formula*

Let X be the space of all minimal majorants of Q, on which G acts transitively. Then X is isomorphic to $G/K\simeq SO_0(p, q)/SO(p)\times SO(q)$, where K is a maximal compact subgroup of G. Thus X is a symmetric space of BD type (cf. Borel [1], Helgason [4]).

Now let us recall the main result of Siegel [11]. Let $d\mu(R)$ be a G-invariant measure on X, which is unique up to constant multiple, where R is a point of X. Let v be the volume

$$\int_{\Gamma_Q\backslash X} d\mu(R),$$

which is known to be finite.

In [11], Siegel considered the integral:

$$I_\theta(z; h)=v^{-1}\int_{\Gamma_Q\backslash X}\theta(z; Q, R; h)d\mu(R),$$

and proved the following formula.

THEOREM (2. 4). (Satz 1 of [11])

$$I_\theta(z;\, h)=\delta_h+\sum_r \gamma_h(r)(z-r)^{-p/2}(\bar{z}-r)^{-q/2}.$$

Here r is taken over all rational numbers. When $n=4$, the summation is first taken over r with a fixed denominator b, and next the sum over all positive b.

2. 6. *A differential operator*

For any differentiable function $F(z)$ on H, we define a differential operator K of Siegel [10] by

$$\{K(F)\}(z)=(z-z)^{1-p/2}\left(\frac{\partial}{\partial z}\right)^{q/2}\{(z-\bar{z})^{n/2-1}F(z)\}.$$

LEMMA (2. 5).

Let Δ be a subgroup of $SL^2(\boldsymbol{R})$. Assume that a differentiable function $F(z)$ on H satisfies

$$F\left(\frac{az+b}{cz+d}\right)=(cz+d)^{p/2}(c\bar{z}+d)^{q/2}F(z)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Δ. Then the function $K(F)$ satisfies

$$\{K(F)\}\left(\frac{az+b}{cz+d}\right)=(cz+d)^{n/2}\{K(F)\}(z)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Δ.

PROOF. For any $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{R})$, we put

$$\varphi(g)=(ci+d)^{-p/2}(-ci+d)^{-q/2}F\left(\frac{ai+b}{ci+d}\right).$$

Then $\varphi(g)$ is a left Δ-invariant function on $SL_2(\boldsymbol{R})$, and satisfies

$$\varphi((g\cdot r(\theta))=\varphi(g)\exp[i\nu\theta],$$

for any $\theta\in\boldsymbol{R}$, where

$$r(\theta)=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and $\nu=(p-q)/2$.

Put

$$g_z=\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}\in SL_2(\boldsymbol{R}), \quad (z=x+\sqrt{-1}\,y)$$

and $f(z)=\varphi(g_z)$.

Then we have

$$f(z)=y^{(p+q)/4}F(z).$$

Now we consider a differential operator $\overline{W}$ corresponding to an element $\begin{pmatrix} 1 & +i \\ +i & -1 \end{pmatrix}$ of the Lie algebra $\mathfrak{ls}_2(\boldsymbol{C})$, which acts on differentiable functions on $SL_2(\boldsymbol{R})$ (cf. Weil [15], p. 76). Then by a result of [15] (see, p. 81), $(\overline{W}^r\varphi)(g_z)$ is given by

$$(4i)^r y^{1-\nu/2} \frac{\partial^r}{\partial z^r}[y^{r-1+\nu/2} f]$$

$$=(4i)^r y^{1-\nu/2} \frac{\partial^r}{\partial z^r}[y^{r-1+p/2} F].$$

Noting that $(W^r\varphi)(g)$ satisfies

$$(\overline{W}^r\varphi)(g\cdot r(\theta))=(\overline{W}^r\varphi)(g)\exp[i(\nu+2r)\theta],$$

for any $\theta\in\boldsymbol{R}$.

Put $r=q/2$. Then

$$(\overline{W}^r\varphi)(g\cdot r(\theta))=(\overline{W}^r\varphi)(g)\exp\left[i\left(\frac{n}{2}\right)\theta\right].$$

Therefore the function

$$y^{-n/4}(\overline{W}^r\varphi)(g_z),$$

which is a constant multiple of $K(F)$, satisfies the conclusion of our proposition.

2. 7. *Proof of the main theorem.*

Assume that $p=2$ and $q=2k$. Let us recall the integral $I_\theta(z;h)$. Then,

$$K(\delta_h)=\delta_h(z-\bar{z})^{1-p/2}\left(\frac{\partial}{\partial z}\right)^{q/2}(z-\bar{z})^{n/2-1}$$

$$=\delta_h\cdot k!$$

and

$$K((z-r)^{-p/2}(\bar{z}-r)^{-q/2})=(\bar{z}-z)^{1-p/2}\left(\frac{\partial}{\partial z}\right)^{q/2}\left\{\frac{(z-\bar{z})^{n/2-1}}{(z-r)^{p/2}(\bar{z}-r)^{q/2}}\right\}$$

$$=k!(z-r)^{-n/2}.$$

Therefore, we have

$$\frac{1}{k!}K(I_\theta(z;h))=\delta_h+\sum_r \gamma_h(r)(z-r)^{-n/2}.$$

Let us evaluate the integral $I_\theta(z;h)$ in a different way. Fix a minimal majorant R_0 of Q, and let K be the maximal compact subgroup of G which is the stabilizer of R_0 in G. Let us identify $X=D$ with G/K and normalize the Haar measures of G and K by $dg=d\mu(R)dk$ and $\int_K dk=1$. Then,

$$I_\theta(z;\,h)\equiv \upsilon^{-1}\int_{\Gamma Q\backslash X}\sum_{\substack{\ell\in L^*\\ \ell\equiv h \bmod L}}\exp\Big[\pi i\Big\{Q(\ell)\frac{z+\bar z}{2}+R(\ell)\frac{z-\bar z}{2}\Big\}\Big]d\mu(R)$$

$$=\upsilon^{-1}\sum_{\substack{\ell\in L^* \bmod L_Q\\ \ell\equiv h \bmod L}}\exp\,[\pi i Q(\ell)x]\int_{\Gamma\ell\backslash G\ell} dg_\ell\int_{G\ell\backslash G/K}d\dot g\,\exp[-\pi R_0(\dot g^{-1}(\ell))y],$$

where $z=x+iy$, and $d\dot g_\ell$ and dg are the Haar measure of G_ℓ and the quasi-invariant measure on $G_\ell\backslash G$ such that $dg=dg_\ell\,d\dot g$.

Put

$$I_\ell=\exp[\pi i\,Q(\ell)x]\int_{\Gamma\ell\backslash G\ell}dg_\ell\int_{G\ell\backslash G/K}d\dot g\,\exp[-\pi R_0(\dot g^{-1}(\ell))y]$$

for each $\ell\in L^*$, and let us evaluate it.

Case (*i*). When $\ell=0$, $G_\ell=G$ and $I_\ell=\upsilon^{-1}\upsilon=1$.

Cace (*ii*). When $\ell\neq 0$ and $Q(\ell)=0$, we have

$$I_\ell=c(\ell)(z-\bar z)^{-k}$$

with some constant $c(\ell)$ depending on ℓ.

Case (*iii*). When $Q(\ell)<0$, we have

$$I_\ell=c'(\ell)(z-\bar z)^{-1}\exp\,[\pi i\,Q(\ell)\,\bar z\,]$$

by a computation similar to that of case (iv). Here $c'(\ell)$ is a constant depending on ℓ.

Case (*iv*). When $Q(\ell)>0$, we consider

$$J_\ell=\int_{\Gamma\ell\backslash G\ell}\Big\{\int_{G\ell\backslash G/K}\exp[-\pi(R_0-Q)(\dot g^{-1}(\ell))y]d\dot g\Big\}dg_\ell\,.$$

Then $I_\ell=J_\ell\,\exp\,[\pi i Q(\ell)z]$. By a computation similar to (4. 36)–(4. 43) of § 4 of [6], we have

$$\int_{G\ell\backslash G/K}\exp\,[-\pi(R_0-Q)(\dot g^{-1}(\ell))y]d\dot g=cQ(\ell)^{-k}y^{-k},$$

with a constant c independent of ℓ. On the other hand, we can check readily that

$$\int_{\Gamma\ell\backslash G\ell}dg_\ell=c'\int_{\Gamma\ell\backslash X\ell}\eta$$

with a constant c' independent of ℓ, comparing the measure forms of the Lie algebras of G, G_ℓ, K, and $K\cap G_\ell$. Hence

$$I_\ell=c''Q(\ell)^{-k}(z-\bar z)^{-k}\Big(\int_{\Gamma\ell\backslash X\ell}\eta\Big)\exp[\pi i\,Q(\ell)z].$$

Now, apply the operator K to each I_ℓ. Then $K(I_\ell)=0$, if $Q(\ell)<0$, or if $Q(\ell)=0$ with $\ell\neq 0$. If $Q(\ell)>0$, we have

$$K(I_\ell)=c(\eta)k!\Big\{\int_{\Gamma\ell\backslash X\ell}\eta\Big\}\exp\,[\pi i\,Q(\ell)z].$$

Therefore, after justification of the change of order of summation and integration in the same way as in [6], we obtain our main theorem.

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