On an automorphism of an *O-space

By

Kichiro TAKAMATSU

(Received December 15, 1964)

S. Tachibana [4] proved that in a compact 2n (n>2) dimensional conformally flat K-space, an almost-analytic transformation is an automorphism and a 2n (n>1) dimensional K-space of positive constant curvature cannot admit a non-trivial automorphism provided that n≠3. In this paper, we shall prove that in a compact 2n (n>1) dimensional *O-space of positive constant curvature, an almost-analytic transformation is an automorphism. An *O-space is a more general almost-Hermitian space than a K-space and its example was given by K. Yano and T. Sumitomo [6].

In §1 we shall give some definitions and propositions. In §2 we shall give some identities in an *O-space. In §3 we shall prepare some lemmas on contravariant almost-analytic vectors in an *O-space. The last §4 will be devoted to discussions on an automorphism of an *O-space of positive constant curvature.

1. Preliminaries

Let X_{2n} be a 2n dimensional almost-Hermitian space which admits an almost-complex structure $\varphi_j{}^i$ and a positive definite Riemannian metric tensor g_{ji} satisfying

$$(1.1)\qquad \varphi_l{}^i\varphi_j{}^l=-\delta_j^i$$

$$(1.2)\qquad g_{lt}\varphi_j{}^l\varphi_i{}^t=g_{ji}.$$

Then from (1.1) and (1.2), we have

$$(1.3)\qquad \varphi_{ji}=-\varphi_{ij}$$

where $\varphi_{ji}=\varphi_j{}^l g_{li}$.

Now in an almost-Hermitian space X_{2n}, we introduce the operators

$$O_{ih}^{ml}=\frac{1}{2}(\delta_i^m\delta_h^l-\varphi_i{}^m\varphi_h{}^l),\quad {}^*O_{ih}^{ml}=\frac{1}{2}(\delta_i^m\delta_h^l+\varphi_i{}^m\varphi_h{}^l)$$

and a tensor is called pure (hybrid) in two indices, if it is annihilated by transvection of *O (O) on these indices.

We have easily the following propositons which will be used in every calculation concerning with purity and hybridity.

PROPOSITION 1. ${}^*O_{ih}^{ab}\nabla_j\varphi_{ab}=0,\quad O_{ib}^{ah}\nabla_j\varphi_a{}^b=0$

where ∇_j *denotes the operator of covariant derivative with respect to the Riemannian connection.*

PROPOSITION 2. *For two tensors* T_{ji} *and* S^{ji}, *if* T_{ji} *is pure in* j, i *and* S^{ji} *is hybrid in* j, i, *then* $T_{ji}\ S^{ji}$ *vanishes.*

PROPOSITION 3. *If* $T_j{}^i$ *is pure (hybrid) in* j, i, *then we have*

$$\varphi_t{}^i T_j{}^t = \varphi_j{}^t T_t{}^i \qquad (\varphi_t{}^i T_j{}^t = -\varphi_j{}^t T_t{}^i).$$

If S^{ji} *is pure (hybrid) in* j, i, *then we have*

$$\varphi_t{}^j S^{ti} = \varphi_t{}^i S^{jt} \qquad (\varphi_t{}^j S^{ti} = -\varphi_t{}^i S^{jt}).$$

2. *O-spaces

An almost-Hermitian space X_{2n} is called an *O-space, if it satisfies

$$(2.\ 1) \qquad {}^*O\,{}^{ab}_{ji}\,\nabla_a \varphi_{bh} = 0^{1)}$$

and it is called a K-space, if it satisfies

$$\nabla_j \varphi_{ih} + \nabla_i \varphi_{jh} = 0.$$

It is easy to show that a K-space is an *O-space.

From (2.1), we have easily

$$(2.\ 2) \qquad \nabla_j \varphi_i{}^j = 0.$$

Let $R_{kji}{}^h$ and $R_{ji} = R_{tji}{}^t$ be Riemannian curvature tensor and Ricci's tensor respectively.

Assuming we are in an *O-space and putting

$$R^*{}_{ji} = \tfrac{1}{2}\varphi^{ab} R_{abti} \varphi_j{}^t,$$

then by the Ricci's identity and (2.2) we get the following

$$(2.\ 3) \qquad \nabla^r \nabla_j \varphi_r{}^i = \tfrac{1}{2}\varphi^{ab} R_{abj}{}^i + R_j{}^r \varphi_r{}^i$$

where $\nabla^r = g^{tr}\nabla_t$ and $\varphi^{ab} = g^{ta}\varphi_t{}^b$.

Transvecting (2.3) with φ_{ik} and taking account of (1.1) we have

$$(2.\ 4) \qquad \varphi_{ik} \nabla^r \nabla_j \varphi_r{}^i = R^*{}_{kj} - R_{jk}.$$

3. Contravariant almost-analytic vectors

In an almost-Hermitian space a contravariant vector v^i is called almost-analytic if it satisfies

$$\underset{v}{\pounds}\, \varphi_j{}^i = v^t \nabla_t \varphi_j{}^i - \varphi_j{}^t \nabla_t v^i + \varphi_t{}^i \nabla_j v^t = 0^{2)}$$

1) S. Koto [1].

2) S. Tachibana [3].

where $\underset{v}{\pounds}$ is the operator of Lie derivative. This is a generalization of the notion of contravariant analytic vectors in a Kählerian space.

The above equation is equivalent to

$$v^t\nabla_t\varphi_{ji}-\varphi_j{}^t\nabla_t v_i-\varphi_i{}^t\nabla_j v_t=0 \tag{3.1}$$

where $v_i=g_{it}v^t$.

Interchanging j and i in (3.1) and adding the equation thus obtained to (3.1), we get

$$\nabla_j v_i+\nabla_i v_j-\varphi_j{}^a\varphi_i{}^b(\nabla_a v_b+\nabla_b V_a)=0 \tag{3.2}$$

i.e.

$$O_{ji}^{ab}\,(\nabla_a v_b+\nabla_b v_a)=0. \tag{3.3}$$

In an almost-Hermitian space we know the following

LEMMA 3.1.[3] *In a compact Einstein *O-space, an almost-analytic vector v^i can be decomposed as*

$$v^i=p^i+r^i \tag{3.4}$$

where p^i is a Killing vector and r^i is a vector such that $r^i=\nabla^i r$ for a certain scalar r.

LEMMA 3.2.[4] *In an almost-Hermitian space, if a tensor S_{jst} is skew-symmetric, then we have*

$$\nabla^s\nabla^t S_{jst}=0. \tag{3.5}$$

Now, we shall prove the following

LEMMA 3.3. *In a compact *O-space, if a contravariant almost-analytic vector v^i can be decomposed as*

$$v^i=p^i+r^i \tag{3.6}$$

where p^i is a Killing vector and r^i is a vector such that $r^i=\nabla^i r$ for a certain scalar r, then we have

$$r^s r^t(R_{st}-R^*{}_{st})=0. \tag{3.7}$$

PROOF Interchanging j and i in (3.1) and subtracting the equation thus obtained from (3.1), we get

$$2v^t\nabla_t\varphi_{ji}-\varphi_j{}^t(\nabla_t v_i-\nabla_i v_t)+\varphi_i{}^t(\nabla_t v_j-\nabla_j v_t)=0. \tag{3.8}$$

Substituting (3.6) into (3.8) and taking account of $\nabla_i p_t=-\nabla_t p_i$ and $\nabla_t r_i=\nabla_i r_t$, we have

$$r^t\nabla_t\varphi_{ji}+p^t\nabla_t\varphi_{ji}+\varphi_{tj}\nabla^t p_i+\varphi_{it}\nabla^t p_j=0.$$

3) S. Sawaki and S. Koto [2].

4) K. Takamatsu [5].

Since $\nabla_i\varphi_j{}^i=0$ and $\nabla_i p^i=0$, this equation can be written as

(3. 9) $$r^t\nabla_t\varphi_{ji}+\nabla^t(p_t\varphi_{ji}+p_i\varphi_{tj}+p_j\varphi_{it})=0.$$

Operating ∇^i to (3.9) we have

(3.10) $$\nabla^i r^t\cdot\nabla_t\varphi_{ji}+r^t\nabla^i\nabla_t\varphi_{ji}+\nabla^i\nabla^t S_{jit}=0$$

where $S_{jit}=p_j\varphi_{it}+p_i\varphi_{tj}+p_t\varphi_{ji}$.

Similarly, substituting (3.6) into (3.2), we have

(3.11) $$\nabla_j r_i-\varphi_j{}^a\varphi_i{}^b\nabla_a r_b=0$$

i. e. $\nabla_j r_i$ is hybrid in j, i and therefore by Proposition 2, we have $\nabla^i r^t\cdot\nabla_t\varphi_{ji}=0$. Moreover, since S_{jit} is skew-symmetric, by Lemma 3.2 we have $\nabla^i\nabla^t S_{jit}=0$.

Hence, from (3.10) we have

(3.12) $$r^t\nabla^j\nabla_t\varphi_{ji}=0.$$

Transvecting (3.12) with $r^s\varphi_s{}^i$, we get

$$r^t r^s\varphi_s{}^i\nabla^j\nabla_t\varphi_{ji}=0.$$

Consequently, by (2.4) it can be written as

(3.13) $$r^t r^s(R_{st}-R^*{}_{st})=0.$$

4. An automorphism of an *O-space of constant curvature

In a 2n dimensional Riemannian space of constant curvature, Riemannian curvature tensor takes the following form

(4. 1) $$R_{kjih}=\frac{R}{2n\ (2n-1)}(g_{kh}g_{ji}-g_{jh}g_{ki})$$

where $R=g^{ji}R_{ji}$.

From (4.1) we have easily

(4. 2) $$R_{ji}=\frac{R}{2n}g_{ji},$$

(4. 3) $$R^*{}_{ji}=\frac{R}{2n\ (2n-1)}g_{ji}.$$

From (4.2), we can see that a space of constant curvature is an Einstein space.

Now, substituting (4.2) and (4.3) into (3.13) we have

$$r^t r^s\{\frac{1}{2n}-\frac{1}{2n\ (2n-1)}\}Rg_{ts}=0$$

or

$$\frac{n-1}{n(2n-1)}Rr^t r_t=0.$$

Therefore if $R \neq 0$ and $n > 1$, then we have $r^t = 0$ and hence from (3.4), we get $v^i = p^i$.
Thus we have the following

THEOREM. *In a compact 2n (n>1) dimensional *O-space of positive constant curvature, an almost-analytic transformation is an automorphism.*

NIIGATA UNIVERSITY

References

1. S. KOTO, ***Some theorems on almost Kählerian spaces***, Journ. Math. Soc. Japan, 12 (1960), 422–433.
2. S. SAWAKI and S. KOTO, ***Almost complex metric spaces***, Sûgaku 16 (1964), 9–18. (Japanese)
3. S. TACHIBANA, ***On almost-analytic vectors in almost-Kählerian manifolds***, Tôhoku Math. Journ., 11 (1959) 247–265.
4. S. TACHIBANA, ***On automorphisms of conformally flat K-spaces***, Journ. Math. Soc. Japan, 13 (1961), 183–188.
5. K. TAKAMATSU, ***On a decomposition of an almost-analytic vector in a K-space with constant scalar curvature***, Tôhoku Math. Journ., 16 (1964), 72–80.
6. K. YANO and T. SUMITOMO, ***Differential geometry on hypersurfaces in a Cayley space***, to appear.