On certain infinitesimal conformal transformations of contact metric spaces

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0. Introduction

In the previous paper [3], we have considered an infinitesimal transformation which leaves $\varphi_j{}^i$ invariant in a contact metric space and obtained the following

THEOREM 0. 1. *In a contact metric space, an infinitesimal transformation which leaves $\varphi_j{}^i$ invariant satisfies*

$$\underset{v}{\pounds} g_{ji}=\rho(g_{ji}+\eta_j\eta_i) \tag{0. 1}$$

$$\underset{v}{\pounds}\eta_i=\rho\eta_i \tag{0. 2}$$

where ρ is a constant. Conversely if v^i satisfies (0. 1) *aud* (0. 2), *then v^i leaves $\varphi_j{}^i$ invariant and consequently ρ is a constant.*

The condition (0. 1) is a formal generalization of an infinitesimal conformal transformation in a Riemannian space. Therefore it is natural that we consider a infinitesimal transformation satifying (0. 1) only where ρ is a scalar function. We shall call such a transformation an infinitesimal η- conformal transformation. In this paper we shall discuss such a transformation in a contact, a K- contact or a normal contact metric space.

1. Preliminaries

An almost contact metric space means an odd dimensional (n=2m+1) differentiable manifold with structure tensors $\varphi_j{}^i$, ξ^i, η_i and g_{ji} satisfying the following relations

$$\begin{cases} \xi^i\eta_i=1, & rank(\varphi_j{}^i)=n-1, \quad \varphi_j{}^i\eta_i=0, \quad \varphi_j{}^i\xi^j=0, \\ \varphi_j{}^r\varphi_r{}^i=-\delta_j^i+\xi^i\eta_j, & g_{ji}\xi^j=\eta_i, \quad g_{ji}\varphi_k{}^j\varphi_h{}^i=g_{kh}-\eta_h\eta_k. \end{cases} \tag{1. 1}$$

[6. 7]. On the other hand if the condition

$$2g_{ir}\varphi_j{}^r=2\varphi_{ji}=\partial_j\eta_i-\partial_i\eta_j \tag{1. 2}$$

hold in an almost contact metric space, the space is called a contact metric space. A contact metric space with a Killing vector ξ^i is called a K-contact metric space. By a normal cormal contact metric space we mean a contact metric space satisfying

(1. 3) $$\nabla_k \varphi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}$$

from which we can deduce that ξ^i is a Killing vector, where ∇_k denotes the covariant differentiation with respect to the Riemannian connection [1].

Let $R_{kji}{}^h$, R_{ji} be the Riemannian curvature tensor and the Ricci tensor respectively and put

(1. 4) $$\begin{cases} H_{ji} = \varphi^{kh} R_{kjih} = -\frac{1}{2} \varphi^{kh} R_{khji}, \\ \tilde{R}_{ji} = \varphi_j{}^r R_{ri}. \end{cases}$$

In a contact metric space, φ_{ji} is a skew symmetric closed tensor and

(1. 5) $$\nabla_r \varphi_j{}^r = (n-1)\eta_j$$

holds good.

In a K-contact metric space the following identities are valid

(1. 6) $$\nabla_j \eta_i = \varphi_{ji},$$

(1. 7) $$\nabla_k \varphi_{ji} + R_{rkji} \xi^r = 0,$$

(1. 8) $$H_{ir} \xi^r = 0,$$

(1. 9) $$R_{ir} \xi^r = (n-1)\eta_i.$$

In a normal contact metric space

(1.10) $$\nabla_k \varphi_{ji} = \eta_j g_{ki} - \eta_i g_{kj},$$

(1.11) $$\eta_r R_{kji}{}^r = \eta_k g_{ji} - \eta_j g_{ki}$$

hold. Operating ∇_l to (1.10) and making use of Ricci's identity and (1.4), we obtain

(1.12) $$\tilde{R}_{ji} - H_{ji} = (n-2)\varphi_{ji}.$$

A vector field v^i is called a Killing vector or an infinitesimal isometry if $\underset{v}{\pounds} g_{ji} = 0$, where $\underset{v}{\pounds}$ denotes the Lie derivative with respect to a vector v^i; an infitesimal conformal transformation if $\underset{v}{\pounds} g_{ji} = 2\rho g_{ji}$ where ρ is a scalar (homothetic, if ρ is a constant); an infintesimal contact transformation if $\underset{v}{\pounds} \eta_i = \sigma \eta_i$ where σ is a scalar; an infinitesimal projetive transformation if $\underset{v}{\pounds} \{^{h}_{ji}\} = \delta_i^j \rho_i + \delta_i^h \rho_j$.

A vector field v^i is called an automorphism if v^i leaves four structure tensors invariant.

A K-contact metric space in which the Ricci tensor takes the form

(1.13) $$R_{ji} = a g_{ji} + b \eta_j \eta_i$$

is called a K-contact η-Einstin space, where a and b become constant (n>3) [4].

2. Infinitesimal η-conformal transformation

It is well known that in a compact Kählerian space, an infinitesimal conformal transformation becomes an infinitesimal isometry [2]. Corresponding theorem to a compact normal contact metric space has not been known. However we can prove the following

THEOREM 2.1. *In a compact normal contact metric space* ($n>3$), *an infinitesimal η-conformal tranformation is necessarily an infinitesimal isometry.*

PROOF. For an infinitesimal η-conformal transformation v^i, we put

$$\underset{v}{\pounds} g_{ji}=\rho(g_{ji}+\eta_j\eta_i) \tag{2.1}$$

where ρ is a scalar function. If we substitute (2.1) into the identity

$$\underset{v}{\pounds}\left\{{h \atop ji}\right\}=\frac{1}{2}g^{hr}(\nabla_j\underset{v}{\pounds}g_{ri}+\nabla_i\underset{v}{\pounds}g_{rj}-\nabla_r\underset{v}{\pounds}g_{ji}),$$

then we have

$$\underset{v}{\pounds}\left\{{h \atop ji}\right\}=\frac{1}{2}\left\{\begin{array}{l}\rho_j(\delta_i^h+\xi^h\eta_i)+\rho_i(\delta_i^h+\xi^h\eta_j)\\ -\rho_h(g_{ji}+\eta_j\eta_i)+2\rho(\varphi_j{}^h\eta_i+\varphi_i{}^h\eta_j)\end{array}\right\}, \quad \rho_j=\partial_j\rho. \tag{2.2}$$

According to the identity

$$\underset{v}{\pounds} R_{kji}{}^h=\nabla_k\underset{v}{\pounds}\left\{{h \atop ji}\right\}-\nabla_j\underset{v}{\pounds}\left\{{h \atop ki}\right\} \tag{2.3}$$

and (2.2), we get

$$\begin{aligned}\eta_r\underset{v}{\pounds}R_{kji}{}^r=\frac{1}{2}[&2\eta_j\nabla_k\rho_i-2\eta_k\nabla_j\rho_i-\eta_r\nabla_k\rho^r(g_{ji}+\eta_j\eta_i)\\&+\eta_r\nabla_j\rho^r(g_{ki}+\eta_k\eta_i)+\varphi_{ki}\rho_j-\varphi_{ji}\rho_k+2\varphi_{ki}\rho_i-\eta_r\rho^r(\varphi_{ki}\eta_j\\&-\varphi_{ji}\eta_k+2\varphi_{kj}\rho_i)+2\rho(g_{ji}\eta_k-g_{ki}\eta_j)]\end{aligned} \tag{2.4}$$

On the other hand, from (1.11) we have

$$R_{kji}{}^r\underset{v}{\pounds}\eta_r+\eta_r\underset{v}{\pounds}R_{kji}{}^r=\rho(g_{ji}\eta_k-g_{ki}\eta_j)+g_{ji}\underset{v}{\pounds}\eta_k-g_{ki}\underset{v}{\pounds}\eta_j. \tag{2.5}$$

From (2.4) and (2.5), we get

$$\begin{aligned}R_{kji}{}^r\underset{v}{\pounds}\eta_r+\frac{1}{2}[&2\eta_j\nabla_k\rho_i-2\eta_k\nabla_j\rho_i-\eta_r\nabla_k\rho^r(g_{ji}+\eta_j\eta_i)\\&+\eta_r\nabla_j\rho^r(g_{ki}+\eta_k\eta_i)+\varphi_{ki}\rho_j-\varphi_{ji}\rho_k+2\varphi_{kj}\rho_i-\eta_r\rho^r(\varphi_{ki}\eta_j\\&-\varphi_{ji}\eta_k+2\varphi_{kj}\eta_i)]=g_{ji}\underset{v}{\pounds}\eta_k-g_{kj}\underset{v}{\pounds}\eta_i.\end{aligned} \tag{2.6}$$

Transvecting (2.6) with φ^{ji} and $\varphi_h{}^k g^{ji}$ respectively, we have

$$H_k{}^r\underset{v}{\pounds}\eta_r+\frac{1}{2}(-\eta_r\varphi_k{}^j\nabla_j\rho^r-n\rho_k+n\eta_r\rho^r\eta_k)=\varphi_k{}^r\underset{v}{\pounds}\eta_r, \tag{2.7}$$

$$\begin{aligned}\tilde{R}_k{}^r\underset{v}{\pounds}\eta_r&+\frac{1}{2}(-(n-2)\eta_r\varphi_k{}^j\nabla_j\rho_r-3\rho_k+3\eta_r\rho^r\eta_k)\\&=(n-1)\varphi_k{}^r\underset{v}{\pounds}\eta_r.\end{aligned} \tag{2.8}$$

Subtracting (2.7) from (2.8) and making use of (1.12), it follows that

(2. 9) $$\rho_k - \eta_r \rho^r \eta_k - \varphi_k{}^r \eta_s \nabla_r \rho^s = 0, \ (n>3)$$

from which we have

(2.10) $$\eta_r \nabla_j \rho^r = a\eta_j - \tilde{\mu}_j$$

where we have put

$$a = \xi^r \xi^s \nabla_r \rho_s, \qquad \tilde{\mu}_j = \varphi_j{}^r \rho_r.$$

Next, if we transvect (2.6) with ξ^k and using (1.11) and (2.10), we get

(2.11) $$2\nabla_j \rho_i = a(-g_{ji} + 3\eta_j \eta_i) - 2(\tilde{\mu}_j \eta_i + \tilde{\mu}_i \eta_j).$$

Differentiating (2.11) covariantry and then transvecting with φ^{kj}, we have from (1.4)

$$2H_i{}^r \rho_r = \varphi_i{}^r \nabla_r a + (3n-5)a\eta_i - 2n\tilde{\mu}_j + 2(\nabla_r \rho^r)\eta_i.$$

Transvecting with ξ^i and taking account of (1.8), we get

(2.12) $$2\nabla_r \rho^r + (3n-5)a = 0.$$

On the other hand, transvecting (2.11) with g_{ji}, we get

(2.13) $$2\nabla_r \rho^r + (n-3)a = 0.$$

Comparing (2.12) and (2.13), we have $\nabla_r \rho^r = 0$.

Consequently by Green's theorem we see that $\rho = 0$, ρ=const.

Lastly from (2.1) we get $\nabla_r v^r = \frac{n+1}{2}\rho$, by Green's theorem we get $\rho = 0$. q. e. d.

In an η-Einstein space, it is known that if $\underset{v}{\pounds} g_{ji} = 0$, then $\underset{v}{\pounds} \eta_i = 0$ holds good [5]. In this case by means of Thorem 0.1, $\underset{v}{\pounds} \varphi_j{}^i = 0$ also holds.

Thus we have the following

COROLLARY. *In a compact normal contact η-Einstein space with $b \neq 0$ $(n>3)$, an infinitesimal η-conformal transformation is an automorophism.*

When the associated function ρ of an η-conformal transformation is a constant, (2.2) becomes

$$\underset{v}{\pounds} \left\{ {h \atop ji} \right\} = \rho(\varphi_j{}^h \eta_i + \varphi_i{}^h \eta_j).$$

By the identity (2.3), it follows that

(2.14) $$\underset{v}{\pounds} R_{ji} = \rho \nabla_r (\varphi_j{}^r \eta_i + \varphi_i{}^r \eta_j),$$

(2.15) $$\underset{v}{\pounds} R = g^{ji} \underset{v}{\pounds} R_{ji} + R_{ji} \underset{v}{\pounds} g^{ji} = -\rho(R + R_{ji}\xi^j \xi^i).$$

If we assume that the space be Einstein that is $R_{ji} = \frac{R}{n} g_{ji}$, we have

$$\underset{v}{\pounds} R = -\frac{n+1}{n}\rho R = 0,$$

from which we get $\rho=0$. Thus we have

THEOREM 2.2. *In an Einstein contact metric space, an infinitesimal η-conformal transformation with $\rho=$ constant is necessarily an infinitesimal isometry.*

If we assume that the space under consideration be K-contact, then (2.15) turns to

$$\underset{v}{\pounds} R = -\rho(R+n-1)$$

because of (1.9). Thus

COROLLARY. *In a K-contact metric space with constant scalar curvature $R \neq -(n-1)$, an infinitesimal η-conformal transformation with $\rho=$ constant is an infinitesimal isometry.*

COROLLARY. *In a K-contact η-Einstein space with $b \neq 0$ $(n>3)$, an infinitesimal η-conformal transformation with $\rho=$constant is an automorphism.*

THEOREM 2.3. *In a contact metric space, if an infinitesimal η-conformal transformation v^i satisfies one of the the following conditions, then v^i is an automorphism.*

$$\text{(i)}\quad \underset{v}{\pounds}\eta_i=0, \qquad \text{(ii)}\quad \underset{v}{\pounds}\xi^i=0, \qquad \text{(iii)}\quad \underset{v}{\pounds}\varphi_{ji}=0.$$

PROOF. (i). From the well known identity

$$\nabla_j \underset{v}{\pounds}\omega_i - \underset{v}{\pounds}\nabla_j\omega_i = \omega_r \underset{v}{\pounds}\left\{{r \atop ji}\right\},$$

we have

$$-\underset{v}{\pounds}\nabla_j\eta_i = \eta_r \underset{v}{\pounds}\left\{{r \atop ji}\right\}$$

from which $\underset{v}{\pounds}\varphi_{ji}=0$ follows. Hence (i) reduces to (iii).

(ii).

$$\underset{v}{\pounds}\xi^i = g^{ji}\underset{v}{\pounds}\eta_j + \eta_j\underset{v}{\pounds}g^{ji} = g_{ji}\underset{v}{\pounds}\eta_j - 2\rho\xi^j$$

$$2\rho = \xi^j\underset{v}{\pounds}\eta_j = -\eta_j\underset{v}{\pounds}\xi^j = 0$$

from which we have $\underset{v}{\pounds}g_{ji}=0$ and $\underset{v}{\pounds}\eta_i=0$. Hence (ii) reduces to (i).

(iii).

$$\underset{v}{\pounds}\varphi_{ji} = g_{ri}\underset{v}{\pounds}\varphi_j{}^r + \rho\varphi_{ji}$$

Transvecting this with g^{ih}, we get

$$\underset{v}{\pounds}\varphi_j{}^h = -\rho\varphi_j{}^h$$

Next, operating $\underset{v}{\pounds}$ to

$$\varphi_j{}^r\varphi_r{}^i = -\delta_j{}^i + \xi^i\eta_j$$

we have

$$2\rho(\delta_j{}^i - \xi^i\eta_j) = \underset{v}{\pounds}(\xi^i\eta_j)$$

from which we have $\rho=0$.

THEOREM 2.4. *In order that a transformation in a contact metric space be an infinitesimal isometry,it is necessary and sufficient that the transformation be infinitesimal η-conformal and infinitesimal affine at the same time.*

PROOF. The necessity is evident. We shall prove the sufficiency.
From (2.2) we have

$$\underset{v}{\pounds}\left\{{h \atop ji}\right\}=\frac{1}{2}\left\{\begin{array}{l}\rho_j(\delta_i{}^h+\xi^h\eta_i)+\rho_i(\delta_j{}^h+\xi^h\eta_j)-\rho_h(g_{ji}+\eta_j\eta_i)\\ +2\rho(\varphi_j{}^h\eta_j+\varphi_i{}^h\eta_j)\end{array}\right\}=0.$$

By contraction with respect to j and h, we get $\rho_i=0$, and

$$\rho(\varphi_j{}^h\eta_i+\varphi_i{}^h\eta_j)=0.$$

Transvecting the last equation with $\varphi_k{}^j\xi^i$, we find $\rho=0$. q. e. d.

More generally, if an infinitesimal η-confornal transformation v^i is an infinitesimal projective transformation, we have $\underset{v}{\pounds}g_{ji}=\rho(g_{ji}+\eta_j\eta_i)$ and $\underset{v}{\pounds}\left\{{h \atop ji}\right\}=\delta_i{}^h\sigma_j+\delta_i{}^h\sigma_j$.

From (2.2), it follows that

$$\delta_j{}^h\sigma_i+\delta_i{}^h\sigma_j=\frac{1}{2}\left\{\begin{array}{l}\rho_j(\delta_i{}^h+\xi^h\eta_i)+\rho_i(\delta_j{}^h+\xi^h\eta_j)\\ -\rho^h(g_{ji}+\eta_j\eta_i)+2\rho(\varphi_j{}^h\eta_i+\varphi_i{}^h\eta_j)\end{array}\right\}. \tag{2.16}$$

Contracting (2.16) with respect to j and h, we get $\rho_i=2\sigma_i$. Next transvecting (2.16) with η_h, we get

$$\eta_j\sigma_i+\eta_i\sigma_j=(\eta_r\sigma^r)(g_{ji}+\eta_j\eta_i)$$

from which we obtain $\eta_r\sigma^r=0$ and $\sigma_i=0$. By virtue of Theorem (2.4), v^i is an infinitesimal isometry. Thus we have

THEOREM 2.5. *In order that a transformation in a contact metric space be an infinitesimal isometry, it is necessarily and sufficient that the transformatiobn be infinitesimal η-conformal and infinitesimal projective at the same time.*

Lastly we shall consider the case that η-conformal transformation is a contact transformation.

LEMMA. *In a contact metric space, if an infinitesimal η-conformal transformation be an infinitesimal contact transformation, that is, $\underset{v}{\pounds}g_{ji}=\rho(g_{ji}+\eta_j\eta_i)$ and $\underset{v}{\pounds}\eta_i=\sigma\eta_i$, then we have $\rho=\sigma$.*

PROOF. $\sigma=\xi^i\underset{v}{\pounds}\eta_i=-\eta_i\underset{v}{\pounds}\xi^i=-\eta_i(g^{ji}\underset{v}{\pounds}\eta_j+\eta_j\underset{v}{\pounds}g^{ji})=2\rho-\sigma$.

Thus taking account of Therem 0.1, we have the following

THEOREM 2.6. *In order that an infinitesimal transformation in a contact metric space leaves $\varphi_j{}^i$ invariant, it is necessarily and sufficient that the transformation be an infinitesimal η-conformal and infinitesimal contact at the same time.*

Moreover the following theorem is known [8.3].

THEOREM. *In a compact contact metric space an infinitesimal transformation which leaves $\varphi_j{}^i$ invariant is an automorphism.*

According to the last two theorems, we have

THEOREM 2.7. *In a compact contact metric space, if an infinitesimal η-conformal transformation be an infinitesimal contact transformation, then it is an automorphism.*

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