Imbedding and immersion of projective spaces

By

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1. Introduction

Let M be a differentiable manifold, and f a differentiable map of M in a euclidean space R^m. We say f an immersion if the differential df has a maximal rank at each point of M and homeomorphic immersion an imbedding. We shall write $M \subset R^m$ or $M \subseteq R^m$ when there exists an imbedding of M in R^m or an immersion of M in R^m, respectively. Let F be one of three basic fields R, C or Q and FP_n the n-dimensional projective space over F.

I. M. James has obtained an imbedding: $FP_n \subset R^{2dn-d+1}$ for every integer $n \geqq 1$, where d is the dimension of F over R. [7].

In this paper we shall prove the following

THEOREM 1. *Let n be any integer which is not power of 2, then $FP_n \subset R^{2dn-d}$.*

This result overlaps with that of [6], [8] and [9].

For the case $F=C$ or Q, we can also prove the following theorems which give us an information on the existence of imbedding of FP_n in lower dimensional euclidean space,

THEOREM 2. *$CP_n \subset R^{4n-3}$ if $CP_n - x \subseteq R^{4n-5}$ and $n \geqq 5$. Moreover if $CP_n - x \subseteq R^{4n-5}$ and n is odd, then $CP_n \subset R^{4n-4}$.*

THEOREM 3. *$QP_n \subset R^{8n-k}$ if $QP_n - x \subseteq R^{8n-k-1}$ and $k \leq n$, where k is 5, 6, or 8.*

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2. Imbeddings

Let $V=F^{n+1}$ be the right F-module and FP_n the associated right projective space. Thus we have a principal F*-bundle π: $V-o \rightarrow FP_n$, where F^* is the multiplicative group of non zero elements of F, and the associated right line bundle (fibre$F$, group F^* operating on F on the left), which we denote by L. We may also consider the left line bundle $L^* = Hom\ (L,F)$. This defines a real vector bundle ξ of dimension d, where d is the dimension of F over R. It is well known that the total space of this bundle is $FP_{n+1} - x$, where x is a point of FR_n. We denote this bundle by ξ. The following lemma is well known.

(2. 1) LEMMA. *Let τ be the tangent bundle of FP_n. Then we have*

$$\tau \oplus \eta = (n+1)\xi$$

where η is the bundle with fibre the Lie algebra F of F^* associated to the principal bundle π: V-$o \rightarrow FP_n$ by the adjoint representation. Moreover if F is commutative, η is a tiivial bundle.

Let κ and μ be the bundles over FP_n whose total space are denoted by $E(\kappa)$, $E(\mu)$, respectively. We call a map $g: E(\kappa) \rightarrow E(\mu)$ a homeomorphism when it satisfies the following properties

(1) g maps each fibre linearly into a fibre,

(2) g induces the identity map over FP_n.

We call g an imbedding if it is one-one. It is clear that an imbedding g: $\kappa \rightarrow \mu$ induces an imbedding of $E(\kappa)$ into $E(\mu)$.

We have the following

(2. 2) PROPOSITION. *Let FP_n be imbedded in R^m with normal vector bundle ν. If there exists an imbeddiug of ξ into ν, then FP_{n+1}-x can be imbedded in R^m.*

PROOF. The assumption implies that FP_{n+1}-x can be imbedded in a tubular neighbourhood of FP_n in R^m. Thus FP_{n+1}-x is imbedded in R^m.

(2. 3) PROPOSITION. *Under tne same assumption as (2.2) FP_{n+1} can be imbedded in R^{m+1} topologically.*

Proof. In view of (2.2), we have an imbedding of FP_{n+1}-x in R^m. Let $S^{dn+d-1} \subset FP_{n+1}$-$x$ be a sphere which is the boundary of ball in FP_{n+1} containing x. The proposition follows by placing a cone on this sphere.

By a result of A. Haeflieger [3], we have

(2. 4) COROLLARY. *If the assumption of (2.3) is satisfied, and if $2m > 3(dn+d)$, then FP_{n+1} can be imbedded in R^{m+1} differentiably.*

Now we shall study veotor bundle over FP_n more closely.

Let κ and μ be k-vector bundle and m-bundle over FP_n, resp. and $Hom(\kappa, \mu)$ the bundle defined bv $Hom(\kappa, \mu)_x = Hom(\kappa_x, \mu_x)$ - group of linear transformations of $\kappa = R^k$ into $\mu = R^m$. We suppose $k < m$. We denote the sub-bundle of $Hom(\kappa, \mu)$ with fibre $L_{m,k}$ -group of linear transformations of R^k into R^m of rank k-by $L(\kappa, \mu)$. It is necessary and sufficient for the existence of an imbedding of κ into μ that $L(\kappa, \mu)$ has a cross section. Since $L_{m,k}$ has the Stiefel manifold $V_{m,k}$ as its deformation retract, the primary obstruction for the existence of a cross section of $L(\kappa, \mu)$ is an element of $H^{m-k+1}(FP_n : \{\pi_{m-k}(V_{m,k})\})$, where $\{\pi_{m-k}(V_{m,k})\}$ denotes the bundle of coefficients with fibre $\pi_{m-k}(V_{m,k})$ which is a product bundle when F is C or Q. We notice that if $k=1$ then $L(\kappa, \mu)$ ia an$(m-1)$-sphere bundle.

The following example shows clearly how we apply (2.3). Consider the imbedding $FP_{n-1} \subset R$, which exists by a result of H. Whitney. Let ν be the normal bundle, whose dimension is dn. Since $\pi_i(L_{dn,d}) = \pi(V_{dn,d}) = o$ for $i < d(n-1)$. $L(\xi, \nu)$ has a cross section over FP_{n-1}. Thus by (2.3), FP_n is imbedded topologically in $R^{2d(n-1)+d+1}$. By (2.4), if $dn > 2(d-l)$, i. e. $n \geqq 2$, this imbedding is approximated by a differentiable one. The exceptional case $d \geqq 2$ and $n=I$ is slso true because $FP = S^d$ is imbedded in R^{d+1}.

Thus we have an imbedding of FP_n in $R^{2dn-d+1}$ fore every integer n and d. This result coincides with that of James' mentioned in Introduction.

3. Immersions

We begin, in this section, with some general theorems about the bundle along the fibres.

Let $\eta=(E,\pi,B)$ be a fibre bundle and $\hat{\eta}$ the bundle along the fibres.
As is well known,

$$\tau(E)=\pi^*\{\tau(B)\}\oplus\hat{\eta}$$

We consider the case $\hat{\eta}$ is a vector bundle, which we shall need in the sequel.
We can prove that the sequence

$$0\to\pi^*(\eta)\to\tau(E)\to\pi^*\tau(B)\to 0$$

is exact, in other words, η is equivalent with $\pi^*(\eta)$.
For each point $x\in B$, we have an inclusion

$$E_x \text{ (fibre of } \eta \text{ at} x)\to E$$

and hence a natural inclusion

$$\tau(E_x)\to\tau(E)$$

It follows from the definition that the total space of $\pi^*\eta$ consists of pair of vectors (v,w) lying over the same base point x: in other words, the fibre of x is $E_x\times E_x$. Since E_x is a euclidean space, $E_x\times E_x$ is naturally identified with $\tau(E\ )$. Hence we have a bijection

$$(\pi^*\eta)_x\to\tau(E\ )$$

for each x. It follows from this that $\pi^*\eta$ and $\hat{\eta}$ are equivalent, or (3.1) is exact. The exactness of (3.1) implies

(3. 2) $$\tau(E)=\pi^*\{\tau(B)\oplus\eta\}$$

We recall some results on regular homotopy classes of immersions of a manifold in a euclidean space R^m.

The following results have been proved by M. W. Hirsch in [5]

(3. 3) *M be an n-manifold. Then the regular homotopy classes of immersions of M in R^m ($m>n$) corresponds injectively with the homotopy classes of cross sections of the bundle associated to the tangent bundle of M with fibre $V_{m,k}$.*

(3. 4) *Two immersions of M in R^{2n+1} dre reguldrly homotopic.*

(3. 5) *Let M be a manifold of even dimension n. Then two immersions of M in R^{2n} are regularly homotopic if and only if they have the same normal class.*

From (3.3), we have the following

(3. 6) LEMMA. *If n is even, two immersions of CP_n in R^{4n-1} are regularly homotopic.*

PROOF. The regular homotopy classes of immerisions of CP_n in R^{4n-1} are in one-one correspondence with the homotopy classes of cross sections of the bundle associated to the tangent bundle of CP_n with fibre $V_{4n-1,2n}$. The obstructions to make two cross sections homotopic lie in the group H^{2n-1} $(CR_n; \pi_{2n-1}(V_{4n-1,2n}))=0$. and H^{2n} $(CP_n; \pi_{2n}(V_{4n-1,2n}))$, which is zero for even n since π_{2n} $(V_{4n-1,2n})=0$, if n is even [11]. Similarly we can prove

(3. 7) LEMMA. *Two immersions of QP_n in R^{8n-1} are regutarly homotopic.*

4. The proof of Theorem 1

We first recall some results on binomial coefficients. Let $a(n)$ be the number of non-zero terms in the dyadic expansion of $n; n=\sum n_i 2^i$ with $n_i=0$, or 1, then $a(n)=\sum ni$.

We have a well known

(4. 1) LEMMA. $\binom{n}{k}$ *is not zero mod 2 if and only if* $a(k)+a(n-k)-a(n)=0$.

PROOF. Recall $\binom{n}{k}=n!/(k!(n-k)!)$. Since $n!=2^{n-a(n)}\,o(n)$, where $o(n)$ is an odd number. We see that

$$\binom{n}{k}=2^{a(k)+a(n-k)-a(n)x}\times(\text{an odd number})$$

Hence $\binom{n}{k}$ is not zero mod $2 \longleftrightarrow \binom{n}{k}$ is odd $\longleftrightarrow a(k)+a(n-k)=a(n)$

(4. 2) $\binom{2n+1}{n} \neq 0$ *mod* $2 \longleftrightarrow n=2^r-1$ *for some integer* r.

(4. 3) $\binom{2n}{n-1} \neq 0$ *mod* $2 \longleftrightarrow n=2^r-1$ *for some integer* r.

PROOF. Let $n=\sum_{i=1}^{s} 2^{r}$. $r_1>r_2>\cdots>r_s\geq 0$. Then $a(n)=s$, $a(2n+1)=s+1$. Hence $\binom{2n+1}{n} \neq 0$ mod $2 \longleftrightarrow a(n)+a(n+1)=a(2n+1) \longleftrightarrow a(2n+1)=1 \longleftrightarrow n=2^r-1$. This implies (4.2). The proof of (4.3) is similar.

We consider first the case $F=C$, Q.

Let FP_n be imbedded in $R^{2dn+d-1}$ with normal vector bundle ν. We can show the following;

(4. 4) *ξ can be imbedded in $\nu\oplus\varepsilon^k$, where k is large enough, in other words, there exists a $(dn+k-1)$ vector bundle $\tilde{\kappa}$ such that $\nu\oplus\varepsilon^k=\xi\oplus\tilde{\kappa}$.*

PROOF. We consider the bundle $L(\xi, \nu\oplus\varepsilon^k)$. Since the fibre of this bundle is $L_{dn+d-1+k,d}$, there is no obstruction for the existence of a cross section of $L(\xi, \nu\oplus\varepsilon^k)$. Hence ξ can be imbedded in $\nu\oplus\varepsilon^k$.

Next we prove

(4. 5) *If $n\neq 2^r-1$, then $\tilde{\kappa}=\kappa\oplus\varepsilon^k$ for some $(dn-1)$ vector bundle κ.*

PROOF. To prove this, it is sufficient to show that the bundle associated to $\tilde{\kappa}$ with fibre $V_{dn-1+k,k}$ has a cross section over FP_n. The only obstruction is an element

$c_{dn} \in H^{dn}(FP_n; \pi_{dn-1}(V_{dn-1+k,k}))$. It is known that c_{dn} is the dn th Stiefel-Whitney class $w_{dn}(\tilde{\kappa})$ of $\tilde{\kappa}$. By (4.4) we have

$$w_{dn}(\tilde{\kappa}) \equiv \binom{2n+1}{n} a^n \mod 2$$

where a is a generator of $H^*(FP_n : Z_2)$. By (4.2), $w_{dn}(\tilde{\kappa})$ is zero if and only if $n \neq 2^r - 1$.

Combining (4.4) and (4.5), we have

(4. 6) $\nu \oplus \varepsilon^k = \xi \oplus \kappa \oplus \varepsilon^k$ *if* $n \neq 2^r - 1$

Hence there is an immersion of FP_n in $R^{2dn+d-1}$ with normal vector bundle $\xi \oplus \kappa$. By (3.4), two immersions of FP_n in $R^{2dn+d-1}$ are regularly homotopic, hence $\nu = \xi \oplus \kappa$. Thus we have

(4. 7) $\nu = \xi \oplus \kappa$ *if* $n \neq 2_n - 1$

From (2.3), (2.4) and (4.7), there exists an imbedding of FP_{n+1} into R^{2dn+d} This completes the proof of Theorem 1.

For the case F=R, see [4].

5. The proof of Theorem 2 and Theorem 3

We recall that the total space of the canonical d-vector bundle ξ over FP_{n-1} is $FP_n - x$. Let τ' be the tangent bundle of $FP_n - x$, and τ be the tangent bundle of FP_{n-1}. Then we have the following

(5. 1) LEMMA. $\tau' | FP_{n-1} = \tau \oplus \xi$

RROOF. Let i be the inclusion of FP_{n-1} in $FP_n - x$. Since $\pi\, i = 1$. $i^* \pi^* = 1$.

By (3.2), we have $\quad \tau' = \pi^*(\tau \oplus \xi)$

Hence we have $\quad \tau' | FP_{n-1} = i^* \tau' = \tau \oplus \xi$

Now let $FP_n - x$ be immersed in R^m with normal bundle ν' and FP_{n-1} imbedded in R^m with normal bundle ν.

Then we have

(5. 2) LEMMA. $\quad \nu \equiv \nu' l FP_{n-1} \oplus \xi$

where the notation" $\equiv$ "means stably equivalent.

PROOF. We have

$$\tau \oplus \nu = \varepsilon^m = (\tau' \oplus \nu') | FP_{n-1} = \nu' | FP_{n-1} \oplus \tau \oplus \xi$$

Hence $\quad \nu \equiv \nu' | FP_{n-1} \oplus \xi$

Now we shall prove Theorem 2. Let $CP_n - x \subseteq R^{4n-5}$ with normal bundle ν' and $CP^{n-1} \subset R^{4n-4}$ with nonmal bundle ν. Then (5.2) implies

$$\nu \equiv (\nu' \oplus \varepsilon') | CP_{n-1} \oplus \xi = \nu' | CP_{n-1} \oplus \xi \oplus \varepsilon^1$$

Since $\quad X(\nu) = X(\nu' | CP_{n-1} \oplus \xi \oplus \varepsilon^1) = 0$. (3.5) implies

$$\nu = \nu' | CP_{n-1} \oplus \xi \oplus \varepsilon^1$$

Hence ξ is imbedded in ν as a sub-bundle. By (2.3) and (2.4) we have Theorem 2

The proof of Theorem 3 is completely similar.

As corollary of Theorem 3, we have

(5. 3) COROLLARY. *If n is integer greater than 9 such that $a(n)=4$, then QP_n is not immersible in R^{8n-9}.*

PROOF. By a result of Atiyah-Hirzebruch, we have

$$QP_n \not\subset R^{8n-8} \text{ for n such that } a(n)=4.$$

Theorem 3 implies that if $QP_n \subseteq R^{8n-9}$, then $QP_n \subset R^{8n-8}$. This completes the proof.

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