On representations of 1-homology classes of closed surfaces II

By

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Let M be an orientable closed surface with genus m, and let α_i, β_i $(i=1,\cdots,m)$ be standard generators of $H_1(M;Z)$. (Note that M has a unique differentiable structure.)

We studied in [1] whether $\xi=\sum_{i=1}^{m}(p_i\alpha_i+q_i\beta_i)\in H_1(M;Z)$ can be represented by a differentiably inbedded 1-sphere or not, and obtained the following results.

THEOREM 1. *When $m=1$, letting α and β be the standard generators of $H_1(M;Z)$, the non-zero homology class $p\alpha+q\beta$ can be represented by a differentiably imbedded 1-sphere if and only if G. C. M. $(p,q)=1$.*

In general cases we have,

THEOREM 2. *The non-zero homology class $\xi=\sum_{j=1}^{m}(p_i\alpha_i+q_i\beta_i)$ can be represented by a differentiably imbedded 1-sphere if G. C. M. $(p_i,q_i)=1$ for some i or G. C. M. $(p_j,q_j)=1+$G. C. M. (p_k,q_k) for some j and k.*

In this paper we improve above results using the instruments prepared in [1].

1. Preliminaries

In this section we recall the graphic representations and some lemmas given in [1]. Let T be a torus and α, β be the standard generators of $H_1(T;Z)$.

We construct T', the connected sum of T and S^2, as follows; take away the neighborhood U of the base point in T and the interior of a disk D in S^2, and identify the boundary of U with the boundary K of D. Then T' is homeomorphic to T, accordingly diffeomorphic to T. We regard $T-U$ to be inside of K and S^2-D to be outside of K, and indicate by vectors which lie across K the differentiable curves which lie across K and have no self intersection and no mutual intersection in $T-U$.

When this graphic representation is consist of p parallel vectors and q other vectors which are orthogonal to the formers, we call it a graph of type (p,q).

We put the base point x_0 of T' outside of K, i. e. in S^2-D, then combining every initial and end points of all vectors to x_0 we obtain a representation of some element of $H_1(T';Z)=H_1(T;Z)$.

In this case we say that the element is represented by a graph of type (p,q). Note that different classes may be represented by graphs of same type.

The element $p\alpha+q\beta$ is naturally represented by a graph of type (p,q). (Cf. Fig. 1 and 2 in [1]). It may be represented by graphs of various types, but the following representations given in [1] are essential.

Let $r=G.\ C.\ M.\ (p,q)$.

LEMMA 1. *$p\alpha+q\beta$ can be represented by a graph of type $(r, 0)$.*

LEMMA 2. *$p\alpha+q\beta$ can be represented by a graph of type (r,r).*

Now we add two lemmas to be used in the next section.

A graph of type $(p/p',q)$ denotes the graph composed of p parallel vectors succeeded by p' inversely oriented vectors and q other vectors orthogonal to them.

LEMMA 3. *The cohomology class which is represented by a graph of type (p,q) can be represented by a graph of type $(p+n/n,q)$ for any integer n.*

PROOF. We can add n horizontal vectors with the same orientation above the given p horizontal vectors and n horizontal vectors with the inverse orientation below the given p vectors, without changing the represented homology class, as follows. When $q=0$, the $2n$ vectors may be those which represent trivial homology class.

When $q\neq 0$, the uppermost and the lowest vectors of p horizontal vectors represent the same homology class. Let the upper n vectors indicate the curves running along the curve which is indicated by the uppermost vector of given horizontal vectors, and let the lower n vectors indicate those running with inverse orientation along the curve which is indicated by the lowest vector of given horizontal vectors, then they mutually subtract and do not change the represented homology class.

LEMMA 4. *The homology class which is represented by a graph of type (p,q) can be represented by a graph of type $(p+q,q)$.*

PROOF. As in the preceding proof, we may add q horizontal vectors above the given p horizontal vectors and add q inversely oriented horizontal vectors below, without changing the represented homology class. Then we combine the end points of lower q horizontal vectors with the initial points of q vertical vectors. This graph may be transformed into a graph of type $(p+q,q)$ without changing the represented homology class. (Cf. Figures in [1]).

2. Theorems

Let M be an orientable closed surface with genus m, and let α_i,β_i $(i=1,\cdots,m)$ be the standard generators of $H_1(M;Z)$. Then we have the following theorem which contains Theorem 2.

THEORMM 3. *The non-zero homology class $\xi=\sum_{i=1}^{m}(p_i\alpha_i+q_i\beta_i)$ can be represented by a differentiably imbedded 1-sphere if $G.\ C.\ M.\ (r_j, r_k)=1$ for some j and k. Where $r_j=0$ if $p_i=q_i=0$, and $r_i=G.\ C.\ M.\ (p_i, q_i)$ otherwise.*

PROOF. We may consider M to be as follows. Let K_i $(i=1,\cdots,m)$ be circles which have mutually no intersection and stand side by side along the equator on S^2. At each K_i, we take away the interior and attach to it a holed torus T_i by the way used in §1. Then the resulting connected sum M is an orientable closed surface with genus m, and has an unique differentiable structure.

At each K_i $(i\neq j, k)$, we represent $p_i\alpha_i+q_i\beta_i$ by a graph of type $(r_i, 0)$ by Lemma 1. We assume that r_k is larger than r_j by s. As r_j and r_k are relatively prime, there are positive integers n_j and n_k such that $n_jr_j=n_kr_k+1$. Using Lemma 2 and Lemma 4, we represent $p_j\alpha_j+q_j\beta_j$ by a graph of type $((s+1)\ n_jr_j+r_j, n_jr_j)$, and $p_k\ \alpha_k+q_k\beta_k$ by a graph of type $((s+1)\ n_kr_k+r_k, -n_kr_k)$. (Note that a graph of type (p,q) may be transformed into a graph of type $(p,-q)$ without changing the represented homology class.)

We may assume without loss of generality that $(s+1)n_kr_k+r_k>\max_i r_i$.

Putting the base point x_0 of M at the north pole of S^2, we give a differentiably imbedded 1-sphere which combines x_0 and all vectors as follows.

We start from x_0 and pass through uppermost horizontal vectors of K_i $(i=1,\cdots,m)$ in order, where if $r=0$ go round the south side of K_i. After going round one time, we go towards the initial point of second horizontal vector of K_1 and pass through all second horizontal vectors as above.

After repeating such process $\{(s+1)n_jr_j+r_j\}$ times, from the end point of the last (i.e. the lowest) horizontal vector of K_j we return towards the initial point of the last (counting from the left) vertical vector of K_j, then we pass through the remaining vertical vectors of K_j and K_k alternately, and then from the end point of the first vertical vector of K_j we return to x_0. (Cf. Figures in [1]).

As easily seen, this process may be carried out such as the resulting curve has no self intersection and is differentiable.

The necessary condition of Theorem 1 is generalized to the case of genus m as follows.

THEORMM 4. *If G. C. M.* $(r_1,\cdots,r_m)\neq 1$, *then* $\xi=\sum_{i=1}^{m}(p_i\,\alpha_i+q_i\,\beta_i)$ *cannot be represented by an imbedded 1-sphere.*

To prove this theorem, we prove the following lemma at first.

LEMMA 5. *For any integer* $r\geqq 2$, *all vectors of graphs of type* (r,r) *and those of type* $(r, 0)$ *on* M *cannot be combined by curves on* $S^2-\bigcup_{i=1}^{m} \mathrm{Int}\ K_i$ *to be an imbedded 1-sphere with same orientation to the vectors, where* $\mathrm{Int}\ K_i$ *means the inside part of* K_i *on* S^2.

PROOF. When all graphs are of type $(r, 0)$, our lemma is trivial. Next we consider the general case. Now we assume that the graph at K_1 is of type (r,r) and all vectors are combined to be an imbedded 1-sphere with compatible orientation. We start from the j-th horizontal vector of K_1 and go along the imbedded 1-sphere. Let the vertical vector of K_1 at which we arrive for the first time be the k-th vertical vector.

In this process, if the curve we have passed has outwardly no self intersection, this outwardly closed curve must outwardly intersect with same number of vectors to go in and to go out. Here we say that two curves outwardly intersect if they outwardly intersect in some K_j, i. e. the vectors of K_j which indicate the parts of curves intersect if they are regarded to be lines on the disk which is bounded by K_j. On the other hand, r vectors with same orientation are arranged in a neighborhood acrossing our curve, with the exception of the vectors of K_1. So the numbers of vectors of K_1 to go in and to go out must be the same. Thus $j=k$.

When the curve has outwardly self intersections, any case may be decomposed into the cases with one outward self intersection. And so the increase of outward self intersections has no influence to the difference of numbers of vectors to go in and those to go out from the outwardly closed curve which passes K_1. Thus in any case, if we start from the j-th horizontal vector of K_1, the vertical vector of K_1 at which we arrive for the first time must be the j-th vertical vector.

This fact proves also that our curve does not pass other horizontal vectors of K_1 on the way. In the same way, if we start from the j-th vertical vector of K_1, we arrive at the j-th horizontal vector of K_1, without passing any other vector of K_1.

These facts contradict with our first assumption, and thus Lemma 5 is proved.

PROOF of Theorem 4. Let $r=G.\ C.\ M.\ (r_1,\cdots,r_m)$, and assume that $r>1$.

1°. When we represent $p_i\alpha_i+q_i\beta_i$ by a natural graph of type (p_i,q_i) for every i. These graphs cannot be combined by curves in $S^2-\bigcup_{i=1}^{m} Int\ K_i$ to be an imbedded 1-sphere as seen below. If they are combined as required, the resulting figure may be regarded to be a figure obtained by combining $\sum_{i=1}^{m} p_iq_i/r^2$ graphs of type (r,r), and this fact contradicts with Lemma 5.

2°. Various ways to combine graphs by curves in M wtihout changing the represented homology class may be considered. Any of such ways may be regarded as follows; transform the graph of type (p_i,q_i) into a graph of suitable type without changing the represented homology class for each i, and combine them by curves in $S^2-\sum_{i=1}^{m} Int\ \ K_i$ to be an imbedded 1-sphere. There are three kinds of transformations. One of them is to combine two vectors with a trivial (in the sense of having no influence to the represented homology class) curve in the neighborhood of T_i and deform it to a curve in T_i. This transformation is used in the proof of Lemma 1, 2 and 4. If the graphs obtained by this transformation can be combined by curves in $S^2-\bigcup_{i=1}^{m} Int\ \ K_i$ to be an imbedded 1-sphere, then the original graphs is the same, because above transformation may be obtained by combining two vectors with a curve in $S^2-\bigcup_{i=1}^{m} Int\ K_i$ and deforming it. The second one is the transformation given in Lemma 3. The proof of this case is covered by 1°, because Lemma 5 can be generalized for this type as easily seen. The third transformation is to add, at an open edge of each K_i, vectors which represent trivial classes. It is trivial that this transformation has no influence to the proof.

Thus every cases reduce to 1°.

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Reference

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