An equilibrium point for N-person stochastic quadratic game

By
Kensuke TANAKA and Nobuo TANABE

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1. Introduction

We consider an N-person game under the evolution of the system described by a stochastic differential equation with a special form. The cost function of each player consists of the expectation of quadratic and bilinear terms of the strategies and quadratic term of state of the system. And, the strategy space available to each player consists of a closed ball in an Euclidean space. Then, each player wishes to minimize his cost function in a finite time interval. So, under the assumption that the quadratic term of strategy chosen by each player in cost function is positive semidefinite and that there exists a solution of the optimality equation, we show that there exists an equilibrium point consisting of a pure multistrategy. Further, we give the necessary and sufficient condition for a pure multistrategy to be an equilibrium point. Next, when the quadratic term may not be positive semidefinite, we can construct an equilibrium point consisting of an atomic probability measure by modifying the term so as to be convex. The essential part of our technique is due to applying the methods of Williams (Ref. 8) and Wilson (Ref. 9) in quadratic game to the optimality equation of N-person stochastic quadratic game.

2. Formulation for N-person stochastic differential game

In this paper, we consider the evolution of the system described by a stochastic differential equation in an n-dimensional Euclidean space R^n of the following form: for $t \varepsilon [0, T]$,

$$dx(t) = [A(t)x(t) + \sum_{i=1}^{N} B_i(t)u_i(t, x)]\, dt + \sigma(t)dB(t) \qquad (1)$$

$$x(0) = x_0 \varepsilon R^n,$$

where

(1) the matrix functions $A(t)$ and $\sigma(t)$ are continuous in t and of orders $n \times n$,

(2) the matrix functions $B_i(t)$, $i=1, 2, \ldots, N$, are continuous in t and of orders $n \times p_i$,

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(3) $u_i(t, x)$ is the strategy used by player i at a pair (t, x) of the time $t \varepsilon [0, T]$ and the value in R^n of x at the time t.

The σ-field $\mathfrak{D}$ of $[0, T]\times R^n$ is $\beta([0, T])\times\beta(R^n)$, where $\beta([0, T])$ denotes the Lebesgue σ-field of $[0, T]$ and $\beta(R^n)$ denotes the Borel σ-field of R^n. Letting $\mathfrak{U}_i$ be the σ-field of Borel subsets of $U_i=\{u_i \varepsilon R^{P_i}: |u_i|^2\leqq M_i\}$ for a positive constant M_i, an admissible pure strategy u_i for each player i is defined by the following measurable function:

$$u_i: ([0, T]\times R^n, \mathfrak{D})\longrightarrow(U_i, \mathfrak{U}_i)$$

So, when player i uses a pure strategy u_i, an action $u_i(t, x)\in U_i$ is chosen by u_i at each $(t, x) \varepsilon [0, T]\times R^n$ without any collaboration to any others. Further, in order to define a mixed strategy, we introduce the notation $P(U_i)$, which denotes the set of all Borel probability measures on $(U_i, \mathfrak{U}_i)$. Letting $C(U_i)$ be the set of all real-valued continuous functions on U_i, $C(U_i)$ is a separable space with metric induced by supnorm. So, by considering $P(U_i)$, endowed with weak topology induced from $C(U_i)$ (see Ref. 4 and 7), we can define an admissible mixed strategy for player i by the following measurable function:

$$u_i: ([0, T]\times R^n, \mathfrak{D})\longrightarrow(P(U_i), W),$$

where $(P(U_i), W)$ denotes $P(U_i)$ endowed with weak topology. So, when player i uses a mixed strategy u_i, an action $u \varepsilon U_i$ is chosen by the probability measure $u_i(t, x) \varepsilon P(U_i)$ corresponding to u_i at each (t, x). Therefore, each term $B_i(t)u_i(t, x)$ in the stochastic differential equation (1) denotes an expectation of $B_i(t)u(u \varepsilon U_i)$ by the probability measure $u_i(t, x)$, i. e.,

$$B_i(t)u_i(t, x)=\int_{U_i}B_i(t)u\, du_i(t, x)(u).$$

Especially, if $u_i(t, x)$ is an atomic probability measure which selects actions u^k, $k=1, 2, \ldots, p$, with probabilities $\nu_i^k(t, x)$, with

$$\nu_i^k(t, x)>0, \qquad \sum_{i=1}^{p}\nu_i^k(t, x)=1,$$

by using the notation $\delta(u)$ denoting the probability measure concentrated at the single action $u \varepsilon U_i$, $u_i(t, x)$ can be written as

$$\sum_{k=1}^{p}\nu_i^k(t, x)\,\delta(u^k)$$

and

$$B_i(t)u_i(t, x)=\sum_{k=1}^{p}\nu_i^k(t, x)B_i(t)u^k.$$

And, an N tuple of admissible strategies $\bar{u}=(u_1, u_2, \ldots, u_N)$ is said to be a multistrategy.

Now, when a multistrategy $\bar{u}=(u_1, u_2, \ldots, u_N)$ is used, each player $i \varepsilon \{1, 2, \ldots, N\}$ receives the expected cost of the following form:

$$J^i(\bar{u})=E_{\bar{u}}\ \{\int_0^T [x(t)'\, Q^i(t)x(t)+u_i(t, x)' \sum_{j=1}^{N} R_{ij}(t)u_j(t, x)]dt+$$

$$+x(T)'\, F^i(T)\ x(T)\}, \tag{2}$$

where

(i) the matrix functions $Q^i(t)$ and $R_{ii}(t)$ are measurable in t, symmetric and of orders $n\times n$,

(ii) the matrix $F^i(T)$ is symmetric and of orders $n\times n$,

(iii) the matrix functions $R_{ij}(t)$, $i\neq j$, are measurable in t and of orders $p_i\times p_j$,

(iv) the notation $E_{\bar{u}}\{\cdot\}$ denotes the expectation by the probability measures corresponding to a multistrategy $\bar{u}$.

We introduce the notation $(\widehat{\bar{u}^i}, v_i)$ for a multistrategy $\bar{u}=(u_1, u_2, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_N)$ that results from changing the strategy u_i of player i to v_i in $\bar{u}=(u_1, u_2, \ldots, u_N)$. Then, if there exists a multistrategy $\bar{u}=(u_1, u_2, \ldots, u_N)$ such that, for all $i \varepsilon \{1, 2, \ldots, N\}$ and all strategies v_i,

$$J^i(\bar{u}) \leqq J^i(\widehat{\bar{u}^i}, v_i), \tag{3}$$

this $\bar{u}$ is said to be *an equilibrium point.*

3. An equilibrium point in pure strategies

We consider the following optimality epuation:

$$W_t^i(t, x)+\frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(t)\frac{\partial^2 W^i(t, x)}{\partial x_j\, \partial x_k}+\min_{\bar{u}}\{<A(t)x(t)+$$

$$+\sum_{j=1}^{N} B_j(t)u_j,\ W_x^i(t, x)>+L^i(t, x, \bar{u})\}=0 \tag{4}$$

with $W^i(T, x)=x(T)'\, F^i(T)x(T)$,

where

(a) $W_t^i(t, x)=\dfrac{\partial W^i(t, x)}{\partial t}$,

(b) $a_{jk}(t)$ is the (j, k) th element of the positive definite matrix function $a(t)=\sigma(t)\sigma(t)'$,

(c) $W_x^i(t, x)=\left(\dfrac{\partial W^i(t, x)}{\partial x_1}, \dfrac{\partial W^i(t, x)}{\partial x_2}, \ldots, \dfrac{\partial W^i(t, x)}{\partial x_n}\right)'$

(d) the notation $<\cdot, \cdot>$ denotes the inner product,

(e) $L^i(t, x, \bar{u}) = x(t)' Q^i(t) x(t) + u_i' \sum_{j=1}^{N} R_{ij}(t) u_j$,

(f) u_i denotes a strategy used by player i.

Then, under the assumption that each matrix function $R_{ii}(t)$, $i=1, 2, \ldots, N$, is positive semidefinite and that there exists a solution $W^i(t, x)$ of the equation (4), we show that the game has an equilibrium point in pure strategies. Further, we give a necessary and sufficient condition for a pure multistrategy to be an equilibrium point. So, introduceing the following notation defined on $\prod_{k=1}^{N} U_k$ for each $(t, x) \in [0, T] \times R^n$:

$$K^i(t, x, \bar{u}) = \langle A(t)x(t) + \sum_{j=1}^{N} B_j(t) u_j, W_x^i(t, x) \rangle + \\ + L^i(t, x, \bar{u}), \tag{5}$$

we can write as follows: for $\bar{u} = (u_1, u_2, \ldots, u_N)$ and $\widehat{\bar{u}^i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$,

$$K^i(t, x, \bar{u}) = u_i' R_{ii}(t) u_i + u_i' S^i(t, x, \widehat{\bar{u}^i}) + T^i(t, x, \widehat{\bar{u}^i}) + Z^i(t, x), \tag{6}$$

where

$$S^i(t, x, \widehat{\bar{u}^i}) = \sum_{j \neq i} R_{ij}(t) u_j + B_i(t)' W_x^i(t, x),$$

$$T^i(t, x, \widehat{\bar{u}^i}) = \sum_{j \neq i} u_j' B_j(t)' W_x^i(t, x)$$

and

$$Z^i(t, x) = x(t)' A(t)' W_x^i(t, x) + x(t)' Q^i(t) x(t).$$

Lemma 3.1 *If the matrix functions $R_{ii}(t)$, $i=1, 2, \ldots, N$, are positive semidefinite, there exists a pure multistrategy $\bar{u}^* = (u_1^*, u_2^*, \ldots, u_N^*)$ such that, for each $(t, x) \in [0, T] \times R^n$ and $i \in \{1, 2, \ldots, N\}$*

$$K^i(t. x, \bar{u}^*) \leqq K^i(t, x, (\widehat{\bar{u}^{*i}}, v_i)) \quad \textit{for any pure strategy } v_i.$$

Proof. For each $i \in \{1, 2, \ldots, N\}$, U_i is convex and compact. Since $R_{ii}(t)$ is positive semidefinite, the function $u_i \rightarrow u_i' R_{ii}(t) u_i$ is convex on U_i. So, $K^i(t, x, \bar{u})$ is convex in u_i for fixed u_j, $i \neq j$ and (t, x). Further, for fixed u_i, (t, x) and each $j \neq i$, $K^i(t, x, \bar{u})$ is linear in u_j and, therefore, is contiuous in u_j. Then, using Theorem 3.1 in Ref. 5, there is a $\bar{u}^0 = (u_1^0, u_2^0, \ldots, u_N^0)$ such that, for each i and (t, x),

$$K^i(t, x, \bar{u}_0) \leqq K^i(t, x, (\widehat{\bar{u}^{0\,i}}, v_i)) \qquad \text{for all } v_i.$$

Next, in order to complete the proof, we show that $\bar{u}^0$ can be replaced by a measurable function $\bar{u}^*$ from $[0, T] \times R^n$ into $\prod_{k=1}^{N} U_k$. Since U_i is separable, it follows that, for any real number a,

$$\{(t, x):\ \min_{v_i} K^i(t, x, (\bar{u}^{0}\hat{\ i}, v_i)) < a\}$$

$$= \bigcup_{v_i \varepsilon \Gamma_i} \{(t, x): K^i(t, x, (\bar{u}^{0}\hat{\ i}, v_i)) < a\}, \tag{7}$$

where Γ_i denotes a countable dense subset of U_i. Then, since, for fixed v_i, $K^i(t, x, (\bar{u}^{0}\hat{\ i}, v_i))$ is measurable with respect to $\mathfrak{D}$, from (7)

$$\min_{v_i} K^i(t, x, (\bar{u}^{0}\hat{\ i}, v_i))$$

is measurable with respect to $\mathfrak{D}$. Consequently, since $K^i(t, \mathrm{x}, (\bar{u}^{0}\hat{\ i}, v_i))$ is continuous in v_i, by using Lemma 1 of Ref. 1, we can prove that there exists a measurable function u_i^* from $[0, \mathrm{T}] \times R^n$ into U_i such that

$$\min_{v_i} K^i(t, x, (\bar{u}^{0}\hat{\ i}, v_i)) = K^i(t, x, (\bar{u}^{0}\hat{\ i}, u_i^*)).$$

Here, $K^i(t, x, (\bar{u}^{0}\hat{\ i}, v_i))$ is convex with respect to v_i and the minimum is attained by u_i^0. So, it follows that the minimum point u_i^0 can be replaced as u_i^* for each i. Hence, for each $i \varepsilon \{1, 2, \ldots, N\}$ and (t, x), we can get

$$\min_{v_i} K^i(t, x, (\bar{u}^{*}\hat{\ i}, v_i)) = K^i(t, x, \bar{u}^*).$$

Thus, the lemma is proved.

Here, applying Ito stochastic differential rule to each solution $W^i(t, x)$ and using

$$E_{\bar{u}}[\int_0^T |W_x^i(t, x)\sigma(t)|^2 dt] < \infty,$$

we can get

$$W^i(0, x_0) = E_{\bar{u}}\{\int_0^T [-W_t^i(t, x) - A(t, x, \bar{u})W^i(t, x)]\, dt + W^i(T, x)\}, \tag{8}$$

where

$$A(t, x, \bar{u})W^i(t, x) = \frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(t)\frac{\partial^2 W^i(t, x)}{\partial x_j \partial x_k} +$$

$$+ K^i(t, x, \bar{u}) - L^i(t, x, \bar{u}). \tag{9}$$

THEOREM 3.1 *Suppose that there exist the solutions $W^i(t, x)$, $i = 1, 2, \ldots, N$, of the optimality equations* (4). *The game has an equilibrium point in pure strategies.*

Proof. From the existence of the solution W^i of (4) and the result of Lemma 3.1, there exists a multistrategy $\bar{u}^*$ such that

$$W_t^i(t, x) + \min_{\bar{u}} \{A(t, x, \bar{u}) W^i(t, x) + L^i(t, x, \bar{u})\} =$$

$$= W_t^i(t, x) + A(t, x, \bar{u}^*) W^i(t, x) + L^i(t, x, \bar{u}^*) = 0. \tag{10}$$

So, by using the equation (8) with $(\widehat{\bar{u}^{*i}}, v_i)$ instead of $\bar{u}$ and an inequality with $(\widehat{\bar{u}^{*i}}, v_i)$ instead of $\bar{u}^*$ in (10), i. e.,

$$W_t^i(t, x) + A(t, x, (\widehat{\bar{u}^{*i}}, v_i)) W^i(t, x) + L^i(t, x, (\widehat{\bar{u}^{*i}}, v_i)) \geqq 0\,,$$

it follows that

$$W^i(0, x_0) \leqq E_{(\widehat{\bar{u}^{*i}}, v_i)}\left[\int_0^T L^i(t, x, (\widehat{\bar{u}^{*i}}, v_i))\, dt + W^i(T, x)\right]. \tag{11}$$

Also, inserting (10) into (8) with $\bar{u}^*$ instead of $\bar{u}$, it follows that

$$W^i(0, x_0) = E_{\bar{u}^*}\left[\int_0^T L^i(t, x, \bar{u}^*)\, dt + W^i(T, x)\right]. \tag{12}$$

Then, from the definition of $L^i(t, x, \bar{u})$, (11) and (12), it holds that, for any pure strategy v_i of player i

$$E_{\bar{u}^*}\left\{\int_0^T \left[x(t)' Q^i(t) x(t) + \sum_{j=1}^N u_i^{*'} R_{ij}(t) u_j^*\right] dt + W^i(T, x)\right\}$$

$$\leqq E_{(\widehat{\bar{u}^{*i}}, v_i)}\left\{\int_0^T \left[x(t)' Q^i(t) x(t) + \sum_{j=1}^N v'_i R_{ij}(t) u_j^*\right] dt + W^i(T, x)\right\}. \tag{13}$$

Consequently, we arrive at

$$J^i(\bar{u}^*) \leqq J^i(\widehat{\bar{u}^{*i}}, v_i) \quad \text{for all pure strategies } v_i.$$

Thus, the theorem is proved.

Throughout this paper, a scalar multiple λI of the identity matrix I is simply denoted by the scalar λ.

From Lemma 2. 2 in Ref. 9 and Theorem 2. 2 in Ref. 8, the necessary and sufficient condition for a pure multistrategy $\bar{u}^*$ to be an equilibrium point is given by the following theorem.

THEOREM 3. 2 *For a pure multistrategy $\bar{u}^*$ to be an equilibrium point, it is necessary and sufficient that, for each $i \in \{1, 2, \ldots, N\}$ and (t, x), there exists $\lambda^i(t, x) \leqq \lambda_*^i(t)$ such that*

$$2(R_{ii}(t) - \lambda^i(t, x))\, u_1^*(t, x) + S^i(t, x, \widehat{\bar{u}^{*i}}(t, x)) = 0,$$

$$\lambda^i(t, x)\,(M_i - |u_i^*(t, x)|) = 0,$$

where

$$\lambda_*^i(t) = \min\{\min_{|u_i|=1} u_i' R_{ii}(t) u_i, 0\}.$$

4. An equilibrium point constructing of a mixed strategy

We consider the general case in which the matrix functions $R_{ii}(t)$, $i=1, 2, \ldots, N$, may not be positive semidefinite. So, we introduce the following modified optimality equation of (4):

$$W_t^i(t, x) + \frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(t)\frac{\partial^2 W^i(t, x)}{\partial x_j \partial x_k} + \min_{\bar{\mu}} K_*^i(t, x, \bar{\mu}) = 0$$

$$\text{with} \quad W^i(T, x) = x(T)' F^i(T) x(T), \tag{14}$$

where for any pure multistrategy $\bar{u} = (u_1, u_2, \ldots, u_N)$

$$\begin{aligned} K_*^i(t, x, \bar{u}) &= K^i(t, x, \bar{u}) - \lambda_*^i(t))|u_i|^2 \\ &= u_i'(R_{ii}(t) - \lambda_*^i(t))u_i + u_i' S^i(t, x, \hat{\bar{u}^i}) + \\ &+ T^i(t, x, \hat{\bar{u}^i}) + Z^i(t, x). \end{aligned} \tag{15}$$

Then, since $R_{ii}(t) - \lambda_*^i(t)$ is positive semidefinite, we show that an equilibrium point is constructed by an atomic probability measure under the assumption that there exists a solution $W^i(t, x)$ of (14). Now, it follows from Lemma 3. 1 that there exists a pure multistrategy $\bar{u}^* = (u_1^*, u_2^*, \ldots, u_N^*)$ such that

$$K_*^i(t, x, \bar{u}^*) \leqq K_*^i(t, x, (\hat{\bar{u}^{*i}}, v_i)) \quad \text{for any pure strategy } v_i. \tag{16}$$

Further, from Theorem 3. 2, it follows that, for this $\bar{u}^*$ and each (t, x), there exists $\lambda^i(t, x) \leqq \lambda_*^i(t)$ satisfying

$$\begin{aligned} &2(R_{ii}(t) - \lambda^i(t, x))u_1^*(t, x) + S^i(t, x, \hat{\bar{u}^{*i}}(t, x)) = 0, \\ &(\lambda^i(t, x) - \lambda_*^i(t))(M_i - |u_i^*(t, x)|) = 0. \end{aligned} \tag{17}$$

Let $\bar{u}^* = (u_1^*, u_2^*, \ldots, u_N^*)$ be the multistrategy in (16). For each $i \in \{1, 2, \ldots, N\}$ and $(t, x) \in [0, T] \times R^n$, we consider two cases:

(a) if $|u_i^*(t, x)| = M_i$ or $\lambda_*^i(t) = 0$, let

$$u_i^1(t, x) = u_i^2(t, x) = u_i^*(t, x)$$

(b) otherwise, i. e., $|u_i^*(t, x)|<M_i$ and $\lambda_*^i(t)<0$. Since $\{u_i \in R^{p_i}: |u_i|=1\}$ is compact subset in R^{p_i}, it follows from the definition of $\lambda_*^i(t)$ that there exists $\tilde{u}_i(t) \in R^{p_i}$ such that

$$|\tilde{u}_i(t)|=1, \quad \tilde{u}_i(t)' R_{ii}(t)\tilde{u}_i(t)=\lambda_*^i(t). \tag{18}$$

Let

$$\beta=\beta_i^1(t, x) \quad \textit{and} \quad \beta=\beta_i^2(t, x)$$

be the solution of an equation

$$|u_i^*(t, x)+\beta\tilde{u}_i(t)|^2=M_i^2 \tag{19}$$

and, then, let

$$u_i^1(t, x)=u_i^*(t, x)+\beta_i^1(t, x)\,\tilde{u}_i(t)$$

$$u_i^2(t, x)=u_i^*(t, x)+\beta_i^2(t, x)\,\tilde{u}_i(t). \tag{20}$$

By making use of $\nu_i^1(t, x)=|\beta_i^2(t, x)|/(|\beta_i^1(t, x)|+|\beta_i^2(t, x)|)$ and $\nu_i^2(t, x)=1-\nu_i^1(t, x)$, we can show the following equations:

$$\sum_{k=1}^{2}\nu_i^k(t, x)=1, \quad \sum_{k=1}^{2}\nu_i^k(t, x)u_i^k(t, x)=u_i^*(t, x). \tag{21}$$

So, we can make the following mixed strategy:

$$\sigma_i(t, x)=\nu_i^1(t, x)\delta(u_i^1(t, x))+\nu_i^2(t, x)\delta(u_i^2(t, x)), \tag{22}$$

but, in the case (a), $\sigma_i(t, x)$ can be chosen as the pure strategy $\delta(u_i^*(t, x))$. If we can understand explicitly (t, x) in (22), we write σ_i instead of $\sigma_i(t, x)$. Also, when a mixed multistrategy $\bar{\mu}=(\mu_1, \mu_2, \ldots, \mu_N)$ is used, the corresponding expected function of $K_*^i(t, x, \bar{u})$ for player i is

$$K_*^i(t, x, \bar{\mu})=\int_{\prod_{k=1}^{N} U_k} K_*^i(t, x, \bar{u})\,d\bar{\mu}(\bar{u})$$

$$=K^i(t, x, \bar{\mu})-\lambda_*^i(t)\int_{U_i}|u_i|^2\,d\mu_i(u_i) \tag{23}$$

where

$$d\bar{\mu}(\bar{u})=d\mu_1(u_1)\,d\mu_2(u_2)\ldots d\mu_N(u_N).$$

LEMMA 4.1 *Let* $i \varepsilon \{1, 2, \ldots, N\}$. *Then, for* $\bar{\sigma}$ *in* (22) *and* $\bar{u}^*$ *in* (16),

$$K_*^i(t, x, (\widehat{\bar{\sigma}^i}, \delta(u_i)) = K_*^i(t, x, (\widehat{\bar{u}^{*i}}, u_i) \quad \text{for all } u_i \varepsilon U_i, \tag{24}$$

where

$$(\widehat{\bar{\sigma}^i}, \delta(u_i)) = (\sigma_1, \ldots, \sigma_{i-1}, \delta(u_i), \sigma_{i+1}, \ldots, \sigma_N).$$

PROOF. By suing σ_i, ν_j^k and u_j^k, $k=1, 2$, omitting (t, x) in (22), it follows that

$$K_*^i(t, x, (\widehat{\bar{\sigma}^i}, \delta(u_i))$$

$$= u_i'(R_{ii}(t) - \lambda_*^i(t))u_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{k=1}^{2} \nu_j^k \{u_i' \, S^i(t, x, u_j^k) +$$

$$+ T^i(t, x, u_j^k)\} + Z^i(t, x)$$

$$= u_i'(R_{ii}(t) - \lambda_*^i(t))u_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} u_i' \, S^i(t, x, u_j^*) +$$

$$+ T^i(t, x, u_j^*) + Z^i(t, x)$$

$$= K_*^i(t, x, (\widehat{\bar{u}^{*i}}, u_i)),$$

where

$$S^i(t, x, u_j^k) = R_{ij}(t)u_j^k + B_i(t)'W_x^i(t, x)$$

and

$$T^i(t, x, u_j^k) = u_j^{k\prime} \, B_j(t)' \, W_x^i(t, x).$$

Thus, the proof is complete.

LEMMA 4.2 *Let* $i \varepsilon \{1, 2, \ldots, N\}$. *Then, for any mixed multistrategy* $\widehat{\bar{\mu}^i} = (\mu_1, \mu_2, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_N) \varepsilon \prod_{\substack{k=1 \\ k\neq i}}^{N} P(U_k)$ *and i th element* σ_i *of* $\bar{\sigma}$ *in* (22), *it follows that*

$$K_*^i(t, x, (\widehat{\bar{\mu}^i}, \sigma_i)) = K_*^i(t, x, (\widehat{\bar{\mu}^i}, \delta(u_i^*)). \tag{25}$$

PROOF. Consider two cases: for any $\bar{u} \varepsilon \prod_{k=1}^{N} U_k$

(a) if $u_i^*(t, x) = M_i$ or $\lambda_*^i(t) = 0$, by using the notation:

$$\delta(\widehat{\bar{u}^i}) = (\delta(u_1), \delta(u_2), \ldots, \delta(u_{i-1}), \delta(u_{i+1}), \ldots, \delta(u_N)),$$

$$K_*^i(t, x, (\delta(\widehat{\bar{u}^i}), \sigma_i)) = K_*^i(t, x, (\widehat{\bar{u}^i}, u_i^*))$$

(b) otherwise,

$$K_*^i(t, x, (\delta(\widehat{\bar{u}^i}), \sigma_i))$$

$$= \sum_{k=1}^{2} \nu_i^k \{ u_1^{k\prime} (R_{ii}(t) - \lambda_*^i(t)) u_i^k + u_1^{k\prime} S^i(t, x, \widehat{\bar{u}^i}) +$$

$$+ T^i(t, x, \widehat{\bar{u}^i}) + Z^i(t, x)$$

$$= u_i^{*\prime} (R_{ii}(t) - \lambda_*^i(t)) u_i^* + u_i^{*\prime} S^i(t, x, \widehat{\bar{u}^i}) + T^i(t, x, \widehat{\bar{u}^i}) +$$

$$+ \sum_{k=1}^{N} \nu_i^k \beta_i^k \widetilde{u_i}' (R_{ii}(t) - \lambda_*^i(t)) u_i^* +$$

$$+ \sum_{k=1}^{N} \nu_i^k (\beta_i^k)^2 \widetilde{u}'_i (R_{ii}(t) - \lambda_*^i(t)) \widetilde{u_i} + Z^i(t, x). \tag{26}$$

From (21), we have

$$\sum_{k=1}^{2} \nu_i^k \beta_i^k = 0 \tag{27}$$

and since by (19)

$$\sum_{k=1}^{2} \nu_i^k (\beta_i^k)^2 = -\beta_i^1 \times \beta_i^2 = M_i^2 - |u_i^*|^2,$$

from (18), it follows that

$$\sum_{k=1}^{2} \nu_i^k (\beta_i^k)^2 \widetilde{u_i}' (R_{ii}(t) - \lambda_*^i(t)) \widetilde{u_i} = 0. \tag{28}$$

By (15), (26), (27) and (28), we get

$$K_*^i(t, x, (\delta(\widehat{\bar{u}^i}), \sigma_i)) = K_*^i(t, x, (\delta(\widehat{\bar{u}^i}), u_i^*)). \tag{29}$$

Integrating both sides of (29) with respect to any $\widehat{\bar{\mu}^i} \varepsilon \prod_{\substack{k=1 \\ k \neq i}}^{N} P(U_k)$, we have

$$K_*^i(t, x, (\widehat{\bar{\mu}^i}, \sigma_i)) = K_*^i(t, x, (\widehat{\bar{\mu}^i}, u_i^*)) \quad \text{for any } \widehat{\bar{\mu}^i} \varepsilon \prod_{\substack{k=1 \\ k \neq i}}^{N} P(U_k).$$

Hence, the lemma is proved.

Now, since from (19) and (20), we have

$$|u_i^k|^2 = M_i^2 \quad \text{for each } k \in \{1, 2\},$$

it follows by Lemma 4. 1 and Lemma 4. 2 that, $\bar{\sigma}$ in (22) and $\bar{u}^*$ in (16),

$$\begin{aligned} K^i(t, x, \bar{\sigma}) &= K_*^i(t, x, \bar{\sigma}) + \lambda_*^i(t) \sum_{k=1}^{2} \nu_i^k |u_i^k|^2 \\ &= K_*^i(t, x, \bar{\sigma}) + \lambda_*^i(t) M_i^2 \qquad \text{by } |u_i^k|^2 = M_i^2 \\ &= K_*^i(t, x, (\bar{\sigma}^{\hat{i}}, \delta(u_i^*))) + \lambda_*^i(t) M_i^2 \\ &\quad (\text{by Lemma 4. 2 with } \bar{\sigma} \text{ instead of } \bar{\mu}) \\ &= K_*^i(t, x, \bar{u}^*) + \lambda_*^i(t) M_i^2 \qquad \text{by Lemma 4. 1.} \end{aligned} \tag{30}$$

On the other hand, it follows from (16) and Lemma 4. 1 that for $\bar{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ in (22),

$$\begin{aligned} K_*^i(t, x, \bar{u}^*) + \lambda_*^i(t) M_i^2 &\leqq K_*^i(t, x, (\bar{u}^{*\hat{i}}, u_i)) + \lambda_*^i(t) M_i^2 \qquad \text{for any } u_i \in U_i \\ &= K_*^i(t, x, (\overline{\sigma}^{\hat{i}}, \delta(u_i))) + \lambda_*^i(t) M_i^2 \\ &= K^i(t, x, (\bar{\sigma}^{\hat{i}}, \delta(u_i))) + \lambda_*^i(t) (M_i^2 - |u_i|^2) \\ &\quad (\text{by } \lambda_*^i(t) \leqq 0 \text{ and } |u_i| \leqq M_i) \\ &\leqq K^i(t, x, (\bar{\sigma}^{\hat{i}}, \delta(u_i))). \end{aligned} \tag{31}$$

Integrating both sides of (31) with respect to $\mu_i \in P(U_i)$, we have

$$K_*^i(t, x, \bar{u}^*) + \lambda_*^i(t) M_i^2 \leqq K^i(t, x, (\bar{\sigma}^{\hat{i}}, \mu_i)) \quad \textit{for any } \mu_i \in P(U_i).$$

THEOREM 4. 1 *Suppose that there exists a solution $W^i(t, x)$ of* (14). *Then, the multistrategy $\bar{\sigma}$ in* (22) *is an equilibrium point, i. e.,*

$$J^i(\bar{\sigma}) \leqq J^i(\bar{\sigma}^{\hat{i}}, \mu_i) \quad \textit{for all mixed strategies } \mu_i.$$

PROOF. From (14) and (30), the following equality holds

$$W_t^i(t, x) + \frac{1}{2} \sum_{j,k=1}^{N} a_{jk}(t) \frac{\partial^2 W^i(t, x)}{\partial x_j \partial x_k} + K^i(t, x, \bar{\sigma})$$

$$=W_t^i(t, x)+\frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(t)\frac{\partial^2 W^i(t, x)}{\partial x_j \partial x_k}+K_*^i(t, x, \bar{u}^*)+$$

$$+\lambda_*^i(t) M_i^2$$

$$=\lambda_*^i(t) M_i^2,$$

that is, by using the notation in (9),

$$W_t^i(t, x)+A(t, x, \bar{\sigma}) W^i(t, x)+L_i(t, x, \bar{\sigma})=\lambda_*^i(t) M_i^2. \tag{32}$$

Inserting (32) into (8) with $\bar{\sigma}$ instead of $\bar{u}$, we have

$$W^i(0, x_0)=E_{\bar{\sigma}}[\int_0^T L^i(t, x, \bar{\sigma})\, dt+W^i(T, x)]-(\int_0^T \lambda_*^i(t)\, dt) M_i^2. \tag{33}$$

On the other hand, from, (14) and (31), the following inequality holds: for any mixed strategy μ_i

$$\lambda_*^i(t) M_i^2 \leqq W_t^i(t, x)+A(t, x, (\widehat{\bar{\sigma}^i}, \mu_i)) W^i(t, x)+$$

$$+L^i(t, x, (\widehat{\bar{\sigma}^i}, \mu_i)).$$

Inserting (34) into (8) with $(\widehat{\bar{\sigma}^i}, \mu_i)$ instead of $\bar{u}$, we get

$$W^i(0, x_0) \leqq E_{(\widehat{\bar{\sigma}^i}, \mu_i)}[\int_0^T L^i(t, x, (\widehat{\bar{\sigma}^i}, \mu_i))\, dt+W^i(T, x)]-$$

$$-(\int_0^T \lambda_*^i(t)\, dt) M_i^2. \tag{35}$$

Combining with (33) and (35), we arrive at, for any mixed strategy μ_i,

$$E_{\bar{\sigma}}[\int_0^T L^i(t, x, \bar{\sigma})\, dt+W^i(T, x)]$$

$$\leqq E_{(\widehat{\bar{\sigma}^i}, \mu_i)}[\int_0^T L^i(t, x, (\widehat{\bar{\sigma}^i}, \mu_i))\, dt+W^i(T, x)]. \tag{36}$$

Since

$$W^i(T, x)=x(T)' F^i(T) x(T),$$

it follows from (2) and (36)

$$J^i(\bar{\sigma}) \leqq J^i(\widehat{\bar{\sigma}^i}, \mu_i) \quad \text{for any mixed strategy } \mu_i.$$

Thus, the thorem is proved.

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Department of Mathematics
Faculty of Science
Niigata University
Niigata, 950–21, Japan