On some compact Riemannian 3-symmetric spaces

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1. Introduction

Let (M, g) be a Riemannian manifold with the Riemannian metric tensor g. We denote by ∇ and R the Riemannian connection and the curvature tensor of (M, g), respectively. The Ricci curvature tensor R_1 of (M, g) is obtained by a contraction of the cur vature tensor R, and (M, g) is called an Einstein space when $R_1 = \alpha g$ holds on M for some constant α. For example, an irreducible Riemannian symmetric space is necessarily an Einstein space. Let (M, J, g) be an almost Hermitian manifold with the almost Hermitian structure (J, g). For (M, J, g) there is another useful contraction of the curvature tensor which is called the Ricci * curvature tensor. The Ricci * curvature tensor $R_1{}^*$ is defined by

$$(1.1) \qquad R_1{}^*(X, Y) = \frac{1}{2}(\mathit{Trace\ of}\ (Z \to R(Y, JX)JZ)),$$

for tangent vectors X, Y of M.
An almost Hermitian manifold (M, J, g) is called nearly Kaehlerian manifold (also known as K-space or almost Tachibana space) provided that the almost Hermitian structure (J, g) satisfies the condition $(\nabla_X J)X = 0$ for any tangent vector X of M. In a nearly Kaehlerian manifold (M, J, g), it is well known that the Ricci curvature tensor R_1 and the Ricci * curvature tensor $R_1{}^*$ satisfy the followings:

$$(1.2) \qquad R_1(X, Y) = R_1(Y, X), \quad R_1(X, Y) = R_1(JX, JY),$$

$$(1.3) \qquad R_1^*(X, Y) = R_1^*(Y, X), \quad R_1^*(X, Y) = R_1^*(JX, JY),$$

for all tangent vectors X, Y of M.

The first Chern form of a nearly Kaehlerian manifold is represented by the 2-form γ_1 (known as the generalized first Chern form) which is defined using the tensor fields R_1, R_1^* and J as follows:

$$(1.4) \qquad 8\pi\gamma_1(X, Y) = 5R_1^*(JX, Y) - R_1(JX, Y),$$

for all tangent vectors X, Y of M (cf. [2], [10]).

In [1], Gray has introduced the notion of Riemannian 3-symmetric space and obtained many interesting results in connection with the geometry of almost Hermitian manifolds. For example, he showed that every Riemannian 3-symmetric space is a homogeneous almost Hermitian manifold together with the canonical almost complex structure associated with the Riemannian 3-symmetric structure, and some of Riemannian 3-symmetric spaces are nearly Kaehlerian manifold. In this paper, we shall calculate the Ricci curvature tensors and the Ricci * curvature tensors of some compact Riemannian 3-symmetric spaces which are defined by the inner automorphisms of compact, simple classical Lie groups of order 3 and state some related results. For example, we may give some new examples of Einstein spaces and show that a complex projective space CP^3 of complex dimension 3 admits an Einstein metric which is not symmetric one. The authors wish to express their hearty thanks to Prof. T. Watabe for his kind advices.

2. The classical Lie algebras

Let G be a connected Lie group and $\mathfrak{g}$ be the Lie algebra of G. We denote by Aut (G) (resp. Aut $(\mathfrak{g})$) the automorphism group of G (resp. $\mathfrak{g}$). Each element $\sigma \in$ Aut (G) induces an element of Aut $(\mathfrak{g})$ in the natural way. So, we also denote by the same letter σ the corresponding element of Aut $(\mathfrak{g})$. We now recall the Lie algebras of the compact classical Lie groups. In the sequel, we denote by $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ the set of all real numbers, complex numbers and quaternionic numbers, respectively, and furthermore by $\mathfrak{gl}(N, \boldsymbol{R})$, $\mathfrak{gl}(N, \boldsymbol{C})$ and $\mathfrak{gl}(N, \boldsymbol{H})$ the sets of all $N \times N$ real matrices, complex matrices and quaternionic matrices, respectively.

$$(A_n) \qquad \mathfrak{g} = \mathfrak{su}(n+1) = \{X \in \mathfrak{gl}(n+1, \boldsymbol{C})\,;\, {}^tX = -\overline{X},\ \mathrm{Trace}\ X = 0\}.$$

We put

$$(2.1) \qquad U_{ij} = E_{ij} - E_{ji},$$
$$U'_{ij} = \sqrt{-1}(E_{ij} + E_{ji}), \quad 1 \leq i, j \leq n+1,$$

where E_{ij} denotes the $(n+1)\times(n+1)$ matrix whose r-th row and s-th column is given by $\delta_{ir}\,\delta_{js}$.

Let $t_i = \sqrt{-1}(E_{ii} - E_{i+1\, i+1})$ $(1 \leq i \leq n)$. Then the Lie subalgebra of $\mathfrak{su}(n+1)$ generated by $\{t_i (1 \leq i \leq n)\}$ over $\boldsymbol{R}$ is a maximal abelian subalgebra of $\mathfrak{su}(n+1)$. We may easily see that $\{(\sqrt{2/(i^2+i)}) \sum_{a=1}^{i} a t_a\ (1 \leq i \leq n);\ U_{ij},\ U'_{ij}\ (1 \leq i < j \leq n+1)\}$ forms an orthonormal basis for $\mathfrak{su}(n+1)$ with respect to the inner product $<, >$ on $\mathfrak{su}(n+1)$ defined by $<X, Y> = -\frac{1}{2}$ *Trace* XY, $X, Y \in \mathfrak{su}(n+1)$. We note that the inner product $<, >$ on $\mathfrak{su}(n+1)$ induces a biinvariant Riemannian metric on the Lie group $G = SU(n+1)$. From (2.1), the Lie multiplication table is given by

$$(2.2) \qquad [U_{ij}, U_{ab}] = \delta_{ja} U_{ib} - \delta_{jb} U_{ia} - \delta_{ia} U_{jb} + \delta_{ib} U_{ja},$$

$[U_{ij}, U'_{ab}]=\delta_{ja}U'_{ib}+\delta_{jb}U'_{ia}-\delta_{ia}U'_{jb}-\delta_{ib}U'_{ja},$

$[U'_{ij}, U'_{ab}]=-\delta_{ja}U_{ib}-\delta_{jb}U_{ia}-\delta_{ia}U_{jb}-\delta_{ib}U_{ja}.$

(B_n) $\quad \mathfrak{g}=\mathfrak{so}(2n+1)=\{X\in\mathfrak{gl}(2n+1, \boldsymbol{R});\ {}^tX=-X\}.$

We put $U_{PQ}=E_{PQ}-E_{QP}$, and

(2.3) $\quad U_i=\dfrac{1}{\sqrt{2}}(u_{1\ n+1+i}+u_{1\ i+1}),\ U'_i=\dfrac{1}{\sqrt{2}}(u_{n+1\ i+1}+u_{n+1+i\ 1}),$

$U_{ij}=\dfrac{1}{\sqrt{2}}(u_{i+1\ j+1}-u_{n+1+i\ n+1+j}),$

$U'_{ij}=\dfrac{1}{\sqrt{2}}(u_{i+1\ n+1+j}-u_{j+1\ n+1+i}),$

$V_{ij}=\dfrac{1}{\sqrt{2}}(u_{i+1\ j+1}+u_{n+1+i\ n+1+j}),$

$V'_{ij}=\dfrac{1}{\sqrt{2}}(u_{i+1+i\ j+1}+u_{n+1+j\ i+1}),\qquad 1\leqq i,\ j\leqq n,$

where E_{PQ} denotes the $(2n+1)\times(2n+1)$ matrix whose S-th row and T-th column is given by $\delta_{PS}\ \delta_{QT}$.

Let $t_i=(1/\sqrt{2})\,V'_{ii}\ (1\leqq i\leqq n)$. Then the Lie subalgebra of $\mathfrak{so}(2n+1)$ generated by $\{t_i\ (1\leqq i\leqq n)\}$ over $\boldsymbol{R}$ is a maximal abelian subalgebra of $\mathfrak{so}(2n+1)$. We may easily see that $\{t_i(1\leqq i\leqq n);\ U_{ij},\ U'_{ij},\ V_{ij},\ V'_{ij}(1\leqq i<j\leqq n)\}$ forms an orthonormal basis for $\mathfrak{so}(2n+1)$ with respect to the inner product $<, >$ defined by $<X, Y>=-\frac{1}{2}\ \mathit{Trace}\ XY,\ X,\ Y\in\mathfrak{so}(2n+1)$. We note that the inner product $<, >$ on $\mathfrak{so}(2n+1)$ induces a biinvariant Riemannian metric on the Lie group $G=SO(2n+1)$. From (2.3), the Lie multiplication table is given by

(2.4) $\quad [U_a, U_b]=(1/\sqrt{2})\,(U_{ba}+V_{ba}),\ [U_a, U'_b]=(1/\sqrt{2})\,(V'_{ab}-U'_{ab}),$

$[U'_a, U'_b]=(1/\sqrt{2})\,(U_{ab}-V_{ab}),\ [U_i, U_{ab}]=(1/\sqrt{2})\,(\delta_{ia}U_b-\delta_{ib}U_a),$

$[U_i, U'_{ab}]=(1/\sqrt{2})\,(\delta_{ia}U'_b-\delta_{ia}U'_b),$

$[U'_i, U_{ab}]=(/\sqrt{2})\,(\delta_{ib}U'_a-\delta_{ia}U'_b),$

$[U'_i, U'_{ab}]=(1/\sqrt{2})\,(\delta_{ia}U_b-\delta_{ib}U_a),$

$[U_i, V_{ab}]=(1/\sqrt{2})\,(\delta_{ia}U_b-\delta_{ib}U_a),$

$[U_i, V'_{ab}]=-(1/\sqrt{2})\,(\delta_{ia}U'_b+\delta_{ib}U'_a),$

$[U'_i, V_{ab}]=(1/\sqrt{2})\,(\delta_{ia}U'_b-\delta_{ib}U'_a),$

$[U'_i, V'_{ab}]=(1/\sqrt{2})\,(\delta_{ia}U_b+\delta_{ib}U_a),$

$[U_{ij}, U_{ab}]=(1/\sqrt{2})\,(\delta_{ja}V_{ib}-\delta_{jb}V_{ia}+\delta_{ib}V_{ja}-\delta_{ia}V_{jb}),$

$[U_{ij}, U'_{ab}]=(1/\sqrt{2})\,(\delta_{ja}V'_{ib}-\delta_{jb}V'_{ia}+\delta_{ib}V'_{ja}-\delta_{ia}V'_{jb}),$

$[U'_{ij}, U'_{ab}]=(1/\sqrt{2})\,(\delta_{ja}V_{ib}-\delta_{jb}V_{ia}+\delta_{ib}V_{ja}-\delta_{ia}V_{jb}),$

$$[U_{ij}, V_{ab}]=(1/\sqrt{2})(\delta_{ja}U_{ib}-\delta_{jb}U_{ia}+\delta_{ib}U_{ja}-\delta_{ia}U_{jb}),$$

$$[U_{ij}, V'_{ab}]=(1/\sqrt{2})(\delta_{ia}U'_{jb}-\delta_{jb}U'_{ia}+\delta_{ib}U'_{ja}-\delta_{ja}U'_{ib}),$$

$$[U'_{ij}, V'_{ab}]=(1/\sqrt{2})(\delta_{ja}U_{ib}+\delta_{ib}U_{ia}-\delta_{ia}U_{jb}-\delta_{ib}U_{ja}),$$

$$[U'_{ij}, V_{ab}]=(1/\sqrt{2})(\delta_{ja}U'_{ib}-\delta_{jb}U'_{ia}+\varrho_{ib}U'_{ja}-\delta_{ia}U'_{jb}),$$

$$[V_{ij}, V_{ab}]=(1/\sqrt{2})((\delta_{ja}V_{ib}-\varrho_{jb}V_{ia}-\delta_{ia}V_{jb}+\delta_{ib}V_{ja}),$$

$$[V_{ij}, V'_{ab}]=(1/\sqrt{2})(\delta_{ja}V'_{ib}+\delta_{jb}V'_{ia}-\delta_{ia}V'_{jb}-\delta_{ib}V'_{ja}),$$

$$[V'_{ij}, V'_{ab}]=-(1/\sqrt{2}(\delta_{ja}V_{ib}+\delta_{ib}V_{ia}+\delta_{ia}V_{jb}+\delta_{ib}V_{ja}),$$

(C_n) $\quad \mathfrak{g}=\mathfrak{sp}(n)=\{X\in\mathfrak{gl}(n, \boldsymbol{H})\,;\,{}^tX=-\overline{X}\}.$

We suppose that $\boldsymbol{H}$ is generated by $\{e_0=1, e_1, e_2, e_3\}$ over $\boldsymbol{R}$. Any quaternion q is written as $q=(a_0+a_1e_1)+(a_2+a_3e_1)e_2$, and $\boldsymbol{R}1+\boldsymbol{R}e_1$ is isomorphic with $\boldsymbol{C}$ by the mapping $\phi: a_0+a_1e_1\rightarrow a_0+\sqrt{-1}a_1$. By this isomorphism ϕ we may identify $\boldsymbol{R}1+\boldsymbol{R}e_1$ with $\boldsymbol{C}$. We not put

$$\mathfrak{g}_0=\left\{X=\begin{pmatrix} A & B \\ -\overline{B} & \overline{A}\end{pmatrix}\in\mathfrak{gl}(2n, \boldsymbol{C})\,;\,{}^tA=-\overline{A},\ {}^tB=B,\ A,\ B\in\mathfrak{gl}(n, \boldsymbol{C})\right\}.$$

Writing $q_i=z_i+z_{n+1}e_2$ $(1\leqq i\leqq n)$, we obtain an isomerphism Ψ: $\mathfrak{sp}(n)\rightarrow\mathfrak{g}_0$ by the cononical way. We may identify $\mathfrak{sp}(n)$ with the Lie algebra $\mathfrak{g}_0$ (and hence the Lie group Sp (n) with the Lie group $\exp\mathfrak{g}_0$ by the canonical isomorphism Ψ which is induced by the iso morphism Ψ). We put

$$(2.5)\qquad W_{ij}=(1/\sqrt{2})(E_{i\ n+i}+E_{j\ n+j}-E_{n+i\ j}-E_{n+j\ i}),$$

$$W'_{ij}=(\sqrt{-1}/\sqrt{2})(E_{i\ n+j}+E_{j\ n+i}+E_{n+i\ j}+E_{n+j\ i}),$$

$$U_{ij}=(1/\sqrt{2})(E_{ij}-E_{ji}+E_{n+i\ n+j}-E_{n+j\ n+i}),$$

$$U'_{ij}=(\sqrt{-1}/\sqrt{2})(E_{ij}+E_{ji}-E_{n+i\ n+j}-E_{n+j\ n+i}),\quad 1\leqq i,\ j\leqq n,$$

where E_{PQ} denotes the $(2n)\times(2n)$ matrix whose S-th row and T-th column is given by $\delta_{PS}\,\delta_{QT}$. Let $t_i=(1/\sqrt{2})\ U'_{ii}(1\leqq i\leqq n)$. Then the Lie subalgebra of $\mathfrak{sp}(n)$ generated by $\{t_i\ (1\leqq i\leqq n)\}$ over $\boldsymbol{R}$ is a maximal abelian subalgebra of $\mathfrak{sp}(n)$. We may easily see that $\{t_i(1\leqq i\leqq n)$: $W_i=(1/\sqrt{2})\ W_{ii}$, $W'_i=(1/\sqrt{2}W'_{ii}\ (1\leqq i\leqq n)$; W_{ij}, $W'_{ij}\ (1\leqq i<j\leqq n)$; U_{ij}, $U'_{ij}\ (1\leqq i<j\leqq n)\}$ forms an orthonormal basis for $\mathfrak{sp}(n)$ with respect to the inner product $<,>$ defined by $<X, Y>=-\frac{1}{2}\ Trace\ XY$, X, $Y\in\mathfrak{sp}(n)$. We note that the inner product $<,>$ on $\mathfrak{sp}(n)$ induces a biinvariant Riemannian metric on the Lie group $G=Sp(n)$. From (2. 5), the Lie multiplication table is given by

$$(2.6)\qquad [U_{ij}, U_{ab}]=(1/\sqrt{2})(\delta_{ja}U_{ib}-\delta_{ia}U_{jb}-\delta_{jb}U_{ia}+\delta_{ib}U_{ja}),$$

$$[U'_{ij}, U_{ab}]=(1/\sqrt{2})(\delta_{ja}U'_{ib}+\delta_{ia}U'_{jb}-\delta_{jb}U'_{ia}-\delta_{ib}U'_{ja}),$$

$$[U'_{ij}, U'_{ab}]=-(1/\sqrt{2})(\delta_{ja}U_{ib}+\delta_{ia}U_{jb}+\delta_{jb}U_{ia}+\delta_{ib}U_{ja}),$$

$$[W_{ij}, W_{ab}]=-(1\sqrt{2})(\delta_{ja}U_{ib}+\delta_{ia}U_{jb}+\delta_{jb}U_{ia}+\delta_{ib}U_{ja}),$$

$$[W'_{ij}, W_{ab}]=-(1/\sqrt{2})(\delta_{ja}U'_{ib}+\delta_{ia}U'_{jb}+\delta_{jb}U'_{ia}+\delta_{ib}U'_{ja}),$$

$$[W'_{ij}, W'_{ab}]=-(1/\sqrt{2})(\delta_{ja}U_{ib}+\delta_{ia}U_{jb}+\delta_{jb}U_{ia}+\delta_{ib}U_{ja}),$$

$$[W_{ij}, U_{ab}]=(1/\sqrt{2})(\delta_{ja}W_{ib}+\delta_{ia}W_{jb}-\delta_{jb}W_{ia}-\delta_{ib}W_{ja}),$$

$$[W'_{ij}, U_{ab}]=(1/\sqrt{2})(\delta_{ja}W'_{ib}+\delta_{ia}W'_{jb}-\delta_{jb}W'_{ia}-\delta_{ib}W_{ja}),$$

$$[W_{ij}, U'_{ab}]=-(1/\sqrt{2})(\delta_{ja}W'_{ib}+\delta_{ia}W'_{jb}+\delta_{jb}W'_{ia}+\delta_{ib}W'_{ja}),$$

$$[W'_{ij}, U'_{ab}]=(1/\sqrt{2})(\delta_{ja}W_{ib}+\delta_{ia}W_{jb}+\delta_{jb}W_{ia}+\delta_{ib}W_{ja}).$$

(D_n) $\mathfrak{g}=\mathfrak{so}(2n)=\{X\in\mathfrak{gl}(2n, \boldsymbol{R});\ {}^tX=-X\}.$

We put $u_{PQ}=E_{PQ}-E_{QP}$, and

(2.7) $U_{ij}=(1/\sqrt{2})(u_{ij}-u_{n+i\ n+j}),$

$U'_{ij}=(1/\sqrt{2})(u_{i\ n+j}-u_{j\ n+i}),$

$V_{ij}=(1/\sqrt{2})(u_{ij}+u_{n+i\ n+j}),$

$V'_{ij}=(1/\sqrt{2})(u_{n+i\ j}+u_{n+j\ i}),$ $\quad 1\leqq i,\ j\leqq n.$

where E_{PQ} denotes the $(2n)\times(2n)$ matrix whose S-th row and T-th colum is given by $\delta_{PS}\delta_{QT}$.

Let $t_i=(1/\sqrt{2})\ V'_{ii}(1\leqq i\leqq n)$. Then the Lie subalgebra of $\mathfrak{so}(2n)$ generated by $\{t_i\ (1\leqq i\leqq n)\}$ over $\boldsymbol{R}$ is a maximal abelian subalgebra of $\mathfrak{so}(2n)$. We may easily see that $\{t_i(1\leqq i\leqq n);\ U_{ij},\ U'_{ij},\ V_{ij},\ V'_{ij}\ (1\leqq i\leqq j\leqq n)\}$ forms an orthonormal basis for $\mathfrak{so}(2n)$ with respect to the inner product $<,>$ defind by $<X, Y>=-\frac{1}{2}\ Trace\ XY,\ X,\ Y\in\mathfrak{so}(2n)$. We note that the inner product $<,>$ on $\mathfrak{so}(2n)$ induces a biinvariant Riemannian metric the on Lie group $G=SO(2n)$. From (2. 7), we see that the Lie multiplication table for $\mathfrak{so}(2n)$ takes the same forms as $(2.\ 4)_{12}\sim(2.\ 4)_{21}$.

3. Riemannian 3-symmetric spaces.

Let (M, g) be a connected Riemannian manifold. We now suppose that (M, g) admits an isometry θ_p of (M, g) for each point $p\in M$ such that

(3. 1) $\theta_p{}^3=1,$

(3. 2) p is an isolated fixed point of θ_p,

(3. 3) the tensor field θ defined by $\theta_p=(d\theta_p)_p$ is of class C^∞.

Then there exists an almost complex structure J on M naturally associated with the family $\{\theta_p\}_{p\in M}$. The tensor field J is given by

(3. 4) $$\frac{\sqrt{3}}{2}\ J=\theta+\frac{1}{2}\ I,$$

and called the canonical almost complex structure associated with the family $\{\theta_p\}_{p\in M}$.

DEFINITION. A Riemannian manifold (M, g) is called a Riemannian 3-symmetric space if it admits a family of isometries $\{\theta_p\}_{p\in M}$ of (M, g) satisfying the conditions (3. 1) $\sim$(3. 3) and furthermore

$$(3.5) \qquad d\theta_p \cdot J = J \cdot d\theta_p, \text{ on } M,$$

where J is the canonical almost complex structure.

Gray [1] showed that a Riemannian 3-symmetric space is characterized by a triple (G, K, σ) satisfying the following conditions (1)$\sim$(3):

(1) G is a connected Lie group and σ is an automorphism of G of order 3,

(2) K is a closed subgroup of G such that $G_0^\sigma \subset K \subset G^\sigma$, where $G^\sigma = \{x\in G;\ \sigma(x)=x\}$ and G_0^σ denotes the identity component of G^σ.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Let algebra of G and K, respectively, and $\mathfrak{m}=\{X\in\mathfrak{g};\ (\sigma^2+\sigma+I)X=0\}$. Then we have the following direct sum decomposition (cf. [1]):

$$(3.6) \qquad \mathfrak{g}=\mathfrak{k}+\mathfrak{m}, \quad Ad(K)\mathfrak{m}=\mathfrak{m}.$$

(3) There exists a positive-definite inner product $<, >$ on $\mathfrak{m}$ which is both $Ad(K)$-invariant and σ-invariant.

The inner product $<, >$ in (3) induces a G-invariant Riemannian metric g on the homogeneous space $M=G/K$, and $(G/K, g)$ is a Riemannian 3-symmetric space. The canonical almost complex structure J on G/K is given by

$$(3.7) \qquad \frac{\sqrt{3}}{2} J = \frac{1}{2} I + \sigma|_{\mathfrak{m}}, \quad \text{at the origin } ek\in G/K.$$

Gray [1] also showed that the corresponding almost Hermitian manifold $(G/K, J, g)$ is a quasi-Kaehlerian manifold (also known sa O^*-space), and the that $(G/K, J, g)$ is a nearly Kaehlerian manifold if and only if $(G/K, g)$ is a naturally reductive Riemannian homogeneous space with respect to the decomposition (3. 6) It is well known that the Riemannian connection and the curvature tensor of a naturally reductive Riemannian homogeneous space are given respectively by the followings at eK:

$$(3.8) \qquad \nabla_X Y = \frac{1}{2}[X, Y]_{\mathfrak{m}},$$

$$(3.9) \qquad R(X, Y)Z = -[[X, Y]_{\mathfrak{k}}, Z] - \frac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} - \frac{1}{4}[[Y, Z]_{\mathfrak{m}}, X]_{\mathfrak{m}} - \frac{1}{4}[[Z, X]_{\mathfrak{m}}, Y]_{\mathfrak{m}},$$

for $X, Y, Z\in\mathfrak{m}$ (cf. [3]).

Wolf and Gray [1] have obtained the complete classification table of indecomposable Riemannian 3-symmetric. Let $(G, K=G^\sigma, \sigma)$ be a triple such that G is a connected, compact classical simple Lie group and σ is an inner automorphism of G of order 3, and g be the G-invariant Riemannian metric on the homogeneous space G/K which is induced by a biinvariant Riemannian metric on G. Then, we may easily see that the corresponding compact Riemannian 3-symmetric space $(G/K, J, g)$ together with the canonical almost complex structure J is a nearly Kaehlerian manifold. From the classification table by Wolf and Gray, we see that if $(G/K, J, g)$ is not Kaehlerian, then the corresponding triple $(G, K=G^{\sigma}, \sigma=Ad\,(\exp(2\pi v)))$ are listed in the following table:

Table 1.

G	v	$K=G^{\sigma}$
$SU(n+1)$	$\frac{\sqrt{-1}}{3(n+1)}((2n+2-h-m)\sum_{a=1}^{h} E_{aa} + (n+1-h-m)\sum_{a=h+1}^{m} E_{aa} - (h+m)\sum_{a=m+1}^{n+1} E_{aa})$ $(1\leqq h<m\leqq n)$	$S(U(h)\times U(m-h)\times U(n-m+1))$
$SO(2n+1)$	$-\frac{1}{3}\sum_{a=1}^{m} u_{1+a\ n+1+a}$ $(2\leqq m\leqq n)$	$SO(2n-2m+1)\times U(m)$
$Sp(n)$	$\frac{\sqrt{-1}}{3}\sum_{a=1}^{m}(E_{aa}-E_{n+a\ n+a})$ $(1\leqq m\leqq n-1)$	$Sp(n-m)\times U(m)$
$SO(2n)$	$-\frac{1}{3}\sum_{a=1}^{m} u_{a\ n+a}$ $(2\leqq m\leqq n-1,\ n\leqq 4)$	$SO(2n-2m)\times U(m)$

4. Some results

In this section, we shall consider the homogeneous spaces listed in Table 1. First, we shall prove the following

THEOREM A. *Let $(G, K=G^{\sigma}, \sigma=Ad\,(\exp(2\pi\, v)))$ be any one of the triples in Table 1 and g be the G-invariant Riemannian metric on the homogeneous space G/K which is induced by a biinvariant Riemannian metric on G. Then the corresponding Riemannian 3-symmetric space $(G/K, J, g)$ is irreducible and not locally symmetric, and furthermore is Einsteinian if and only if G/K is one of the followings*:

(i) $SU(3m)/S(U(m)\times U(m)\times U(m))$, $m\geqq 1$,

(ii) $SO(3m-1)/(SO(m-1)\times U(m))$, $m\geqq 2$,

(iii) $Sp(3m-1)/(Sp(m)\times U(2m-1))$, $m\geqq 1$.

If G/K is one of the spaces in (i)$\sim$(iii), *then* $R_1-5R_1{}^*=0$ *holds on* G/K, *and hence the generalized first Chern form of the corresponding nearly Kaehlerian manifold* $(G/K, J, g)$ *vanishes, where J denotes the canonical almost complex structure.*

We now recall the definitions of the Ricci curvature tensor and the Ricci * curvature tensor of a $2N$-dimensinnal almost Hermitian manifold (M, J, g). Let $\{E_1 \ldots, E_N, JE_1, \ldots, JE_N\}$ be an orthonormal basis of the tangent space to M at a point $p \in M$. Then the Ricci curvature tensor R_1 and the Ricci * curvature tensor R_1^* are given respectively by

$$(4.1)\qquad R_1(X, Y)=\sum_{i=1}^{N}(R(E_i, X, Y, E_i)+R(JE_i, X, Y, JE_i)),$$

$$(4.2)\qquad R_1^*(X, Y)=\sum_{i=1}^{N} R(Y, JX, JE_i, E_i),$$

where we put $R(X, Y, Z, W)=g(R(X, Y)Z, W)$, for all tangent vectors X, Y, Z, W at p.

For the proof of Theorem A, it suffices to check the following four cases (I)$\sim$(IV).

(I) $G/K=SU(n+1)/S(U(h)\times U(m-h)\times U(n-m+1))$.

Then, from Table 1 and (2. 1), we may easily see that the Lie subalgebra $\mathfrak{su}(n+1)^\sigma$ of $\mathfrak{su}(n+1)$ is given by the linear span of $\{t_i(1\leqq i\leqq n);\ U_{ij},\ U'_{ij}(1\leqq i\leqq j\leqq h);\ U_{ij},\ U'_{ij}(h+1\leqq i<j\leqq m);\ U_{ij},\ U'_{ij}(m+1\leqq i<j\leqq n+1)\}$ over $\boldsymbol{R}$, and hence the subspace$\mathfrak{m}$ of $\mathfrak{su}(n+1)$ in the decomposition (3. 6) is given by $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3$ (directsum), where $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{m}_3$ are the subspaces of $\mathfrak{m}$ generated respectively by $\{U_{ij},\ U'_{ij}(1\leqq i\leqq h,\ h+1\leqq j\leqq m)\}$, $\{U_{ij},\ U'_{ij}(1\leqq i\leqq h,\ m+1\leqq j\leqq n+1)\}$ and $\{U_{ij},\ U'_{ij}(h+1\leqq i\leqq m,\ m+1\leqq j\leqq n+1)\}$ over $\boldsymbol{R}$ Taking account of (2. 2), we may easily see that the linear isotropy action of $S(U(h)\times U(m-h)\times U(n-m+1))$ on each space $\mathfrak{m}_s$ gives rise to an irreducible representation $(s=1, 2, 3)$. From (3. 7) and Table 1, we see that the canonical almost complex structure J is given as follows:

$$(4.3)\qquad \begin{aligned} &JU_{ij}=U'_{ij},\ JU'_{ij}=-U_{ij},\ \text{for } U_{ij},\ U'_{ij}\in\mathfrak{m}_1,\\ &JU_{ij}=-U_{ij},\ JU'_{ij}=U_{ij},\ \text{for } U_{ij},\ U'_{ij}\in\mathfrak{m}_2,\\ &JU_{ij}=U'_{ij},\ JU'_{ij}=-U_{ij},\ \text{for } U_{ij},\ U'_{ij}\in\mathfrak{m}_3. \end{aligned}$$

From the hypothesis of Theorem A, without loss of essentiality, we may assume that Riemannian metric on the homogeneous space $SU(n+1)/S(U(h)\times U(m-h)\times U(n-m+1))$ is induced by the inner product $\langle X, Y\rangle=-\frac{1}{2}\,\mathit{Trace}\, XY, X, Y\in\mathfrak{su}(n+1)$. First, we calculate the Ricci tensor R_1. Let $U_{ab}\in\mathfrak{m}_1$. Then, from (3. 9), taking account of (2. 2), we get

$$(4.4)\qquad \begin{aligned} &R(U_{ij},\ U_{ab})U_{ij}=-\delta_{ia}U_{ib}+2\delta_{ia}\delta_{jb}U_{ij}-\delta_{jb}U_{aj},\\ &R(U'_{ij},\ U_{ab})U'_{ij}=-\delta_{ia}U_{ib}-2\delta_{ia}\delta_{jb}U_{ij}-\delta_{jb}U_{aj}, \end{aligned}$$

for $U_{ij},\ U'_{ij}\in\mathfrak{m}_1$.

Similarly, we get

$$(4.5)\qquad R(U_{ij}, U_{ab})U_{ij} = -(1/4)\delta_{ia} U_{ib},$$

$$R(U'_{ij}, U_{ab})U'_{ij} = -(1/4)\delta_{ia} U_{ib}, \quad \text{for } U_{ij}, U'_{ij} \in \mathfrak{m}_2,$$

$$(4.6)\qquad R(U_{ij}, U_{ab})U_{ij} = -(1/4)\delta_{ib} U_{ai},$$

$$R(U'_{ij}, U_{ab})U'_{ij} = -(1/4)\delta_{ib} U_{ai}, \quad \text{for } U_{ij}, U'_{ij} \in \mathfrak{m}_3.$$

From (4. 1), (4. 3)～(4. 6), we get

$$(4.7)\qquad R_1(U_{ab}, U_{cd}) = R_1(U_{ab}, JU_{cd}) = 0, \quad \text{for } U_{cd} \in \mathfrak{m}_s (s=2, 3).$$

Similarly, we get

$$(4.8)\qquad R_1(U_{ab}, U_{cd}) = R_1(U_{ab}, JU_{cd}) = 0, \quad \text{for } U_{ab} \in \mathfrak{m}_s, U_{cd} \in \mathfrak{m}_t$$

$(s \neq t)$.

Let $U_{ab}, U_{cd} \in \mathfrak{m}_1$. Then for (4. 4)～(4. 6), we get

$$(4.9)\qquad \sum_{\substack{1\leq i\leq h\\ h+1\leq j\leq m}} R(U_{ij}, U_{ab}, U_{ij}, U_{cd}) = -(m-2)\delta_{ac}\delta_{bd},$$

$$\sum_{\substack{1\leq i\leq h\\ h+1\leq j\leq m}} R(U'_{ij}, U_{ab}, U'_{ij}, U_{cd}) = -(m+2)\delta_{ac}\delta_{bd},$$

$$\sum_{\substack{1\leq i\leq h\\ m+1\leq j\leq n+1}} R(U_{ij}, U_{ab}, U_{ij}, U_{cd}) = -(1/4)(n+1-m)\delta_{ac}\delta_{bd},$$

$$\sum_{\substack{1\leq i\leq h\\ m+1\leq j\leq n+1}} R(U'_{ij}, U_{ab}, U'_{ij}, U_{cd}) = -(1/4)(n+1-m)\delta_{ac}\delta_{bd},$$

$$\sum_{\substack{h+1\leq i\leq m\\ m+1\leq j\leq n+1}} R(U_{ij}, U_{ab}, U_{ij}, U_{cd}) = -(1/4)(n+1-m)\delta_{ac}\delta_{bd},$$

$$\sum_{\substack{h+1\leq i\leq m\\ m+1\leq j\leq n+1}} R(U'_{ij}, U_{ab}, U'_{ij}, U_{cd}) = -(1/4)(n+1-m)\delta_{ac}\delta_{bd}.$$

From (4. 1), (4. 3) and (4. 9), we get

$$(4.10)\qquad R_1(U_{ab}, U_{cd}) = (n+m+1)\delta_{ac}\delta_{bd}, \quad \text{for } U_{ab}, U_{cd} \in \mathfrak{m}.$$

Similarly, we get

$$(4.11)\qquad R_1(U_{ab}, U_{cd}) = (2(n+1)-m+h)\delta_{ac}\delta_{bd},$$

for $U_{ab}, U_{cd} \in \mathfrak{m}_2$,

$$(4.12)\qquad R_1(U_{ab}, U_{cd}) = (2n+2-h)\delta_{ac}\delta_{bd},$$

for $U_{ab}, U_{cd} \in \mathfrak{m}_3$.

From (4. 4)～(4. 5), we get easily

$$(4.13)\qquad R_1(U_{ab}, JU_{cd}) = 0, \quad \text{for } U_{ab}, U_{cd} \in \mathfrak{m}_1.$$

Similarly, we get

(4.14) $\quad R_1(U_{ab}, JU_{cd})=0$, for $U_{ab}, U_{cd}\in \mathfrak{m}_s$ $(s=2, 3)$.

Next, we calculate the Ricci * curvature tensor R_1^*. Let $U_{ab}\in\mathfrak{m}_1$ and $U_{cd}\in\mathfrak{m}_2$. Then, from (3.9), taking account of (2.2) and (4.3), we get

$$(4.15)\qquad R(U_{ab}, JU_{cd})JU_{ij}=-R(U_{ab}, U'_{cd})U'_{ij}$$

$$=-\frac{1}{2}\delta_{ac}\delta_{bj}U_{id}-\frac{1}{4}\delta_{ci}\delta_{jb}U_{ad},\quad \text{for } U_{ij}\in\mathfrak{m}_1,$$

$$(4.16)\qquad R(U_{ab}, JU_{cd})JU_{ij}=\frac{1}{2}\delta_{ac}\delta_{dj}U_{ib}-\frac{1}{4}\delta_{ai}\delta_{jd}U_{cb},\quad \text{for } U_{ij}\in\mathfrak{m}_2,$$

$$(4.17)\qquad R(U_{ab}, JU_{cd})JU_{ij}=0,\quad \text{for } U_{ij}\in\mathfrak{m}_3.$$

From (4.2), (4.15)~(4.17), we get

(4.18) $\quad R_1^*(U_{ab}, U_{cd})=0$, for $U_{ab}\in\mathfrak{m}_1$, $U_{cd}\in\mathfrak{m}_2$.

Similarly, we get

(4.19) $\quad R_1^*(U_{ab}, JU_{cd})=0$, for $U_{ab}\in\mathfrak{m}_1$, $U_{cd}\in\mathfrak{m}_2$.

Moreover, we get generally

(4.20) $\quad R_1^*(U_{ab}, U_{cd})=R_1^*(U_{ab}, JU_{cd})=0$,

for $U_{ab}\in\mathfrak{m}_s$, $U_{cd}\in\mathfrak{m}_t$ $(s\neq t)$.

Let $U_{ab}, U_{cd}\in\mathfrak{m}_1$. Then, from (3.9), taking account of (2.2) and (4.3), we get

$$(4.21)\qquad \sum_{\substack{1\leq i\leq h\\ h+1\leq j\leq m}} R(U_{ab}, JU_{cd}, JU_{ij}, U_{ij})=2m\delta_{ac}\delta_{bd},$$

$$\sum_{\substack{1\leq i\leq h\\ h+1\leq j\leq m}} R(U_{ab}, JU_{cd}, JU_{ij}, U_{ij})=-\frac{3}{2}(n+1-m)\delta_{ac}\delta_{bd},$$

$$\sum_{\substack{h+1\leq i\leq m\\ m+1\leq j\leq n+1}} R(U_{ab}, JU_{cd}, JU_{ij}, U_{ij})=-\frac{3}{2}(n+1-m)\delta_{ac}\delta_{bd}.$$

From (4.2) and (4.21), we get

(4.22) $\quad R_1^*(U_{ab}, U_{cd})=(5m-3n-3)\delta_{ac}\delta_{bd}$, for $U_{ab}, U_{cd}\in\mathfrak{m}_1$.

Similarly, we get

(4.23) $\quad R_1^*(U_{ab}, U_{cd})=(2n-5m+5h+2)\delta_{ac}\delta_{bd}$, for $U_{ab}, U_{cd}\in\mathfrak{m}_2$,

(4.24) $\quad R_1^*(U_{ab}, U_{cd})=(2n-5h+2)\delta_{ac}\delta_{bd}$, for $U_{ab}, U_{cd}\in\mathfrak{m}_3$.

Furthermore, we get also

(4.25) $\quad R_1^*(U_{ab}, JU_{cd})=0$, for $U_{ab}, U_{cd}\in\mathfrak{m}_s$ $(1\leq s\leq 3)$

Thus, from (4. 7), (4. 8), (4. 10)~(4. 14) and (4. 18)~(4. 20), (4. 22)~(4. 24), taking account of (1. 2) and (1. 3), we see that the Riemannian 3-symmetric space $(SU(n+1)/S(U(h)\times U(m-h)\times U(n-m+1)), J, g)$ is an Einstein space if and only if $m=2h$ and $n+1=3h$. Furthermore, $R_1-5R_1^*=0$ holds for $(SU(3h)/S(U(h)\times U(h)\times U(h)), J, g)$ $(h\geqq 1)$. From (3. 8) and (3. 9), taking account of (2. 2), we may easily see that $(SU(n+1)/S(U(h)\times U(m-h)\times U(n-m+1)), g)$ is irreducible and not locally symmetric.

(II) $G/K=SO(2n+1)/(SO(2n-2m+1)\times U(m))$.

Then, from (2. 3) and Table 1, we may easily see that the Lie subalgebra $\mathfrak{so}(2n+1)^\sigma$ of $\mathfrak{so}(2n+1)$ is given by the linear span of $\{t_i(1\leqq i\leqq n);\ U_i,\ U'_i(m+1\leqq i\leqq n);\ U_{ij},\ U'_{ij}(m+1\leqq i\leqq n);\ V_{ij},\ V'_{ij}(1\leqq i<j\leqq m);\ V_{ij},\ V'_{ij}(m+1\leqq i<j\leqq n)\}$ over $\boldsymbol{R}$, and hence the subspace $\mathfrak{m}$ in the decomposition (3. 6) is given by $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3+\mathfrak{m}_4$ (direct sum), where $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$ and $\mathfrak{m}_4$ are the subspaces of $\mathfrak{m}$ generated respectively by $\{U_i,\ U'_i(1\leqq i\leqq m)\}$, $\{U_{ij},\ U'_{ij}(1\leqq i<j\leqq m)\}$, $\{U_{ij},\ U'_{ij}(1\leqq i\leqq m,\ m+1\leqq j\leqq n)\}$, and $\{V_{ij},\ V'_{ij}(1\leqq i\leqq m,\ m+1\leqq j\leqq n)\}$ over $\boldsymbol{R}$. Taking account of (2. 4), we may easily see that the linear isotropy action of $SO(2n-2m+1)\times U(m)$ on the space $\mathfrak{m}_1+\mathfrak{m}_3+\mathfrak{m}_4$ (or $\mathfrak{m}_2$) gives rise to an irreducible representation over $\boldsymbol{R}$. From (3. 7), taking account of (2. 3) and Table 1, we see that the canonical almost complex structure J is given as follows:

(4.26) $JU_i=U'_i,\ JU'_i=-U_i,$ for $U_i,\ U'_i\in\mathfrak{m}_1,$

$JU_{ij}=-U'_{ij},\ JU'_{ij}=U_{ij},$ for $U_{ij},\ U'_{ij}\in\mathfrak{m}_2,$

$JU_{ij}=U'_{ij},\ JU'_{ij},=U_{ij},$ for $U_{ij},\ U'_{ij}\in\mathfrak{m}_3,$

$JV_{ij}=V'_{ij},\ JV'_{ij}=-V_{ij},$ for $V_{ij},\ V'_{ij}\in\mathfrak{m}_4.$

From the hypothesis of Theorem A, without loss of essentiality, we may assume that the Riemannian metric g on the homogeneous space $SO(2n+1)/SO(2n-2m+1)\times U(m))$ is induced by the inner product $<X,\ Y>=-\frac{1}{2}\ Trace\ XY,\ X,\ Y\in\mathfrak{so}(2n+1)$. First, we calculate the Ricci curvature tensor R_1. From (2. 4), (3. 9) and (4. 1), by the similar calculations as in the case (I), we get

(4.27) $R_1(X, Y)=0,$ for $X\in\mathfrak{m}_s,\ Y\in\mathfrak{m}_t\ (s\neq t),$

and furthermore

(4.28) $R_1(U_c, U_d)=(1/2)(4n-m-1)\,\delta_{cd},$

$R_1(U_c, JU_d)=0,$ for $U_c,\ U_d\in\mathfrak{m}_1,$

(4.29) $R_1(U_{ab}, U_{cd})=(1/2)(2n+2m-3)\,\delta_{ac}\,\delta_{bd},$

$R_1(U_{ab}, JU_{cd})=0,$ for $U_{ab},\ U_{cd}\in\mathfrak{m}_2,$

(4.30) $R_1(U_{ab}, U_{cd})=(1/2)(4n-m-1)\,\delta_{ac}\,\delta_{bd},$

$R_1(U_{ab}, JU_{cd})=0,$ for $U_{ab},\ U_{cd}\in\mathfrak{m}_3,$

(4.31) $R_1(V_{ab}, V_{cd}) = (1/2)(4n-m-1)\delta_{ac}\delta_{bd}$,

$R_1(V_{ab}, JV_{cd}) = 0$, for $V_{ab}, V_{cd} \in \mathfrak{m}_4$.

Next, we calculate the Ricci * curvature tensor R_1^*. From (2.4), (3.9), (4.1) and (4.26), by the similar calculations as in the case (I), we get

(4.32) $R_1^*(X, Y)=0$, for $X \in \mathfrak{m}_s$, $Y \in \mathfrak{m}_t$ $(s \neq t)$,

and furthermore

(4.33) $R_1^*(U_c, U_d) = (1/2)(4n-5m+3)\delta_{cd}$,

$R_1^*(U_c, JU_d)=0$, for $U_c, U_d \in \mathfrak{m}_1$,

(4.34) $R_1^*(U_{ab}, U_{cd}) = (1/2)(10m-6n+7)\delta_{ac}\delta_{bd}$,

$R_1^*(U_{ab}, JU_{cd}) = 0$, for $U_{ab}, U_{cd} \in \mathfrak{m}_2$,

(4.35) $R_1^*(U_{ab}, U_{cd}) = (1/2)(4n-5m+3)\delta_{ac}\delta_{bd}$,

$R_1^*(U_{ab}, JU_{cd}) = 0$, for $U_{ab}, U_{cd} \in \mathfrak{m}_3$,

(4.36) $R_1^*(V_{ab}, V_{cd}) = (1/2)(4n-5m+3)\delta_{ac}\delta_{bd}$,

$R_1^*(V_{ab}, JV_{cd}) = 0$, for $V_{ab}, V_{cd} \in \mathfrak{m}_4$.

Thus, from (1.2), (4.27)~(4.31) and (1.3), (4.32)~(4.36), we see that the Riemannian 3-symmetric space $(SO(2n+1)/(SO(2n-2m+1) \times U(m)), J, g)$ is an Einstein space if and only if $2(n+1)=3m$. Furthermore, $R_1 - 5R_1^* = 0$ holds for $(SO(3m-1)/(SO(m-1) \times U(m)), J, g)$ (m is even). From (3.8) and (3.9), taking account of (2.4), we may easily see that $(SO(2n+1)/(SO(2n-2m+1) \times U(m)), g)$ is irreducible and not locally symmetric.

(III) $G/K = Sp(n)/(Sp(n-m) \times U(m))$.

Then, from (2.5) and Table 1, we may easily see that the Lie subalgebra $\mathfrak{sp}(n)^\sigma$ of $\mathfrak{sp}(n)$ is given by the linear span of $\{t_i (1 \leq i \leq n)$; $W_i, W'_i (m+1 \leq i \leq n)$; $W_{ij}, W'_{ij} (m+1 \leq i < j \leq n)$; $U_{ij}, U'_{ij} (1 \leq i < j \leq m)$; $U_{ij}, U'_{ij} (m+1 \leq i < j \leq n)\}$ over $\boldsymbol{R}$, and hence the subspace $\mathfrak{m}$ of $\mathfrak{sp}(n)$ in the decomposition (3.6) is given by $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \mathfrak{m}_4$ (direct sum), where $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$ and $\mathfrak{m}_4$ are the subspaces of $\mathfrak{m}$ generated respectively by $\{W_i, W'_i (1 \leq i \leq m\}$, $\{W_{ij}, W'_{ij} (1 \leq i \leq j \leq m)\}$, $\{W_{ij}, W'_{ij} (1 \leq i \leq m, m+1 \leq j \leq n)\}$, $\{U_{ij}, U'_{ij} (1 \leq i \leq m, m+1 \leq j \leq n)\}$, over $\boldsymbol{R}$. Taking account of (2.6), we may easily see that the linear isotropy action of $Sp(n-m) \times U(m)$ on the space $\mathfrak{m}_1 + \mathfrak{m}_2$ (or $\mathfrak{m}_3 + \mathfrak{m}_4$) gives rise to an irreducible representation over $\boldsymbol{R}$. From (3.7), taking account of (2.5) and Table 1, we see that the canonical almost complex structure J is given as follows:

(4.37) $JW_i = -W'_i$, $JW'_i = W_i$, for $W_i, W'_i \in \mathfrak{m}_1$,

$JW_{ij} = -W'_{ij}$, $JW'_{ij} = W_{ij}$, for $W_{ij}, W'_{ij} \in \mathfrak{m}_2$,

$JW_{ij}=W'_{ij}, JW'_{ij}=-W_{ij},$ for $W_{ij}, W'_{ij}\in\mathfrak{m}_3,$

$JU_{ij}=U'_{ij}, JU'_{ij}=-U_{ij},$ for $U_{ij}, U'_{ij}\in\mathfrak{m}_4.$

Form the hypothesis of Theorem A, without loss of essentiality, we may assume that the Riemannian metric g on the homogeneous space $Sp(n)/(Sp(n-m)\times U(m))$ is induced by the inner product $<X, Y>=-\frac{1}{2}$ $Trace\ XY$, $X, Y\in\mathfrak{sp}(n)$. By the similar calculations as in the previous cases, we see that the Ricci curvature tensor R_1 and the Ricci * curvature tensor R_1^* are given as follows.

(4. 38) $R_1(X, Y)=0,$ for $X\in\mathfrak{m}_s, Y\in\mathfrak{m}_t\ (s\neq t),$

and furthermore

(4. 39) $R_1(W_a, W_c)=(n+m+2)\ \delta_{ac},$

$R_1(W_a, JW_c)=0,$ for $W_a, W_c\in\mathfrak{m}_1,$

(4. 40) $R_1(W_{ab}, W_{cd})=(n+m+2)\ \delta_{ac}\,\delta_{bd},$

$R_1(W_{ab}, JW_{cd})=0,$ for $W_{ab}, W_{cd}\in\mathfrak{m}_2,$

(4. 41) $R_1(W_{ab}, W_{cd})=(1/2)\ (4n-m+3)\ \delta_{ac}\,\delta_{bd},$

$R_1(W_{ab}, JW_{cd})=0,$ for $W_{ab}, W_{cd}\in\mathfrak{m}_3,$

(4. 42) $R_1(U_{ab}, U_{cd})=(1/2)\ (4n-m+3)\ \delta_{ac}\,\delta_{bd},$

$R_1(U_{ab}, JU_{cd})=0,$ for $U_{ab}, U_{cd}\in\mathfrak{m}_4.$

(4. 43) $R_1^*(X, Y)=0,$ for $X\in\mathfrak{m}_s, Y\in\mathfrak{m}_t\ (s\neq t),$

and furthermore

(4. 44) $R_1^*(W_a, W_c)=(-3n+5m+2)\ \delta_{ac},$

$R_1^*(W_a, JW_c)=0,$ for $W_a, W_c\in\mathfrak{m}_1,$

(4. 45) $R_1^*(W_{ab}, W_{cd})=(-3n+5m+2)\ \delta_{ac}\,\delta_{bd},$

$R_1^*(W_{ab}, JW_{cd})=0,$ for $W_{ab}, W_{cd}\in\mathfrak{m}_2,$

(4. 46) $R_1^*(W_{ab}, W_{cd})=(1/2)\ (4n-5m-1)\ \delta_{ac}\,\delta_{bd},$

$R_1^*(W_{ab}, JW_{cd})=0,$ for $W_{ab}, W_{cd}\in\mathfrak{m}_3,$

(4. 47) $R_1^*(U_{ab}, U_{cd})=(1/2)\ (4n-m+3)\ \delta_{ac}\,\delta_{bd},$

$R_1^*(U_{ab}, JU_{cd})=0,$ for $U_{ab}, U_{cd}\in\mathfrak{m}_4.$

Thus, from (1. 2), (4. 38)~(4. 42) and (1. 3), (4. 43)~(4. 47), we see that the Riemannian 3-symmetric space $(Sp(n)/(Sp(n-m)\times U(m)), J, g)$ is an Einstein space if and only if $2n=$

$3m+1$. Furthermore, $R_1-5R_1^*=0$ holds for $(Sp(3m-1)/(Sp(m)\times U(2m-1)), J, g)$ $(m\geqq 1)$. From (3. 8) and (3. 9), taking account of (2. 6), we may easily see that $(Sp(n)/(Sp(n-m)\times U(m)), g)$ is irreducible and not locally symmetric.

(IV) $G/K=SO(2n)/(SO(2n-2m)\times U(m))$.

Then, from (2. 7) and Table 1, we may easily see that the Lie subalgebra $\mathfrak{so}(2n)^{\sigma}$ of $\mathfrak{so}(2n)$ is given by the linear span of $\{t_i(1\leqq i\leqq n);\ U_{ij},\ U'_{ij}(m+1\leqq i<j\leqq n);\ V_{ij},\ V'_{ij}(1\leqq i<j\leqq m);\ V_{ij},\ V'_{ij}(m+1\leqq i<j\leqq n)\}$ over $\boldsymbol{R}$, and hence the subspace $\mathfrak{m}$ of $\mathfrak{so}(2n)$ in the decomposition (3. 6) is given by $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3$ (direct sum), where $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are the subspaces of $\mathfrak{m}$ generated respectively by $\{U_{ij},\ U'_{ij}(1\leqq i<j\leqq m)\}$, $\{U_{ij},\ U'_{ij}(1\leqq i\leqq m,\ m+1\leqq j\leqq n)\}$ and $\{V_{ij},\ V'_{ij}(1\leqq i\leqq m,\ m+1\leqq j\leqq n)\}$ over $\boldsymbol{R}$. Taking account of (2. 4), we may easily see that the linear isotropy action of $SO(2n-2m)\times U(m)$ on the space $\mathfrak{m}_1$ (or $\mathfrak{m}_2+\mathfrak{m}_3$) gives rise to an irreducible representation over $\boldsymbol{R}$. From (3. 7), taking account of (2. 7) and Table 1, we see that the canonical almost complex structure J is given as follows:

$$
\begin{aligned}
(4.48)\qquad & JU_{ij}=-U'_{ij},\quad JU'_{ij}=U_{ij}, && \text{for } U_{ij},\ U'_{ij}\in\mathfrak{m}_1,\\
& JU_{ij}=U'_{ij},\quad JU'_{ij}=-U_{ij}, && \text{for } U_{ij},\ U'_{ij}\in\mathfrak{m}_2,\\
& JV_{ij}=V'_{ij},\quad JV'_{ij}=-V_{ij}, && \text{for } V_{ij},\ V'_{ij}\in\mathfrak{m}_3.
\end{aligned}
$$

From the hypothesis of Theorem A, without loss of essentiality, we may assume that the Riemannian metric g on the homogeneous space $SO(2n)/(SO(2n-2m)\times U(m))$ is induced by the inner product $\langle X, Y\rangle=-\frac{1}{2}\, Trace\, XY$, $X, Y\in\mathfrak{so}(2n)$. By the similar calculations as in the previous cases, we see that the Ricci curvature tensor R_1 and the Ricci $*$ curvature tensor R_1^* are given as follows.

$$(4.49)\qquad R_1(X, Y)=0,\qquad \text{for } X\in\mathfrak{m}_s,\ Y\in\mathfrak{m}_t\ (s\neq t),$$

and furthermore

$$
\begin{aligned}
(4.50)\qquad & R_1(U_{ab}, U_{cd})=(n+m-2)\,\delta_{ac}\,\delta_{bd},\\
& R_1(U_{ab}, JU_{cd})=0,\qquad \text{for } U_{ab},\ U_{cd}\in\mathfrak{m}_1,\\
(4.51)\qquad & R_1(U_{ab}, U_{cd})=(1/2)\,(4n-m-3)\,\delta_{ac}\,\delta_{bd},\\
& R_1(U_{ab}, JU_{cd})=0,\qquad \text{for } U_{ab},\ U_{cd}\in\mathfrak{m}_2,\\
(4.52)\qquad & R_1(V_{ab}, U_{cd})=(1/2)\,(4n-m-3)\,\delta_{ac}\,\delta_{bd},\\
& R_1(V_{ab}, JV_{cd})=0,\qquad \text{for } V_{ab},\ V_{cd}\in\mathfrak{m}_3.
\end{aligned}
$$

$$(4.53)\qquad \overset{*}{R_1}(X, Y)=0,\qquad \text{for } X\in\mathfrak{m}_s,\ Y\in\mathfrak{m}_t\ (s\neq t),$$

and furthermore

$$(4.54)\qquad R_1^*(U_{ab}, U_{cd})=(5m-3n-2)\,\delta_{ac}\delta_{bd},$$

$R_1^*(U_{ab}, JU_{cd})=0, \quad$ for $U_{ab}, U_{cd}\in\mathfrak{m}_1$

(4. 55) $\quad R_1^*(U_{ab}, U_{cd})=(1/2)(4n-5m+1)\delta_{ac}\delta_{bd},$

$R_1^*(U_{ab}, JU_{cd})=0, \quad$ for $U_{ab}, U_{cd}\in\mathfrak{m}_2,$

(4. 56) $\quad R_1^*(V_{ab}, V_{cd})=(1/2)(4n-5m+1)\delta_{ac}\delta_{bd},$

$R_1^*(V_{ab}, JV_{cd})=0, \quad$ for $U_{ab}, V_{cd}\in\mathfrak{m}_3.$

Thus, from (1. 2), (4. 49)~(4. 52) and (1. 3), (4. 53)~(4. 56), we see that the Riemannian 3-symmetric space $(SO(2n)/(SO(2n-2m)\times U(m)), J, g)$ is an Einstein space if and only if $2n=3m-1$. Furthermore, $R_1-5R_1^*=0$ holds for $(SO(3m-1)/(SO(m-1)\times U(m)), J, g)$ (m is odd). From (3. 8) and (3. 9), taking account of (2. 4), we may easily see that $(SO(2n)/(SO(2n-2m)\times U(m)), g)$ is irreducible and not locally symmetric. Summing up the above arguments in (I)~(IV), we have finally Theorem A. Let (M, J, g) be a nearly Kaehlerian manifold and S_1 be the tensor field on M of type (0, 2) given by $S_1=R_1-R_1^*$. Then it is known that, the tensor field S_1 gives rise to a symmetric (by (1. 2), (1. 3), positive semi-definite bilinear form on each tangent space of M (cf. [10]). We denote by S^1 the field of symmetric endomorphism which corresponds to the tensor field S_1, that is, $g(S^1X, Y)=S_1(X, Y)$, for all tangent vectors X, Y of M. From the arguments in the proof of Theorem A, we have easily the following (for the related results, see [1], [4], [5], [6]).

THEOREM B. *Let* $(G, K=G^\sigma, \sigma=Ad(\exp 2\pi v))$ *be any one of the triples listed in the* Table 1 *and* g *be the* G*-invariant Riemannian metric on the space* G/K *which is determined by the inner product* $\langle , \rangle$ *on the Lie algebra* $\mathfrak{g}$ *of* G *defined by* $\langle X, Y\rangle=-\frac{1}{2}\,\mathrm{Trace}\, XY$, *for* $X, Y\in\mathfrak{g}$. *Then the eigenvalues* $\{\lambda\}$ *of the symmetric endomorphism* S^1 *of the corresponding Riemannian* 3*-symmetric space* $(G/K, J, g)$ *are given as follows:*

G/K	λ	*Multiplicities*
$SU(n+1)/S(U(h)\times U(m-h)\times U(n-m+1))$ $(1\leq h<m\leq n)$	$4(n-m+1)$	$2h(m-h)$
	$4(m-h)$	$2h(n-m+1)$
	$4h$	$2(m-h)(n-m+1)$
$SO(2n+1)/(SO(2n-2m+1)\times U(m))$ $(1<m\leq n)$	$2(m-1)$	$2m(2n-2m+1)$
	$2(2n-2m+1)$	$m(m+1)$
$Sp(n)/(Sp(n-m)\times U(m))$ $(1\leq m\leq n-1)$	$4(n-m)$	$m(m+1)$
	$2(m+1)$	$4m(n-m)$
$SO(2n)/(SO(2n-2m)\times U(m))$ $(2\leq m\leq n-1,\ n\geq 4)$	$4(n-m)$	$m(m-1)$
	$2(m-2)$	$4m(n-m)$

5. On the space Sp(2)/(Sp(1)×U(1))

Let $(SP(2)/(Sp(1)\times U(1)), J, g)$ be the Riemannian 3-symmetric space appeared in the proof of Theorem A. Then, by Theorem A, we see that $(Sp(2)/(Sp(1)\times U(1)), g)$ is an Einstein space. We now show that the homogeneous space $Sp(2)/(Sp(1)\times U(1))$ is diffeomorphic with a complex projective space CP^3 of complex dimension 3. Let $\mathfrak{g}=\mathfrak{sp}(2)$ and $\mathfrak{k}=\mathfrak{sp}(2)^{\sigma}=\mathfrak{sp}(1)+\boldsymbol{R}$ (direct sum). Then, from the argumenst developed in the case (III) (with $n=2$, $m=1$) of the proof of Theorem A, we may easily see that $\mathfrak{k}$ is given by the linear span of

$$\left\{\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e_3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}\right\} \text{ over } \boldsymbol{R}.$$

Thus, we have

$$(5.1) \qquad \mathfrak{k}=\mathfrak{sp}(1)+\boldsymbol{R}e_1 \xrightarrow{\iota_1} \mathfrak{sp}(1)+\mathfrak{sp}(1) \xrightarrow{\iota_2} \mathfrak{sp}(2),$$

where ι_1, ι_2 denote the respective natural inculusions. Taking account of (5. 1). we have the following fibration:

$$(5.2) \qquad \{1\}\times U(1) \longrightarrow Sp(2)/(Sp(1)\times\{1\}) \longrightarrow Sp(2)/(Sp(1)\times U(1)).$$

In the above fibration (5. 2), the action Φ of the group $\{1\}\times U(1)$ on the space $Sp(2)/(Sp(1)\times\{1\})=S^7$ is given as follows:

$$(5.3) \qquad (\Phi(\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}))(\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix})(Sp(1)\times\{1\})$$
$$=(\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix})(Sp(1)\times\{1\}),$$

where $q=\cos u+(\sin u)e_1$, $u\in\boldsymbol{R}$.

Taking account of (5. 2) and (5. 3), we may easily see that the space $Sp(2)/(Sp(1)\times U(1))$ is diffeomorphic with a 3-dimensional complex projective space CP^3.

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