Strong uniform consistency of recursive kernel density estimators*

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1. Introduction

Let $f(x)$ be a (unknown) probability density function (p.d.f.) on the p-dimensional Euclidean space R^p with respect to Lebesgue measure. Based on a sequence $X_1, X_2, \dots$ of independent identically distributed p-dimensional random vectors having the common p.d.f. $f(x)$, we wish to estimate the p.d.f. $f(x)$. Yamato [8] proposed recursive kernel estimators of the form

$$\tilde{f}_0(x) \equiv 0$$

(1.1) $$\tilde{f}_n(x)=\tilde{f}_{n-1}(x)+n^{-1}\{K_n(x, X_n)-\tilde{f}_{n-1}(x)\} \qquad \text{for each } n \geqq 1,$$

where

(1.2) $$K_n(x, y)=h_n^{-p}K((x-y)/h_n) \qquad \text{for } x, y \in R^p \text{ and each } n \geqq 1,$$

$\{h_n\}$ is a sequence of positive numbers and $K(x)$ is a real-valued Borel measurable function on R^p, on which certain properties were imposed. He showed the weak uniform consistency of these estimators as well as the weak pointwise consistency. DEVROYE [4] discussed several results related to the weak or the strong pointwise consistency of $\tilde{f}_n(x)$. DAVIES [3] showed the strong uniform consistency of $\tilde{f}_n(x)$ as well as the strong pointwise consistency.

In this paper we consider a class of recursive kernel estimators of the form

$$f_0(x) \equiv K(x)$$

(1.3) $$f_n(x)=f_{n-1}(x)+a_n\{K_n(x, X_n)-f_{n-1}(x)\} \qquad \text{for each } n \geqq 1,$$

or equivalently,

$$f_n(x)=\sum_{m=0}^{n} a_m \beta_{mn} K_m(x, X_m) \qquad \text{for each } n \geqq 0,$$

where $K_n(x, y)$ is defined by (1.2), $K_0(x,X_0) \equiv K(x)$, $\{a_n\}$ is a sequence of positive numbers satisfying

(1.4) $$a_0=1,\ 0<a_n \leqq 1 \text{ for all } n \geqq 1,\ \lim_{n\to\infty} a_n=0 \text{ and } \sum_{n=1}^{\infty} a_n=\infty,$$

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and let

$$(1.5) \qquad \beta_{mn} = \prod_{j=m+1}^{n} (1-a_j) \qquad \text{if } n > m \geqq 0$$
$$= 1 \qquad \text{if } n = m \geqq 0.$$

We note that if $K(x)$ is a p.d.f., then $f_n(x)$ is a p.d.f. for each $n \geqq 0$. It is easy to see that the class of our estimators contains both the estimators of YAMATO (1.1) and the estimators of ISOGAI [5] with $a_n = an^{-1}$ for $2^{-1} < a \leqq 1$ and each $n \geqq 1$. Putting $a_n = an^{-1}$ for $2^{-1} < a \leqq 1$, the differences of properties between the estimators of (1.1) and (1.3) were discussed in ISOGAI [5].

In Section 2 we shall make some preparations for sections that follow. In Section 3 we shall show the weak or the strong pointwise consistency of $f_n(x)$. In Section 4 we shall show that $f_n(x)$ is strongly uniformly consistent and that $E[\sup_{x \in R^p} |f_n(x) - f(x)|^2]$ converges to 0 as n tends to infinity. We also consider a problem of estimating the mode θ of $f(x)$.

2. Preliminaries and auxiliary results

Let $K(x)$ in (1.2) be a bounded, integrable, real-valued Borel measurable function on R^p satisfying

$$(\text{K } 1) \qquad \int K(x)\, dx = 1,$$

where all integrals with respect to Lebesgue measure are over R^p, unless otherwise specified. Let $\{h_n\}$ in (1.2) be a sequence of positive numbers satisfying $\lim_{n \to \infty} h_n = 0$. On this sequence $\{h_n\}$ we shall impose some of the following conditions:

$$(\text{H } 1) \qquad a_n h_n^{-p} \xrightarrow{n} 0,$$

$$(\text{H } 2) \qquad \sum_{n=1}^{\infty} a_n^2 h_n^{-p} < \infty,$$

$$(\text{H } 3) \qquad h_1 \geqq h_2 \geqq h_3 \geqq \dots,$$

$$(\text{H } 4) \qquad h_n / h_{n+1} \xrightarrow{n} 1,$$

$$(\text{H } 5) \qquad \sum_{n=1}^{\infty} (a_n h_n^{-p})^2 < \infty,$$

$$(\text{H } 6) \qquad \sum_{n=1}^{\infty} \frac{A_n}{h_n^{2p-1}} \left| \frac{1}{h_{n+1}} - \frac{1}{h_n} \right| < \infty,$$

where a_n and β_{mn} are defined by (1.4) and (1.5) respectively, and let $A_n = \sum_{m=1}^{n} a_m^2 \beta_{mn}^2$ for each $n \geqq 1$. Throughout this paper $C_1, C_2, \dots$ denote positive constants, and let $\|g\|_\infty = \sup |g(x)|$ for any real-valued function g on R^p, where supremum is taken over R^p, unless otherwise specified.

DEFINITION 2.1 *Let $g(x)$ be a real-valued Borel measurable function on R^p. Then a point x is said to be a Lebesgue point of g if it holds that*

$$\rho^{-p}\int_{S(x,\rho)}|g(y)-g(x)|\,dy \longrightarrow 0 \qquad \text{as } \rho\to 0,$$

where $S(x, \rho)$ denotes a closed sphere in R^p centered at x with radius ρ.

REMARK 2.1. If $g(x)$ is integrable then almost every x is a Lebesgue point of g (see STEIN [7], p. 25). If x is a continuity point of g then x is a Lebesgue point of g.

The following lemma is a modification of Lemma 2 of DEVROYE [4].

LEMMA 2.1. *Let $g(x)$ be a real-valued Borel measurable function on R^p, let $K(x)$ be a bounded, integrable, real-valued Borel measurable function on R^p, let $\{h_n\}$ be a sequence of positive numbers converging to zero, and let* Condition $A(x, g, K)$ *hold.*

Condition $A(x, g, K)$. *One of the following is true:*

(i) *x is a continuity point of g, g is integrable and*

(K 2) $\|y\|^p|K(y)|\longrightarrow 0$ *as* $\|y\|\to\infty$,

where $\|\cdot\|$ denotes the Euclidean norm on R^p,

(ii) *x is a Lebesgue point of g and g is bounded on R^p,*

(iii) *x is a Lebesgue point of g and K has compact support.*

Then

$$\int h_n^{-p}K((x-y)/h_n)g(y)\,dy \longrightarrow g(x)\int K(y)\,dy.$$

PROOF. In case (i) the lemma holds by Theorem 2.1 of CACOULLOS [2]. Thus we need only show that in both cases (ii) and (iii)

$$\left|\int h_n^{-p}K((x-y)/h_n)g(y)\,dy-g(x)\int K(y)\,dy\right|$$

$$\leqq \int h_n^{-p}|K((x-y)/h_n)|\;|g(y)-g(x)|\,dy \overset{n}{\longrightarrow} 0.$$

Let $U_n=\int h_n^{-p}|K((x-y)/h_n)|\;|g(y)-g(x)|\,dy$. Thus we need only show

(2.1) $\lim_{n\to\infty} U_n=0.$

For any $\rho>0$

$$(2.2)\qquad U_n=\int_{S^c(x,\rho h_n)} h_n^{-p}|K((x-y)/h_n)|\,|g(y)-g(x)|\,dy$$

$$+\int_{S(x,\rho h_n)} h_n^{-p}|K((x-y)/h_n)|\,|g(y)-g(x)|\,dy$$

$$=V_n+W_n, \text{ say},$$

where S^c denotes the complement of a set S. First we consider case (ii). Let any $\varepsilon>0$

be fixed. Since $K(x)$ is integrable, there exists a positive constant ρ such that $\int_{S^c(0,\rho)} |K(y)|dy < \varepsilon$. Such ρ is fixed. It is easy to see that

$$(2.3) \qquad V_n \leqq 2\|g\|_\infty \int_{S^c(0,\rho)} |K(y)|dy \leqq 2\|g\|_\infty \varepsilon \qquad \text{for all } n \geqq 1.$$

Since x is a Lebesgue point of g and $\lim_{n\to\infty} h_n=0$, we get

$$W_n \leqq \|K\|_\infty \rho^p (\rho h_n)^{-p} \int_{S^c(0,\rho)} |g(y)-g(x)|dy \xrightarrow{n} 0,$$

which implies that

$$(2.4) \qquad \lim_{n\to\infty} W_n=0.$$

Combining (2. 2), (2. 3) and (2. 4) we obtain (2. 1). We next consider case (iii). Since $K(x)$ has compact support, there exists a positive constant ρ such that $K(y)=0$ for all $y \in S^c(0, \rho)$, which implies that

$$(2.5) \qquad V_n= \int_{S^c(0,\rho)} |K(y)|\,|g(x-h_n y)-g(x)|dy=0 \qquad \text{for all } n \geqq 1.$$

Thus by (2. 2), (2. 4) and (2. 5) we get (2. 1). This completes the proof.

LEMMA 2. 2. Condition (iii) *in* Condition $A(x, g, K)$ *is replaced by the following*:

(iii)′ *x is a Lebesgue point of g, g is integrable and K has compact support.*

Then, under all assumptions of Lemma 2. 1

$$\sup_{n\geqq 1} \int h_n^{-p} |K((x-y)/h_n)|\,|g(y)|dy < \infty.$$

PROOF. If $g(x)$ is integrable we get that

$$U_n \equiv \int h_n^{-p} |K((x-y)/h_n)|\,|g(y)|dy$$
$$\leqq \|K\|_\infty h_n^{-p} \int |g(y)|dy < \infty \qquad \text{for all } n \geqq 1.$$

If $g(x)$ is bounded we have

$$U_n \leqq \|g\|_\infty \int |K(y)|dy < \infty \qquad \text{for all } n \geqq 1.$$

Thus, under Condition $A(x, g, K)$

$$(2.6) \qquad U_n < \infty \qquad \text{for all } n \geqq 1.$$

Replacing $g(x)$ and $K(x)$ of Lemma 2. 1 by $|g(x)|$ and $|K(x)|$ respectively, and using Lemma 2. 1, we get

$$U_n= \int h_n^{-p} |K((x-y)/h_n)|\,|g(y)|dy \xrightarrow{n} |g(x)| \int |K(y)|dy,$$

which, together with (2. 6), yields the lemma. The proof is complete.

3. Weak or strong pointwise consistency

In this section we shall show the weak or the strong pointwise consistency of $f_n(x)$.

THEOREM 3. 1. *Suppose that* Condition $A(x, f, K)$ *holds. If* $\{h_n\}$ *satisfies* (H 1), *then*

$$E[(f_n(x)-f(x))^2] \xrightarrow{n} 0,$$

which implies that $f_n(x) \xrightarrow{n} f(x)$ *in probability.*
If (H2) *is true, then*

$$f_n(x) \xrightarrow{n} f(x) \qquad \textit{with probability one } (w.p.\ 1)$$

and

$$E[(f_n(x)-f(x))^2] \xrightarrow{n} 0.$$

PROOF. We note that

$$(3.1) \qquad |f_n(x)-f(x)| \leqq |f_n(x)-E[f_n(x)]| + |E[f_n(x)]-f(x)|$$

and

$$(3.2) \qquad E[(f_n(x)-f(x))^2] = E[(f_n(x)-E[f_n(x)])^2] + (E[f_n(x)]-f(x))^2.$$

It follows from (1. 4) and (1. 5) that

$$a_m \beta_{mn} \geqq 0 \text{ for all } m=1,\dots, n,\ n=1, 2, \dots, \qquad \lim_{n\to\infty} \beta_{mn}=0 \text{ for each } m \geqq 1,$$

$$\sum_{m=1}^{n} a_m \beta_{mn} \leqq 1 \text{ for all } n \geqq 1, \qquad \sum_{m=1}^{n} a_m \beta_{mn} \xrightarrow{n} 1.$$

Thus from Lemma 2. 1 and the Toeplitz lemma (see LOÈVE [6], p. 238), we get

$$(3.3) \qquad E[f_n(x)] = \sum_{m=1}^{n} a_m \beta_{mn} E[K_m(x, X_m)] + \beta_{0n} K(x) \xrightarrow{n} f(x).$$

Since $f_n(x) - E[f_n(x)] = \sum_{m=1}^{n} a_m\beta_{mn} \{K_m(x, X_m) - E[K_m(x, X_m)]\}$,

$$E[(f_n(x)-E[f_n(x)])^2]$$

$$\leqq \sum_{m=1}^{n} a_m^2 \beta_{mn}^2 E[K_m^2(x, X_m)]$$

$$\leqq \|K\|_\infty \sum_{m=1}^{n} a_m^2 \beta_{mn}^2 h_m^{-p} \int h_m^{-p} |K((x-y)/h_m)| f(y)\, dy.$$

From Lemma 2. 2 we get

$$\int h_m^{-p} |K((x-y)/h_m)| f(y)\, dy \leqq C_1 \qquad \text{for all } m \geqq 1.$$

Hence

$$(3.4)\qquad E[(f_n(x)-E[f_n(x)])^2]\leqq C_2\sum_{m=1}^{n}a_m^2\beta_{mn}^2h_m^{-p}$$

$$\leqq C_2\sum_{m=1}^{n}a_m\beta_{mn}(a_mh_m^{-p}),$$

because of the fact that $0\leqq\beta_{mn}\leqq 1$. By (H1) and the Toeplitz lemma the last term of (3.4) approaches to 0 as n tends to infinity.
Hence

$$(3.5)\qquad E[(f_n(x)-E[f_n(x)])^2]\xrightarrow{n}0,$$

which, together with (3.2) and (3.3), yields the first assertion of the lemma. The proof of the second assertion parallels the proof of Theorem 3.1 of Isogai [5]. Since

$$E[(f_n(x)-E[f_n(x)])^2|X_1,\dots,X_{n-1}]$$
$$\leqq(1-a_n)(f_{n-1}(x)-E[f_{n-1}(x)])^2$$
$$+\|K\|_\infty a_n^2h_n^{-p}\int h_n^{-p}|K((x-y)/h_n)|f(y)\,dy\qquad\text{w.p.1},$$

where $E[\cdot|\cdot]$ denotes a conditional expectation, we get, by Lemma 2.2,

$$E[(f_n(x)-E[f_n(x)])^2|X_1,\dots,X_{n-1}]$$
$$\leqq(1-a_n)(f_{n-1}(x)-E[f_{n-1}(x)])^2+C_3a_n^2h_n^{-p}\qquad\text{w.p.1}.$$

Hence by (H2) and Proposition 2.4 of Isogai [5] we have (3.5) and

$$(3.6)\qquad |f_n(x)-E[f_n(x)]|\xrightarrow{n}0\qquad\text{w.p.1}.$$

Thus combining (3.1)~(3.3), (3.5) and (3.6), the second assertion is established. The proof of the lemma is complete.

Remark 3.1. The conditions (H1) and (H2) do not imply each other. Let

$$a_n=n^{-1},$$
$$h_n^p=n^{-1}\qquad\text{if } n=2^k\qquad\text{for } k=0,1,2,\dots$$
$$=n^{-\frac{1}{2}}\quad\text{if } n\neq 2^k,$$

whose sequences are given by Devroye [4]. These sequences satisfy (1.4) and (H2) but not (H1). Let

$$a_1=1,\ a_n=(n(\log n)^{\frac{1}{2}})^{-1}\qquad\text{for } n\geqq 2,$$
$$h_n^p=n^{-1}.$$

Then, these sequences satisfy (1.4) and (H1) but not (H2). Putting $a_n=an^{-1}$ with $0<a\leqq 1$ in (1.4), the conditions (H1) and (H2) coincide with (3) and (12) of Devroye [4] respectively. The following sequences satisfy (1.4), (H1) and (H2):

$$a_1=1,\ a_n=(n\log n)^{-1}\qquad\text{for } n\geqq 2,\qquad h_n^p=n^{-1}.$$

4. Strong uniform consistency

In this section we shall show the strong uniform consistency of $f_n(x)$ and show that $E[\sup|f_n(x)-f(x)|^2]$ converges to 0 as n tends to infinity. Furthermore, the strong consistency of the mode estimator will be shown.

First we shall show the strong uniform consistency of $f_n(x)$ and the convergence of $E[\sup|f_n(x)-f(x)|^2]$ to 0. The method of proof is similar to that of DAVIES [3]. Let $k(t)=\int e^{it\cdot u}K(u)du$ for $t\in R^p$ be the Fourier transform of $K(u)$, where $t\cdot u=\sum_{j=1}^{p} t_j u_j$ for $t=(t_1,\dots,t_p)$ and $u=(u_1,\dots,u_p)$, and $i^2=-1$. Also let $\varphi(t)=E[e^{it\cdot X_1}]$ for $t\in R^p$ be the characteristic function of the random vector X_1. The next theorem is concerned with the strong uniform consistency of $f_n(x)$.

THEOREM 4.1. *In addition to the conditions* (H3)~(H6), *suppose that* $K(x)$ *is continuous on* R^p *and satisfies* (K2) *and*

(K 3) $\qquad \int |k(t)|dt<\infty,$

where $|k(t)|$ *is non-increasing on a ray* $R(u)=\{v=qu;\ q>0\}$ *for each* $u\neq 0\in R^p$. *(That is,* $|k(v_1)|\geqq|k(v_2)|$ *for* $v_1, v_2\in R(u)$ *with* $\|v_1\|\leqq\|v_2\|$*). If* $f(x)$ *is uniformly continuous on* R^p, *then it holds that*

$$\sup|f_n(x)-f(x)|\xrightarrow{n} 0 \qquad w.p.1$$

and

$$E[\sup|f_n(x)-f(x)|^2]\xrightarrow{n} 0.$$

PROOF. It is easy to see that the theorem holds if we show that

$$\sup|E[f_n(x)]-f(x)|\xrightarrow{n} 0, \tag{4.1}$$

$$\sup|f_n(x)-E[f_n(x)]|\xrightarrow{n} 0 \qquad \text{w.p.1} \tag{4.2}$$

and

$$E[(\sup|f_n(x)-E[f_n(x)]|)^2]\xrightarrow{n} 0, \tag{4.3}$$

because of the fact that $E[\sup|f_n(x)-f(x)|^2]=E[(\sup|f_n(x)-f(x)|)^2]$. First we shall show (4.1). Note that since $f(x)$ is uniformly continuous p.d.f., $f(x)$ is bounded. Since by Lemma 1 of YAMATO [8] $\sup|E[K_n(x, X_n)]-f(x)|\longrightarrow 0$, using the Toeplitz lemma we get

$$\sup|E[f_n(x)]-f(x)|$$
$$\leqq \sum_{m=1}^{n} a_m\beta_{mn}(\sup|E[K_m(x, X_m)]-f(x)|)+\beta_{0n}(\|K\|_\infty+\|f\|_\infty)\xrightarrow{n} 0,$$

which implies (4.1). Next we shall show (4.2) and (4.3). Since both $K(u)$ and $k(u)$ are integrable and $K(u)$ is continuous on R^p, we have, by using the inversion theorem for the Fourier transform (see Bochner and Chandrasekharan [1], p. 66),

$$K_m(x, X_m)-E[K_m(x, X_m)]$$
$$=(2\pi)^{-p}\int [e^{iu\cdot X_m}-\varphi(u)]e^{-iu\cdot x}k(h_m u)du \qquad \text{for all } x\in R^p \qquad \text{w.p.1.}$$

Hence

$$\text{(4.4)} \qquad \sup|f_n(x)-E[f_n(x)]|$$
$$\leqq (2\pi)^{-p}\int |\sum_{m=1}^{n} a_m\beta_{mn}[e^{iu\cdot X_m}-\varphi(u)]k(h_m u)|du \qquad \text{w.p.1.}$$

For simplicity we shall introduce the following notations:

$$S_n=\{u\in R^p;\ k(h_n u)\neq 0\},$$
$$\xi_{mn}(u)=a_m\beta_{mn}[e^{iu\cdot X_m}-\varphi(u)] \qquad \text{for } 1\leqq m\leqq n,$$
$$g_{mn}(u)=k(h_m u)/k(h_n u) \qquad \text{if } u\in S_n$$
$$=0 \quad \text{if } u\in S_n^c \qquad \text{for } 1\leqq m\leqq n$$

and

$$Z_{mn}(u)=\xi_{mn}(u)\,g_{mn}(u) \qquad \text{for } 1\leqq m\leqq n.$$

By the assumption on $|k(u)|$ and Condition (H3) we get

$$|k(h_m u)|\leqq|k(h_n u)| \qquad \text{for } u\in R^p \text{ and } 1\leqq m\leqq n,$$

which implies that

$$\text{(4.5)} \qquad S_n^c\subset S_m^c \qquad \text{for } 1\leqq m\leqq n$$

and

$$\text{(4.6)} \qquad |g_{mn}(u)|\leqq 1 \qquad \text{for } 1\leqq m\leqq n \text{ and } u\in R^p.$$

Hence by the definition of $g_{mn}(u)$ and (4.5) we have

$$\text{(4.7)} \qquad \int|\sum_{m=1}^{n}\xi_{mn}(u)k(h_m u)|du=\int|\sum_{m=1}^{n}Z_{mn}(u)|\,|k(h_n u)|du \qquad \text{w.p.1.}$$

In view of (4.4) and (4.7) we obtain, by the Schwarz inequality,

$$\sup|f_n(x)-E[f_n(x)]|$$
$$\leqq (2\pi)^{-p}[\int|k(h_n u)|du]^{\frac{1}{2}}\ [\int|\sum_{m=1}^{n}Z_{mn}(u)|^2|k(h_n u)|du]^{\frac{1}{2}}$$
$$=(2\pi)^{-p}[\int|k(u)|du]^{\frac{1}{2}}\ [h_n^{-p}\int|\sum_{m=1}^{n}Z_{mn}(u)|^2|k(h_n u)|du]^{\frac{1}{2}}$$
$$=C_1 Y_n^{\frac{1}{2}} \qquad \text{w.p.1,}$$

where $Y_n = h_n^{-p} \int |\sum_{m=1}^{n} Z_{mn}(u)|^2 |k(h_n u)| du.$

Thus, in order to prove (4. 2) and (4. 3) it suffices to show that

$$Y_n \xrightarrow{n} 0 \quad \text{w.p.1} \tag{4.8}$$

and

$$E[Y_n] \xrightarrow{n} 0. \tag{4.9}$$

It is clear that

$$\begin{aligned} &Y_{n+1} \\ &= (1-a_{n+1})^2 h_{n+1}^{-p} \int |\eta_n(u)|^2 |k(h_{n+1}u)| du \\ &\quad + a_{n+1}(1-a_{n+1}) h_{n+1}^{-p} \{ \int \eta_n(u) [e^{-iu \cdot X_{n+1}} - \varphi(-u)] \overline{g}_{n+1\, n+1}(u) |k(h_{n+1}u)| du \\ &\quad + \int \overline{\eta_n}(u) [e^{iu \cdot X_{n+1}} - \varphi(u)] g_{n+1\, n+1}(u) |k(h_{n+1}u)| du \} \\ &\quad + a_{n+1}^2 h_{n+1}^{-p} \int |e^{iu \cdot X_{n+1}} - \varphi(u)|^2 |g_{n+1\, n+1}(u)|^2 |k(h_{n+1}u)| du, \end{aligned} \tag{4.10}$$

where $\eta_n(u) = \sum_{m=1}^{n} \xi_{mn}(u) g_{m\, n+1}(u)$ and $\overline{b}$ denotes the conjugate of a complex number b. It follows from the independence of X_n's that

$$E[e^{iu \cdot X_{n+1}} - \varphi(u) | X_1, \ldots, X_n] = 0 \quad \text{w.p.1}$$

and

$$E[|e^{iu \cdot X_{n+1}} - \varphi(u)|^2 | X_1, \ldots, X_n] = 1 - |\varphi(u)|^2 \leqq 1 \quad \text{w.p.1.}$$

Hence, taking conditional expectations of both sides of (4. 10) and using (4. 6) we have

$$\begin{aligned} &E[Y_{n+1} | X_1, \ldots, X_n] \\ &\leqq (1-a_{n+1}) h_{n+1}^{-p} \int |\eta_n(u)|^2 |k(h_{n+1}u)| du + C_2 (a_{n+1} h_{n+1}^{-p})^2 \quad \text{w.p.1.} \end{aligned} \tag{4.11}$$

By the definition of $g_{mn}(u)$ and (4.5) we can prove that

$$\begin{aligned} &\int_{S_n} |\sum_{m=1}^{n} \xi_{mn}(u) g_{m\, n+1}(u)|^2 |k(h_{n+1}u)| du \\ &= \int_{S_n} |\sum_{m=1}^{n} \xi_{mn}(u) g_{mn}(u)|^2 |g_{n\, n+1}(u)|\, |k(h_n u)| du \quad \text{w.p.1} \end{aligned}$$

and

$$\begin{aligned} &\int_{S_n^c} |\sum_{m=1}^{n} \xi_{mn}(u) g_{m\, n+1}(u)|^2 |k(h_{n+1}u)| du \\ &= \int_{S_n^c} |\sum_{m=1}^{n} \xi_{mn}(u) g_{mn}(u)|^2 |g_{n\, n+1}(u)|\, |k(h_n u)| du = 0 \quad \text{w.p.1,} \end{aligned}$$

which, together with (4. 6) and (4. 11), yields that

$$E[Y_{n+1}|X_1,\dots,X_n]$$

$$(4.12)\qquad \leqq (1-a_{n+1})h_n^p h_{n+1}^{-p} Y_n + C_2(a_{n+1}h_{n+1}^{-p})^2$$

$$\leqq (1-a_{n+1})Y_n + h_n^p\,|h_{n+1}^{-p}-h_n^{-p}|\,Y_n + C_2(a_{n+1}h_{n+1}^{-p})^2 \qquad \text{w.p.1.}$$

Since by (H4) $h_{n+1}^{-p}-h_n^{-p} \sim ph_n^{1-p}(h_{n+1}^{-1}-h_n^{-1})$ as $n\to\infty$, where "$a_n \sim b_n$ as $n\to\infty$" means that $a_n/b_n \to 1$ as $n\to\infty$, we get

$$h_n^p\,|h_{n+1}^{-p}-h_n^{-p}| \leqq C_3 h_n\,|h_{n+1}^{-1}-h_n^{-1}| \qquad \text{for all } n\geqq 1,$$

which, together with (4.12), yields that

$$(4.13)\qquad E[Y_{n+1}|X_1,\dots,X_n]$$

$$\leqq (1-a_{n+1})\,Y_n + C_3h_n\,|h_{n+1}^{-1}-h_n^{-1}|\,Y_n + C_2(a_{n+1}h_{n+1}^{-p})^2 \qquad \text{w.p.1.}$$

It follows from (4.6) that

$$E[|Z_{mn}(u)|^2] \leqq a_m^2\beta_{mn}^2 E[|e^{iu\cdot X_m}-\varphi(u)|^2]$$

$$\leqq a_m^2\beta_{mn}^2 \qquad \text{for } 1\leqq m\leqq n \text{ and } u\in R^p.$$

Thus, by the independence of $Z_{1n}(u),\dots,Z_{nn}(u)$ with $E[Z_{mn}(u)]=0$ and Fubini's theorem we get

$$E[Y_n]=h_n^{-p}\sum_{m=1}^{n}\int E[|Z_{mn}(u)|^2]\,|k(h_nu)|\,du$$

$$\leqq C_3 h_n^{-2p}\sum_{m=1}^{n} a_m^2\,\beta_{mn}^2,$$

which, together with Condition (H6), implies that

$$(4.14)\qquad \sum_{n=1}^{\infty} h_n\,|h_{n+1}^{-1}-h_n^{-1}|\,E[Y_n] < \infty.$$

Thus, by the use of (4.13), (4.14), Condition (H5) and Proposition 2.4 of Isogai [5], we obtain (4.8) and (4.9). The proof is complete.

We shall present the strong uniform consistency of the estimators of Isogai [5] which contains the estimators of Yamato [8] in a special case.

Corollary 4.1. In (1.4) *we put* $a_n=an^{-1}$ *with* $0<a\leqq 1$. *Suppose that instead of* Condition (H6) *the sequence* $\{h_n\}$ *satisfies*

$$(\text{H }7)\qquad \sum_{n=1}^{\infty}\frac{d_n}{n^b h_n^{2p-1}}\left|\frac{1}{h_{n+1}}-\frac{1}{h_n}\right| < \infty,$$

where $b=\min(1, 2a)$, $d_1=d_2=1$, *and*

$$d_n=\log n \qquad \textit{if } a=2^{-1}$$

$$=1 \qquad \textit{otherwise} \qquad \textit{for } n=3,4,\dots.$$

Then, under all conditions of Theorem 4.1 *we obtain*

$$\sup|f_n(x)-f(x)|\xrightarrow{n}0 \qquad w.p.1$$

and

$$E[\sup|f_n(x)-f(x)|^2]\xrightarrow{n}0.$$

PROOF. In order to prove the corollary it suffices to verify Condition (H6). It is easy to see that $0<\beta_{mn}\leqq 2m^a n^{-a}$ for $1\leqq m\leqq n$, which implies that

(4.15) $$A_n\leqq C_1 n^{-2a}\sum_{m=1}^{n} m^{2(a-1)} \qquad \text{for each } n.$$

After some calculations we can show the following inequality:

(4.16) $$n^{-2a}\sum_{m=1}^{n} m^{2(a-1)}\leqq C_2(d_n/n^b) \qquad \text{for each } n.$$

Thus, combining (4.15), (4.16) and Condition (H7) we obtain Condition (H6). This completes the proof.

REMARK 4.1. If we put $a=1$ in (H7) then Condition (H7) coincides with Condition (11) of DAVIES [3]. Thus, Corollary 4.1 contains Theorem 2 of DAVIES [3] in a special case.

We shall give an example of $\{h_n\}$ satisfying all conditions of Corollary 4.1.

EXAMPLE 4.1.
Let $a_n=an^{-1}$ with $0<a\leqq 1$ and $h_n=n^{-r/p}$ with $0<r<1$. Then Conditions (H1)~(H5) and (H7) are fulfilled if we choose $0<r<\min(2^{-1}, a)$ for fixed a with $0<a\leqq 1$.

Now, we consider a problem of estimating a mode θ of the continuous p.d.f. $f(x)$ defined by $f(\theta)=\max_{x\in R^p} f(x)$. We assume that θ is unique. If in Theorem 4.1 all conditions on $K(x)$ are fulfilled, there exists a random vector θ_n (called the sample mode) which satisfies

(4.17) $$f_n(\theta_n)=\max_{x\in R^p} f_n(x) \qquad \text{for each } n,$$

where $f_n(x)$ is defined by (1.3).
The following theorem is concerned with strong consistency of the sample mode θ_n.

THEOREM 4.2. *If all conditions of Theorem* 4.1 *or Corollary* 4.1 *are satisfied, then*

$$\|\theta_n-\theta\|\xrightarrow{n}0 \qquad w.p.1.$$

PROOF. Since the p.d.f. $f(x)$ is uniformly continuous on R^p and the mode θ is unique, for arbitrary $\varepsilon>0$ there exists $\eta=\eta(\varepsilon)>0$ such that

(4.18) $$\|x-\theta\|\geqq\varepsilon \text{ implies } |f(x)-f(\theta)|\geqq\eta.$$

It is easily verified from (4.17) that

(4.19) $\quad |f(\theta_n)-f(\theta)|\leqq 2\sup_x|f_n(x)-f(x)| \qquad$ for each n.

Let any $\varepsilon>\delta_k>\tilde{\delta}>0$ be fixed. By (4.18) and (4.19) we have

$$\begin{aligned}&P\{\sup_{m\geqq n}\|\theta_m-\theta\|>\delta_k\}\\&\leqq P\{\sup_{m\geqq n}\|\theta_m-\theta\|>\tilde{\delta}\}\\&\leqq P\{\sup_{m\geqq n}\sup_x|f_m(x)-f(x)|\geqq\eta(\tilde{\delta})/2\} \qquad \text{for each } n,\end{aligned}$$

which implies that as $\delta_k\uparrow\varepsilon$

$$\begin{aligned}&P\{\sup_{m\geqq n}\|\theta_m-\theta\|\geqq\varepsilon\}\\&\leqq P\{\sup_{m\geqq n}\sup_x|f_m(x)-f(x)|\geqq\eta(\tilde{\delta})/2\} \qquad \text{for each } n.\end{aligned}$$

By Theorem 4.1 or Corollary 4.1 the second term in the above inequality converges to zero as n tends to infinity. Thus

$$P\{\sup_{m\geqq n}\|\theta_m-\theta\|\geqq\varepsilon\}\xrightarrow{n}0 \qquad \text{for each } \varepsilon>0,$$

which implies that $\|\theta_n-\theta\|\xrightarrow{n}0$ w.p.1. This completes the proof.

We close this section with a presentation of sufficient conditions on $K(x)$ which are required in Theorem 4.1.

PROPOSITION 4.1. *Let $K_j(x)$ for $1\leqq j\leqq p$ be bounded, integrable, continuous functions on the real line R satisfying*

(4.20) $\quad |x|^p|K_j(x)|\longrightarrow 0 \qquad$ as $|x|\to\infty$

and

$$\int_R K_j(x)dx=1.$$

Suppose that for each $j=1,\dots,p$

(4.21) $\quad \int_R |k_j(u)|du<\infty,$

where $|k_j(u)|=|\int_R e^{iux}K_j(x)dx|$ is non-decreasing for $u<0$ and non-incereasing for $u\geqq 0$.

Let $K(x)=\prod_{j=1}^p K_j(x_j)$ for $x=(x_1,\dots,x_p)$. Then $K(x)$ satisfies all conditions of Theorem 4.1.

PROOF. It is easy to see that $K(x)$ is a bounded, integrable, continuous function on R^p satisfying (K1). We shall verify Condition (K2). For $x=(x_1,\dots,x_p)$ let $j(x)$ be the smallest integer j such that $\max\{|x_i|;\ 1\leqq i\leqq p\}=|x_j|$. Then we get

(4.22) $\quad \|x\|^2\leqq p|x_{j(x)}|^2.$

Let $C=\max\{\prod_{\substack{i=1\\ i\neq q}}^{p}\sup_{y\in R}|K_i(y)|;\ 1\leqq q\leqq p\}$. By the boundedness of $K_j(y)$'s, C is finite. It follows from (4.22) that

$$\|x\|^p|K(x)|\leqq p^{p/2}C|x_{j(x)}|^p|K_{j(x)}(x_{j(x)})|. \tag{4.23}$$

Thus, by (4.20) and (4.23) we have Condition (K2), since $|x_{j(x)}|\to\infty$ as $\|x\|\to\infty$ by (4.22). From (4.21) and the fact that

$$|k(t)|=\prod_{j=1}^{p}|k_j(t_j)| \qquad \text{for } t=(t_1,\ldots,t_p), \tag{4.24}$$

it holds that $\int|k(t)|dt<\infty$, that is, Condition (K3) holds. Let $s=(s_1,\ldots,s_p)$ and $t=(t_1,\ldots,t_p)$ in a ray $R(u)$ for $u\neq 0\in R^p$ with $\|s\|\leqq\|t\|$ be fixed. Then we can write s and t as $s=q_1u$ and $t=q_2u$ with $0<q_1\leqq q_2$, respectively. Hence, for each $j=1,\ldots,p$ we get $|k_j(s_j)|\geqq|k_j(t_j)|$ by the property of $|k_j(y)|$. Thus by (4.24) we have $|k(s)|\geqq|k(t)|$, that is, $|k(t)|$ is non-increasing on the ray $R(u)$. This completes the proof.

EXAMPLE 4.2.

The following functions satisfy all conditions of Proposition 4.1: For $x=(x_1,\ldots,x_p)$

(i) $K(x)=2^{-p}\exp(-\sum_{j=1}^{p}|x_j|)$ for $p\geqq 1$

(ii) $K(x)=(2\pi)^{-p/2}\exp(-\|x\|^2/2)$ for $p\geqq 1$

(iii) $K(x)=(1/\pi)(1+x^2)^{-1}$ for $p=1$

(iv) $K(x)=(1/2\pi)(\sin(x/2)/(x/2))^2$ for $p=1$.

References

[1] BOCHNER, S., and CHANDRASEKHARAN, K. *Fourier transforms.* Princeton University Press, 1949.

[2] CACOULLOS, T. *Estimation of a multivariate density.* Ann. Inst. Statist. Math., 18 (1966), 179–189.

[3] DAVIES, H. I. *Strong consistency of a sequential estimator of a probability density function.* Bull. Math. Statist., 15 (1973), No. 3~4, 49–54.

[4] DEVROYE, L. P. *On the pointwise and the integral convergence of recursive kernel estimates of probability densities.* Util. Math., 15 (1979), 113–128.

[5] ISOGAI, E. *Strong consistency and optimality of a sequential density estimator.* Bull. Math. Statist., 19 (1980), No. 1~2, 55–69.

[6] LOÈVE, M. *Probability theory.* 3rd Edition. D. Van Nostrand, Princeton, 1963.

[7] STEIN. E. M. *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, New Jersey, 1970.

[8] YAMATO, H. *Sequential estimation of a continuous probability density function and mode.* Bull. Math. Statist., 14 (1971). No 1~2, 1–12.

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