On the limit behavior of some branching processes with immigration

By

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1. Introduction

We have considered in [1], Galton Watson processes in two districts, A and B, and a simplified type of immigration from B into A, and studied some limit behaviors.

In A-district, the primary Galton Watron process $\{X_n; n=0, 1, 2, \dots\}$ is considered to be based on a probabirity distribution $\{p_k; k=0, 1, 2, \dots\}$; i.e., after one unit of time each particle splits independently of others into a random number of offsprings according to the probability distribution $\{p_k\}$. Similarly in B-district, the primary Galton Watson process $\{Y_n; n=0, 1, 2, \dots\}$ is considered to be based on a probability distribution $\{q_k\}$. Now we assume that when every particle in the n-th generation in A-district has no offspring then instantly one particle in B-district (if exists) immigrates into A-district. That is, let the number of particles in the n-th generation in A-district be $\bar{X}_n$, the number of their offsprings be $\tilde{X}_{n+1}$, and let $\bar{Y}_n$, $\tilde{Y}_{n+1}$ be those in B-district ($n=0, 1, 2, \dots$), then

$$\bar{X}_{n+1}=1,\ \bar{Y}_{n+1}=\tilde{Y}_{n+1}-1 \quad \text{if } \tilde{X}_{n+1}=0,\ \tilde{Y}_{n+1}>0,$$

$$\bar{X}_{n+1}=\tilde{X}_{n+1},\ \bar{Y}_{n+1}=\tilde{Y}_{n+1} \quad \text{otherwise.}$$

$\{Z_n=(\bar{X}_n, \bar{Y}_n); n=0, 1, 2, \dots\}$ is a Markov chain on the pairs of nonnegative integers with homogenious transition probabilities

$$\pi(i. j\longrightarrow k, l)=P(Z_{n+1}=(k, l)\,|\,Z_n=(i, j)).$$

We assume $X_0=1$, $p_0+p_1<1$, $Y_0=1$, and define the generating functions of X_1, Y_1 by

$$f(s)=\sum_{k=0}^{\infty} p_k s^k,\quad g(s)=\sum_{k=0}^{\infty} q_k s^k,\quad |s|<1,$$

and those of X_n, Y_n by

$$f_{(n)}(s)=\sum_{k=0}^{\infty} P(X_n=k)s^k,\quad g_{(n)}(s)=\sum_{k=0}^{\infty} P(Y_n=k)s^k$$

and denote the extinction probabilities of X_n and Y_n by Q_X and Q_Y respectively.

We denote the n-step transition functions of $\{Z_n\}$ by

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$$\pi_{(n)}(i, j \longrightarrow k, l) = P\{Z_{m+n} = (k, l) \mid Z_m = (i, j)\} \quad (n = 1, 2, \dots),$$

and define the generating function of Z_n by

$$\varphi_{(n)}(s, t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \pi_{(n)}(1, 1 \longrightarrow k, l) s^k t^l \quad (\text{n}=1, 2, \dots), \ |s|, \ |t| \leqq 1,$$

and $\varphi_{(0)}(s, t) = st$, and denote $\varphi_{(1)}(s, t)$ by $\varphi(s, t)$.

We obtained in [1] the following two results.

(i) In the subcritical case, i.e., when the mean M_X of $X_1 < 1$, if we denote the generating function of $\bar{X}_n$ by $\bar{f}_{(n)}$, and assume that $\{\bar{f}_{(n)}(p_0) - \varphi_{(n)}(p_0, q_0)\}$ converges to α as $n \to \infty$, then we have

$$\lim_n E(\bar{X}_n) = \frac{\alpha}{1 - M_X}.$$

(ii) In the supercritical case, i.e. when $M_X > 1$, denoting $\lim_n \bar{f}_{(n)}(Q_X) = R_X$, we have

$$P(\bar{X}_n = k) \longrightarrow 0 \ (n \longrightarrow \infty) \ \text{for } k = 1, 2, \dots,$$

$$P(\bar{X}_n = 0) \longrightarrow R_X \ (n \longrightarrow \infty),$$

$$P(\bar{X}_n \longrightarrow \infty (n \longrightarrow \infty)) = 1 - R_X. \quad \text{([1, Theorem 3])}$$

In this paper we study the limit behavior of $\varphi_{(n)}(s, t)$ and $\bar{Y}_n$ in the supercritical case, and give a complementary note to the subcritical case.

2. The limit behavior of $\varphi_{(n)}$ (s. t) in the supercritical case

When $M_X > 1$, as we have seen in [1, p. 53],

$$\varphi_{(n)}(s, 1) = \bar{f}_{(n)}(s) \longrightarrow R_X \ (n \longrightarrow \infty) \ \text{for } 0 \leqq s < 1 \quad (1)$$

$$\varphi_{(n)}(0, t) = \sum_{l=0}^{\infty} \pi_{(n)}(1, 1 \longrightarrow 0, l) t^l$$

$$= \pi_{(n)}(1, 1 \longrightarrow 0, 0) \longrightarrow R_X \ (n \longrightarrow \infty). \quad (2)$$

When $0 < s < 1$ and $0 \leqq t \leqq 1$,

$$\varphi_{(n)}(0, t) \leqq \varphi_{(n)}(s, t) \leqq \varphi_{(n)}(s, 1).$$

Therefore by (1) and (2),

$$\lim_n \varphi_{(n)}(s, t) = R_X.$$

$$\bar{g}_{(n)}(t) = \varphi_{(n)}(1, t)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \pi_{(n)}(1, 1 \longrightarrow k, l) t^l$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_{(n-1)}(1, 1 \longrightarrow i, j) \, \pi(i, j \longrightarrow k, l) t^l$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\pi_{(n-1)}(1,1\longrightarrow i.\ j)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(i,j\longrightarrow k,l)t^l.$$

We denote the transition probabilities of the processes $\{X_n\}$ and $\{Y_n\}$ by $\{p_{ij}\}$ and $\{q_{ij}\}$ respectively. Then

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi(i,j\longrightarrow k,l)t^l$$

$$=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}p_{ik}\,q_{jl}t^l+p_{i0}\,q_{j0}+p_{i0}\sum_{l=1}^{\infty}q_{jl}t^{l-1}$$

$$\geqq\sum_{l=0}^{\infty}q_{jl}\,t^l=[g(t)]^j.$$

Accordingly

$$\bar{g}_{(n)}(t)\geqq\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\pi_{(n-1)}(1,1\longrightarrow i,j)[g(t)]^j$$

$$=\bar{g}_{(n-1)}[g(t)]. \tag{3}$$

When $0\leqq t\leqq Q_Y$, since $g(t)\geqq t$, we have

$$\bar{g}_{(n)}(t)\geqq\bar{g}_{(n-1)}(t),$$

and so $\bar{g}_{(n)}(t)$ converges.
Let $R_Y=\lim_n\ \bar{g}_{(n)}(Q_Y)$, then $R_Y\geqq Q_Y$. We can see by (3),

$$\bar{g}_{(n)}(t)\geqq\bar{g}_{(k)}(g_{(n-k)}(t))\ \text{if}\ 0\leqq k\leqq n.$$

Here if $n\longrightarrow\infty$, then $g_{(n-k)}(t)\longrightarrow Q_Y$ and so

$$\lim_n\ \bar{g}_{(n)}(t)\geqq g_{(k)}(Q_Y)\ \text{for every}\ k.$$

Accordingly

$$\lim_n\ \bar{g}_{(n)}(t)\geqq R_Y\ \text{if}\ 0\leqq t\leqq Q_Y.$$

On the otherhand, if $0\leqq t\leqq Q_Y$

$$\bar{g}_{(n)}(t)\leqq\bar{g}_{(n)}(Q_Y)\leqq R_Y.$$

Thus we have

$$\lim_n\ \bar{g}_{(n)}(t)=R_Y\ \text{if}\ 0\leqq t\leqq Q_Y. \tag{4}$$

Next we prove that when $Q_Y>0$ this equality holds for $0\leqq t<1$.
Let $g_{(n)}(t)=\sum_{k=0}^{\infty}a_k(n)t^n$, then

$$a_k(n)\geqq 0\ \text{for}\ k=0,1,2,\ldots,\ n=0,1,2,\ldots.$$

and (4) implies

$$\lim_n a_0(n)=R_Y$$

$$\lim_n a_k(n)=0 \quad k=1, 2, \dots . \tag{5}$$

Now we put $t\in[0, 1)$. For arbitrary positive number ε, we can put sufficiently large number K such that

$$\sum_{k=K+1}^{\infty} a_k(n)t^k<\varepsilon \text{ for } n=1, 2, \dots.$$

For another positive number δ, since (5) holds, we can put N such that if $n>N$ then

$$\sum_{k=0}^{K} a_k(n)t^k< R_Y+\delta.$$

Thus we have for sufficiently large n

$$R_Y\leqq\sum_{k=0}^{\infty} a_k(n)t^k<R_Y+\varepsilon+\delta.$$

Here ε and δ are arbitrary positive numbers, so we have

$$\lim_n \bar{g}_{(n)}(t)=R_Y \text{ if } 0\leqq t<1.$$

Thus we have;

THEOREM 1. *If $M_X>1$ and $Q_Y>0$, then*

$$\lim_n \varphi_{(n)}(s, t)=\begin{cases} R_X & (0\leqq t<1) \\ R_Y & (s=1, 0\leqq t<1) \\ 1 & (s=t=1)\end{cases}$$

COROLLARY. *If $M_X>1$ and $Q_Y>0$, then*

1) $P(\bar{Y}_n=k)\longrightarrow 0\ (n\longrightarrow\infty)$ *for* $k=1, 2, \dots.$

2) $P(\bar{Y}_n=0)\longrightarrow R_Y(n\longrightarrow\infty)$

3) $P(\bar{Y}_n\longrightarrow\infty(n\longrightarrow\infty))=1-R_Y.$

It is easily known that $R_X\leqq Q_X$, $R_Y\geqq Q_Y$ and $R_X\leqq R_Y$, and that $R_X=R_Y$ if and only if $Q_X\geqq Q_Y$.

3. The limit behavior of $E(\bar{X}_n)$ in subcritical case

In the Theorem 2 of [1], we have assumed the convergence of $\{\bar{f}_{(n)}(p_0)-\varphi_{(n)}(p_0, q_0)\}$. But yet now we cannot know whether it converges or not. So in this section we give some complementary notes.

LEMMA. If $0\leqq s, t\leqq 1$ and N is a positive integr, then

$$\bar{f}_{(N)}(s)-\varphi_{(N)}(s, t) \leqq f_{(N)}(s)(1-g_{(N)}(t)). \tag{1}$$

PROOF. We prove it by the induction.

(I) When $N=1$,

$$\bar{f}_{(1)}(s)-\varphi_{(1)}(s, t)=\varphi_{(1)}(s, 1)-\varphi_{(1)}(s, t)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi(1,1\longrightarrow m,n)s^m(1-t^n)$$

$$=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}p_{1m}q_{1n}s^m(1-t^n)+p_{10}\sum_{n=1}^{\infty}q_{1n}s(1-t^{n-1})$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p_{1m}q_{1n}s^m(1-t^n)-p_{10}\sum_{n=0}^{\infty}q_{1n}(1-t^n)+p_{10}\sum_{n=1}^{\infty}q_{1n}s(1-t^n)$$

$$=f(s)-f(s)g(t)-p_{10}\sum_{n=1}^{\infty}q_{1n}\{1-s-t^{n-1}(t-s)\}$$

$$\leqq f(s)(1-g(t)).$$

(II) Now we assume

$$\bar{f}_{(N)}(s)-\varphi_{(N)}(s,t)\leqq f_{(N)}(s)(1-g_{(N)}(t))\ (0\leqq s,t\leqq 1).$$

As in (I)

$$\bar{f}_{(N+1)}(s)-\varphi_{(N+1)}(s,t)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi_{(N+1)}(1,1\longrightarrow m,n)s^m(1-t^n)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi_{(N)}(1,1\longrightarrow k,l)\pi(k,l\longrightarrow m,n)s^m(1-t^n)$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi_{(N)}(1,1\longrightarrow k,l)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi(k,l\longrightarrow m,n)s^m(1-t^n)$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi_{(N)}(1,1\longrightarrow k,l)\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p_{km}q_{ln}s^m(1-t^n)-p_{k0}\sum_{n=1}^{\infty}q_{ln}(1-s-t^{n-1}(t-s))\Big\}. \quad (2)$$

Here we can see by the assumption,

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi_{(N)}(1,1\longrightarrow k,l)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p_{km}q_{ln}s^m(1-t^n)$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi_{(N)}(1,1\longrightarrow k,l)\Big\{f(s)^k-f(s)^k g(t)^l\Big\}$$

$$=\bar{f}_{(N)}(f(s))-\varphi_{(N)}(f(s),g(t))$$

$$\leqq f_{(N)}(f(s))(1-g_{(N)}(g(t)))$$

$$=f_{(N+1)}(s)(1-g_{(N+1)}(t)).$$

Therefore we have

$$\bar{f}_{(N+1)}(s)-\varphi_{(N+1)}(s,t)\leqq f_{(N+1)}(s)(1-g_{(N+1)}(t)).$$

(I) and (II) prove (1).

If we consider $N\longrightarrow\infty$ in (1), then we have

$$\overline{\lim_n} \left\{ \bar{f}_{(n)}(s) - \varphi_{(n)}(s, t) \right\} \leqq Q_X(1-Q_Y). \tag{2}$$

THEOREM 2. *If $M_X > 1$, then*

$$\overline{\lim_n}\ E(\bar{X}_n) \leqq \frac{1-Q_Y}{1-M_X} . \tag{3}$$

PROOF. Since $M_X < 1$, we have $Q_X = 1$, hence by (2)

$$\overline{\lim_n} \left\{ \bar{f}_{(n)}(s) - \varphi_{(n)}(s, t) \right\} \leqq 1 - Q_Y .$$

Thus by the expansion of $E(\bar{X}_n)$ in [1, p. 51] and [2, p. 22, lemma A], we get (3).

References

[1] KANEKO, T. *The immigration between branching processes.* Sci. Rep. Niigata Univ. Ser. A, 12 (1975), 47–53.

[2] CHUNG, K. L. Markov Chains with Stationary Transition Probabilities. Springer-Verlag.