On a type I factor direct summand of a W^*-tensor product

By

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As pointed out by A. Wulfsohn in Zbl. 372 #46061, the argument of the theorem of [3] is incomplete. We give a correct proof (Theorem 5) as a consequence of a characterization of a type I factor direct summand of a W^*-tensor product of two W^*-algebras. The author wishes to take this opportunity to deeply thank Professor A. Wulfsohn for his useful suggestions.

1. Tensor products of abelian W^*-algebras

For a locally compact Hausdorff space X with a Radon measure m let $L^\infty(X, m)$ be the algebra of all essentially bounded measurable functions on X, and let $L^2(X, m)$ be the Hilbert space of all measurable square integrable functions on X. Each function $f \in L^\infty(X, m)$ gives rise to a multiplication operator $\pi(f) \in B(L^2(X, m))$, defined by $(\pi(f)g)(s) = f(s)g(s)$ ($g \in L^2(X, m)$, $s \in X$). We may identify $f \in L^\infty(X, m)$ with $\pi(f)$, and $L^\infty(X, m)$ with $\pi(L^\infty(X, m))$ [1, I, §7, Théorème 2].

Let X and Y be compact Hausdorff spaces with Radon measures m and n respectively. Then we have canonically $L^2(X \times Y, m \otimes n) = L^2(X, m) \otimes L^2(Y, n)$, the Hilbert space tensor product. In this situation, we have the following two lemmas.

LEMMA 1. *Let $L^\infty(X, m) \bar{\otimes} L^\infty(Y, n)$ be the W^*-tensor product of $L^\infty(X, m)$ and $L^\infty(Y, n)$. Then $L^\infty(X \times Y, m \otimes n) = L^\infty(X, m) \bar{\otimes} L^\infty(Y, n)$.*

PROOF. For each $f \in B(L^2(X, m))_*$, the predual of $B(L^2(X, m))$, let $R_f : B(L^2(X, m)) \bar{\otimes} B(L^2(Y, n)) \longrightarrow B(L^2(Y, n))$ be a unique σ-weakly continuous linear map satisfying $R_f(a \otimes b) = \langle f, a \rangle b$ ($a \in B(L^2(X, m))$, $b \in B(L^2(Y, n))$). Let $g \in L^\infty(X \times Y, m \otimes n)$ with $g \geq 0$. For a vector state $f : a \longrightarrow (a\xi | \xi)$ ($a \in B(L^2(X, m))$, $\xi \in L^2(X, m)$), we have $R_f(g) \in L^\infty(Y, n)$. Then for a normal state f, $R_f(g) \in L^\infty(Y, n)$, and for $f \in B(L^2(X, m))_*$, $R_f(g) \in L^\infty(Y, n)$. Hence $R_f(g) \in L^\infty(Y, n)$ for any $g \in L^\infty(X \times Y, m \otimes n)$ and $f \in B(L^2(X, m))_*$. Similarly, for each $f \in B(L^2(Y, n))_*$ let $L_f : B(L^2(X, m)) \bar{\otimes} B(L^2(Y, n)) \longrightarrow B(L^2(X, m))$ be a unique σ-weakly continuous linear map satisfying $L_f(a \otimes b) = \langle f, b \rangle a$ ($a \in B(L^2(X, m))$, $b \in B(L^2(Y, n))$). Let $g \in L^\infty(X \times Y, m \otimes n)$. For each $f \in B(L^2(Y, n))_*$ we have $L_f(g) \in$

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$L^{\infty}(X, m)$. Since $L^{\infty}(X\times Y, m\otimes n) \supseteq L^{\infty}(X, m)\bar{\otimes} L^{\infty}(Y, n)$, by [5, Theorem 2. 1] $L^{\infty}(X\times Y, m\otimes n)=L^{\infty}(X, m)\bar{\otimes} L^{\infty}(Y, n)$.

LEMMA 2. *If p is a minimal projection in $L^{\infty}(X, m)\bar{\otimes} L^{\infty}(Y, n)$, then there are minimal projections p_1 and p_2 in $L^{\infty}(X, m)$ and $L^{\infty}(Y, n)$ respectively such that $p=p_1\otimes p_2$.*

PROOF. Let $N_1=\{s\in X, m(\{s\})\neq 0\}$, $N_2=\{t\in Y, n(\{t\})\neq 0\}$. Then N_1 and N_2 are at most countable. Let m_1 be the atomic part of m, defined by $m_1(E)=m(E\cap N_1)$ for each measurable set $E\subseteq X$, and put $m_2=m-m_1$. Then $L^{\infty}(X, m)=L^{\infty}(X, m_1)\oplus L^{\infty}(X, m_2)$. Similarly, let n_1 be the atomic part of n, defined by $n_1(F)=n(F\cap N_2)$ for each measurable set $F\subseteq Y$, and put $n_2=n-n_1$. Then $L^{\infty}(Y, n)=L^{\infty}(Y, n_1)\oplus L^{\infty}(Y, n_2)$. Since m_2 satisfies $m_2(\{s\})=0$ for each $s\in X$, $m_2\otimes n$ also satisfies $m_2\otimes n(\{s\times t\})=0$ for each $s\times t\in X\times Y$. Hence for each $\varepsilon>0$ and $s\times t\in X\times Y$ there is a neighborhood $U(s\times t)$ of $s\times t$ such that $m_2\otimes n(U(s\times t))<\varepsilon$. Then there is a finite open covering $\{U_i\}_{i=1}^{n}$ of $X\times Y$ with $m_2\otimes n(U_i)<\varepsilon$ $(i=1,\dots, n)$. If q is a miminal projection in $L^{\infty}(X, m_2)\bar{\otimes} L^{\infty}(Y, n)$, by Lemma 1 we have $q\in L^{\infty}(X\times Y, m_2\otimes n)$. Hence there is a measurable subset E of $X\times Y$ such that $\pi(\chi_E)=q$, where $\pi(\chi_E)$ is the multiplication operator of the characteristic function χ_E of E. Then there is a subset U in the above covering such that $\pi(\chi_{E\cap U})\neq 0$. Since q is a minimal projection, $\pi(\chi_E)=q\leq\pi(\chi_{E\cap U})$. Hence $m_2\otimes n(E)\leq m_2\otimes n(E\cap U)\leq m_2\otimes n(U)<\varepsilon$. Since ε is arbitrary, $m_2\otimes n(E)=0$, and so $\pi(\chi_E)=q=0$. This is a contradiction. Thus there is no minimal projection in $L^{\infty}(X, m_2)\bar{\otimes} L^{\infty}(Y, n)$. Similarly, there is no minimal projection in $L^{\infty}(X, m_1)\bar{\otimes} L^{\infty}(Y, n_2)$. Consequently, we have $p\in L^{\infty}(X, m_1)\bar{\otimes} L^{\infty}(Y, n_1)$.

The algebra $L^{\infty}(X, m_1)\bar{\otimes} L^{\infty}(Y, n_1)$ is *-isomorphic to the algebra $L^{\infty}(N_1\times N_2)$ of all bounded functions on $N_1\times N_2$. Since each minimal projection in $L^{\infty}(N_1\times N_2)$ is the characteristic function of a point of $N_1\times N_2$, p can be written in the form: $p=p_1\otimes p_2$, where p_1 and p_2 are minimal projections in $L^{\infty}(X, m)$ and $L^{\infty}(Y, n)$. This completes the proof.

LEMMA 3. *Let A and B be abelian W^*-algebras. Let p be a minimal projection in the W^*-tensor product $A\bar{\otimes}B$. Then there are minimal projections p_1 and p_2 in A and B respectively such that $p=p_1\otimes p_2$.*

PROOF. There is a locally compact Hausdorff space X with a Radon measure m such that $\cup X_i=X$, $X_i\cap X_j=\phi$ for $i\neq j$, each X_i is compact and open, and $L^{\infty}(X, m)$ is *-isomorphic to A; there is a locally compact Hausdorff space Y with a Radon measure n such that $\cup Y_k=Y$, $Y_k\cap Y_j=\phi$ for $k\neq j$, each Y_k is compact and open, and $L^{\infty}(Y, n)$ is *-isomorphic to B [1, I, §7, 2–3]. Let m_i be the restriction of m to X_i and let n_k be the restriction of n to Y_k. Since $L^{\infty}(X, m)=\Sigma_i\oplus L^{\infty}(X_i, m_i)$ and $L^{\infty}(Y, n)=\Sigma_k\oplus L^{\infty}(Y_k, n_k)$, we have $L^{\infty}(X, m)\bar{\otimes} L^{\infty}(Y, n)=\Sigma_{i,k}\oplus L^{\infty}(X_i, m_i)\bar{\otimes} L^{\infty}(Y_k, n_k)$. Since p is a minimal projection, there is a W^*-subalgebra $L^{\infty}(X_i, m_i)\bar{\otimes} L^{\infty}(Y_k, n_k)$ which contains p. By Lemma 2 there are minimal projections p_1 and p_2 in $L^{\infty}(X_i, m_i)$ and $L^{\infty}(Y_k, n_k)$ such that $p=p_1\otimes p_2$.

2. The main results

THEOREM 4. *Let M and N be W^*-algebras. If z is a central projection in $M\bar{\otimes} N$ such that $(M\bar{\otimes} N)_z$ is a type I factor. Then there are central projections p and q in M and N respectively such that $(M\bar{\otimes} N)_z=M_p\bar{\otimes} N_q$.*

PROOF. By [4, Proposition 2. 2. 10] M and N can be written as follows: $M=M_d\oplus M_c$, $N=N_d\oplus N_c$, where M_d, N_d are of type I and M_c, N_c are continuous. By [4, Theorem 2. 6. 6] $M_d\bar{\otimes} N_d$ is the type I direct summand of $M\bar{\otimes} N$. Hence $z\in M_d\bar{\otimes} N_d$, and $(M\bar{\otimes} N)_z=(M_d\bar{\otimes} N_d)_z$; so we may assume that M and N are of type I.

By [4, Theorems 2. 3. 2 and 2. 3. 3] M can be written as follows: $M=\Sigma_i\oplus A_i\bar{\otimes} L(H_i)$, where A_i is an abelian W^*-algebra and H_i is an i-dimensional Hilbert space. Similarly, we have $N=\Sigma_j\oplus B_j\bar{\otimes} L(K_j)$, where B_j is an abelian W^*-algebra and K_j is a j-dimensional Hilbert space. Then there is a canonical $*$-isomorphism of $M\bar{\otimes} N$ onto $\Sigma_{i,j}\oplus(A_i\bar{\otimes} L(H_i))\bar{\otimes}(B_j\bar{\otimes} L(K_j))$. Since each $(A_i\bar{\otimes} L(H_i))\bar{\otimes}(B_j\bar{\otimes} L(K_j))$ is $*$-isomorphic to $(A_i\bar{\otimes} B_j)\bar{\otimes} L(H_i\otimes K_j)$, there is a $*$-isomorphism of $M\bar{\otimes} N$ onto $\Sigma_{i,j}\oplus(A_i\bar{\otimes} B_j)\bar{\otimes} L(H_i\otimes K_j)$. Hence there is a canonical $*$-isomorphism Φ of the center of $M\bar{\otimes} N$ onto $\Sigma_{i,j}\oplus A_i\bar{\otimes} B_j$.

Since $(M\bar{\otimes} N)_z$ is a factor, there is a pair (i, j) of cardinal numbers such that $\Phi(z)\in A_i\bar{\otimes} B_j$ and $\Phi(z)$ is a minimal projection in $A_i\bar{\otimes} B_j$. By Lemma 3 there are minimal projections $p_i\in A_i$ and $q_j\in B_j$ such that $\Phi(z)=p_i\otimes q_j$. Hence there are central projections p and q in M and N such that $z=p\otimes q$, so that $(M\bar{\otimes} N)_z=M_p\bar{\otimes} N_q$.

Let A and B be C^*-algebras and let A^{**} and B^{**} be second duals of A and B. The spatial C^*-tensor product $A\otimes_\alpha B$ is canonically embedded in $A^{**}\bar{\otimes} B^{**}$ by [6, Théorème 1].

THEOREM 5. *In the above situation, let π be an irreducible representation of $A\otimes_\alpha B$ on a Hilbert space H. Suppose that* a state $x\longrightarrow(\pi(x)\xi\,|\,\xi)$ $(\xi\in H)$ *on $A\otimes_\alpha B$ has a normal extension g to $A^{**}\bar{\otimes} B^{**}$. Then there are representations π_1 and π_2 of A and B respectively such that π is equivalent to $\pi_1\otimes\pi_2$.*

PROOF. Let (ρ, η) be the representation associated with g. Since $\rho(A\otimes_\alpha B)\eta$ is dense in the representation space of ρ, and $\|\rho(x)\eta\|=\|\pi(x)\xi\|$ for $x\in A\otimes_\alpha B$, we may assume that ρ is a normal extension of π to $A^{**}\bar{\otimes} B^{**}$ on H and $\eta=\xi$. Hence ρ is irreducible. Then there is a central projection z in $A^{**}\bar{\otimes} B^{**}$ such that $(A^{**}\bar{\otimes} B^{**})_z$ is $*$-isomorphic to $\rho(A^{**}\bar{\otimes} B^{**})$, so that $(A^{**}\bar{\otimes} B^{**})_z$ is a type I factor. By Theorem 4 there are central projections p and q in A^{**} and B^{**} such that $(A^{**}\bar{\otimes} B^{**})_z=A^{**}_p\bar{\otimes} B^{**}_q$. By [4, Theorem 2. 6. 6] factors A^{**}_p and B^{**}_q are of type I. Let $\tilde{\pi}$ be the restriction of π to A ([2, p. 9, Definiton 3]). Then the weak closure of $\tilde{\pi}(A)$ is $\rho(A^{**}\otimes I)$, and is $*$-isomorphic to A^{**}_p. Hence $\tilde{\pi}$ is a type I factor representation. By [2, p. 7, Proposition 2] there are representations π_1 and π_2 of A and B respectively such that $\pi\simeq\pi_1\otimes\pi_2$.

EXAMPLE 6. *Let A and B be UHF algebras. Under the embedding $A\otimes_\alpha B\subseteq A^{**}\bar{\otimes} B^{**}$,* *the canonical injection Ψ of $A \otimes_\alpha B$ into $(A \otimes_\alpha B)^{**}$ has no normal extension to $A^{**} \bar{\otimes} B^{**}$.*

PROOF. By [2, p. 20, Proposition 7] the spatial C^*-tensor product $A \otimes_\alpha B$ is a unique C^*-tensor product of A and B. Then, by [2, p. 32, Theorem 6] and Theorem 5, there is a pure state f on $A \otimes_\alpha B$ which has no normal extension to $A^{**} \bar{\otimes} B^{**}$.

Suppose that Ψ has a normal extension $\bar{\Psi}$ to $A^{**} \bar{\otimes} B^{**}$. Since f may be regarded as an element $\bar{f}$ of the predual of $(A \otimes_\alpha B)^{**}$, we have

$$f(x) = \bar{f}(\bar{\Psi}(x)) \quad (x \in A \otimes_\alpha B).$$

Hence f has a normal extension to $A^{**} \bar{\otimes} B^{**}$. This is a contradiction, and completes the proof.

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