On energy inequalities for the mixed problems for the wave equation in a domain with a corner

By

Kazuo ASANO*

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1. Introduction

Recently the mixed problems for hyperbolic equations in domains with corners have been investigated (see, for example, [6], [7], [10], [12], [13], [14]). We also consider the mixed problems for the wave equation:

$$(1)\qquad \square u=\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=f(t,x,y) \qquad \text{in} \qquad (0,T)\times\Omega,$$

$$(2)\begin{cases} \text{(a)}\quad \mathfrak{B}_1 u=\dfrac{\partial u}{\partial x}-b\dfrac{\partial u}{\partial y}-c\dfrac{\partial u}{\partial t}=0 & \text{on} \quad (0,T)\times B_1,\\ \text{(b)}\quad \mathfrak{B}_2 u=0 & \text{on} \quad (0,T)\times B_2 \end{cases}$$

where $\mathfrak{B}_2 u=u$ or $\dfrac{\partial u}{\partial y}$, $\Omega=\{(x,y)\in\boldsymbol{R}^2;\ x>0,\ y>0\}$, $B_1=\{(x,y)\in\boldsymbol{R}^2;\ x=0,\ y>0\}$, $B_2=\{(x,y)\in\boldsymbol{R}^2;\ x>0,\ y=0\}$ and b, c are real constants. Furthermore we assume the following condition:

$$(3)\qquad |b|\leqq c.$$

Here we remark that the condition (3) is the necessary and sufficient condition to be L^2-well-posed for the mixed problem: Equation (1) in a domain $0<t<T$, $0<x<\infty$, $-\infty<y<\infty$ with boundary condition (2(a)) (see [2–I]).

Ibuki [6] proved the existence and the regularity of the solution for the mixed problem with boundary conditions $\mathfrak{B}_1 u=\dfrac{\partial u}{\partial x}$, $\mathfrak{B}_2 u=u$, which is only the available case when we restrict his methods of considerations to our problems (1) and (2). Taniguchi [14] showed the energy estimate for (1) with boundary conditions $\mathfrak{B}_1 u=\dfrac{\partial u}{\partial x}+b\dfrac{\partial u}{\partial y}-\dfrac{\partial u}{\partial t}=g_1(t,y)$, $\mathfrak{B}_2 u=\dfrac{\partial u}{\partial y}+\bar{b}\dfrac{\partial u}{\partial x}-\bar{b}\dfrac{\partial u}{\partial t}=g_2(t,x)$, where b is a complex constant with $|b|=1$, $Re\, b>0$. Kojima and Taniguchi [7] dealed with the existence and the energy estimate

* Niigata University

of the solution for (1) with boundary conditions $\mathcal{B}_1 u=\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}-c\frac{\partial u}{\partial t}=g_1(t, y)$, $\mathcal{B}_2 u=\frac{\partial u}{\partial y}+\frac{1}{b}\frac{\partial u}{\partial x}-\frac{c}{b}\frac{\partial u}{\partial t}=g_2(t, x)$, where b and c are complex constants such that $(c+1)z^2+2bz+(c-1)=0$ has two different roots in the domain D or has the double roots in its interior, where $D=\{z\in C^1;\ |z|\leqq 1,\ Re\, z\leqq 0,\ z\neq\pm i\}$. Concerning with the existence of the solution, it is assumed that b and c are real constants. They also dealed with the energy estimate for the mixed problems for hyperbolic symmetric systems in a domain $x, y, t>0$, $-\infty<z<\infty$ with constant coefficients. Osher [10] considered the energy estimate for the mixed problems for hyperbolic symmetric systems in a domain $x, y, t>0$, $-\infty<z_j<\infty$, $j=3,\dots, n$ with constant coefficients by constructing a symmetrizer under Kreiss' condition for two half space problems whose domains are $x, t>0$, $-\infty<y, z_j<\infty$ and $y, t>0$, $-\infty<x, z_j<\infty$, respectively. Sarason and Smoller [13] proved the necessary condition for certain a priori estimate for the mixed problems for strictly hyperbolic systems from the point of view of geometrical optics. And Sarason [12] discussed the mixed problems for hyperbolic symmetrizable systems in a corner domain.

In this paper we give the sufficient condition to obtain the energy inequalities for the mixed problem (1) and (2), that is, for the solution $u(t, x, y)$ of the mixed problem (1) and (2) which belongs to $H^2((0, T)\times\Omega)$, there exists a positive constant K such that the following energy inequality holds: for any $t (0<t<T)$

$$(4)\qquad \left|\left|\left| u(t,\cdot)\right|\right|\right|_1^2 \leqq K\left\{\int_0^t \left\|(\Box u)(s,\cdot)\right\|^2 ds+\left|\left|\left| u(0,\cdot)\right|\right|\right|_1^2\right\},$$

where $\|u(\cdot)\|^2=\|u(\cdot)\|^2_{L^2(\Omega)}$, $|||u(t,\cdot)|||_1^2=\|u(t,\cdot)\|^2+\left\|\frac{\partial u}{\partial t}(t,\cdot)\right\|^2+\left\|\frac{\partial u}{\partial x}(t,\cdot)\right\|^2+\left\|\frac{\partial u}{\partial y}(t,\cdot)\right\|^2$ and K is independent of u.

We set

$$Q_1=\{(b, c)\in \mathbf{R}^2;\ |b|\leqq c,\ b\geqq -1\}-(-1, 1)$$

and

$$Q_2=\{(b, c)\in \mathbf{R}^2;\ |b|\leqq c,\ |b|\leqq 1\}-\{(-1, 1)\cup(1, 1)\}.$$

Then we have the following

THEOREM.

(i) *Let* $(b, c)\in Q_1$. *Then the solution* $u(t, x, y)$ $(\in H^2((0, T)\times\Omega))$ *of the mixed problem* (1) *and* (2) *with* $\mathcal{B}_2 u=u$ *has the energy inequality* (4).

(ii) *Let* $(b, c)\in Q_2$. *Then the solution* $u(t, x, y)(\in H^2((0, T)\times\Omega))$ *of the mixed problem* (1) *and* (2) *with* $\mathcal{B}_2 u=\frac{\partial u}{\partial y}$ *has also the energy inequality* (4).

To show Theorem we apply the methods of the consideration used by Agemi [1].

It is easily seen by the proof of (4) that we have the same results even if we add (lower order terms)u in the left hand sides in (1) and (2(a)), here all coefficients in (lower order terms) are sufficiently smooth and constant except a compact set.

2. Proof of Theorem

By $(u(t, \cdot), v(t, \cdot))$, $\langle u(t, 0, y), v(t, 0, y)\rangle_{B_1}$ and $\langle u(t, x, 0), v(t, x, 0)\rangle_{B_2}$ we denote $\int_0^\infty\int_0^\infty u(t, x, y)\overline{v(t, x, y)}dxdy$, $\int_0^\infty u(t, 0, y)\overline{v(t, 0, y)}dy$ and $\int_0^\infty u(t, x, 0)\overline{v(t, x, 0)}dx$, respectively. We set

$$G_1(t)=\left(A_0\frac{\partial u}{\partial t}(t, \cdot), \frac{\partial u}{\partial t}(t, \cdot)\right)+2Re\left(\frac{\partial u}{\partial t}(t, \cdot), A_1\frac{\partial u}{\partial x}(t, \cdot)\right.$$

$$\left.+A_2\frac{\partial u}{\partial y}(t, \cdot)\right)+\left(A_0\frac{\partial u}{\partial x}(t, \cdot), \frac{\partial u}{\partial x}(t, \cdot)\right)+\left(A_0\frac{\partial u}{\partial y}(t, \cdot), \frac{\partial u}{\partial y}(t, \cdot)\right),$$

$$G_2(t)= Re\left\{2\left\langle\frac{\partial u}{\partial x}(t, 0, y), A_0\frac{\partial u}{\partial t}(t, 0, y)\right\rangle_{B_1}+\left\langle\frac{\partial u}{\partial t}(t, 0, y),\right.\right.$$

$$\left.A_1\frac{\partial u}{\partial t}(t, 0, y)\right\rangle_{B_1}+\left\langle\frac{\partial u}{\partial x}(t, 0, y), A_1\frac{\partial u}{\partial x}(t, 0, y)\right\rangle_{B_1}$$

$$\left.-\left\langle\frac{\partial u}{\partial y}(t, 0, y), A_1\frac{\partial u}{\partial x}(t, 0, y)\right\rangle_{B_1}+2\left\langle\frac{\partial u}{\partial y}(t, 0, y), A_2\frac{\partial u}{\partial x}(t, 0, y)\right\rangle_{B_1}\right\}$$

and

$$G_3(t)= Re\left\{2\left\langle\frac{\partial u}{\partial y}(t, x, 0), A_0\frac{\partial u}{\partial t}(t, x, 0)\right\rangle_{B_2}+\left\langle\frac{\partial u}{\partial t}(t, x, 0),\right.\right.$$

$$\left.A_2\frac{\partial u}{\partial t}(t, x, 0)\right\rangle_{B_2}+\left\langle\frac{\partial u}{\partial y}(t, x, 0), A_2\frac{\partial u}{\partial y}(t, x, 0)\right\rangle_{B_2}$$

$$\left.-\left\langle\frac{\partial u}{\partial x}(t, x, 0), A_2\frac{\partial u}{\partial x}(t, x, 0)\right\rangle_{B_2}+2\left\langle\frac{\partial u}{\partial x}(t, x, 0), A_1\frac{\partial u}{\partial y}(t, x, 0)\right\rangle_{B_2}\right\},$$

here A_j $(j=0, 1, 2)$ are real constants.

LEMMA 1. *Let $u(t, x, y)\in H^2((0, T)\times\Omega)$. Then we have the following equality.*

$$2\,Re\int_0^t\left((\Box u)(s, \cdot), A_0\frac{\partial u}{\partial t}(s, \cdot)+A_1\frac{\partial u}{\partial x}(s, \cdot)+A_2\frac{\partial u}{\partial y}(s, \cdot)\right)ds$$

$$= G_1(s)\Big|_0^t+\int_0^t G_2(s)ds+\int_0^t G_3(s)ds$$

for any $t(0<t<T)$.

PROOF. Using the integration by parts we obtain that for any t $(0<t<T)$

$$2\,Re\int_0^t\left((\Box u)(s, \cdot), \frac{\partial u}{\partial t}(s, \cdot)\right)ds$$

$$=\left[\left\|\frac{\partial u}{\partial t}(s, \cdot)\right\|^2+\left\|\frac{\partial u}{\partial x}(s, \cdot)\right\|^2+\left\|\frac{\partial u}{\partial y}(s, \cdot)\right\|^2\right]\Big|_0^t$$

$$+\int_0^t Re\left\{2\left\langle\frac{\partial u}{\partial x}(s, 0, y), \frac{\partial u}{\partial t}(s, 0, y)\right\rangle_{B_1}+2\left\langle\frac{\partial u}{\partial y}(s, x, 0),\right.\right.$$

$$\frac{\partial u}{\partial t}(s, x, 0)\Big\rangle_{B_2}\Big\}\, ds,$$

$$2\, Re \int_0^t \Big((\Box u)(s, \cdot), \frac{\partial u}{\partial x}(s, \cdot)\Big)\, ds = 2\, Re\Big(\frac{\partial u}{\partial t}(s, \cdot), \frac{\partial u}{\partial x}(s, \cdot)\Big)\Big|_0^t$$

$$+ \int_0^t \Big\{ \Big\langle \frac{\partial u}{\partial t}(s, 0, y), \frac{\partial u}{\partial t}(s, 0, y)\Big\rangle_{B_1} + \Big\langle \frac{\partial u}{\partial x}(s, 0, y), \frac{\partial u}{\partial x}(s, 0, y)\Big\rangle_{B_1}$$

$$- \Big\langle \frac{\partial u}{\partial y}(s, 0, y), \frac{\partial u}{\partial y}(s, 0, y)\Big\rangle_{B_1} + 2\, Re \Big\langle \frac{\partial u}{\partial x}(s, x, 0), \frac{\partial u}{\partial y}(s, x, 0)\Big\rangle_{B_2}\Big\}\, ds$$

and

$$2\, Re \int_0^t \Big((\Box u)(s, \cdot), \frac{\partial u}{\partial y}(s, \cdot)\Big)\, ds = 2\, Re\Big(\frac{\partial u}{\partial t}(s, \cdot), \frac{\partial u}{\partial y}(s, \cdot)\Big)\Big|_0^t$$

$$+ \int_0^t \Big\{ 2\, Re \Big\langle \frac{\partial u}{\partial y}(s, 0, y), \frac{\partial u}{\partial x}(s, 0, y)\Big\rangle_{B_1}$$

$$+ \Big\langle \frac{\partial u}{\partial t}(s, x, 0), \frac{\partial u}{\partial t}(s, x, 0)\Big\rangle_{B_2} + \Big\langle \frac{\partial u}{\partial y}(s, x, 0), \frac{\partial u}{\partial y}(s, x, 0)\Big\rangle_{B_2}$$

$$- \Big\langle \frac{\partial u}{\partial x}(s, x, 0), \frac{\partial u}{\partial x}(s, x, 0)\Big\rangle_{B_2}\Big\}\, ds.$$

From these equalities we get this lemma.

LEMMA 2. *Using* $\mathcal{B}_1 u=0$ *on* B_1 *and* $\mathcal{B}_2 u=0$ *on* B_2 *in* $G_2(t)$ *and* $G_3(t)$, *we suppose that, for some* A_j $(j=0, 1, 2)$ *in Lemma* 1, *the quadratic form corresponding to* G_1 *is positive definite and those corresponding to* G_2 *and* G_3 *are both positive semi-definite. Then the energy inequality* (4) *holds for* $u(t, x, y) \in H^2((0, T)\times\Omega)$ *with* $\mathcal{B}_1 u=0$ *on* B_1 *and* $\mathcal{B}_2 u=0$ *on* B_2.

PROOF. From Lemma 1, for any t $(0<t<T)$ we have

$$(5)\qquad K_1\Big(\Big\|\frac{\partial u}{\partial t}(t, \cdot)\Big\|^2 + \Big\|\frac{\partial u}{\partial x}(t, \cdot)\Big\|^2 + \Big\|\frac{\partial u}{\partial y}(t, \cdot)\Big\|^2\Big)$$

$$- K_2\Big(\Big\|\frac{\partial u}{\partial t}(0, \cdot)\Big\|^2 + \Big\|\frac{\partial u}{\partial x}(0, \cdot)\Big\|^2 + \Big\|\frac{\partial u}{\partial y}(0, \cdot)\Big\|^2\Big)$$

$$\leqq \Big| \int_0^t \Big((\Box u)(s, \cdot), A_0\frac{\partial u}{\partial t}(s, \cdot) + A_1\frac{\partial u}{\partial x}(s, \cdot) + A_2\frac{\partial u}{\partial y}(s, \cdot)\Big)\, ds\Big|,$$

where A_j $(j=0, 1, 2)$ are chosen such that the hypotheses of this lemma are satisfied and the constants K_1 and K_2 are independent of u. From (5) and the inequality

$$\Big(\| u(s, \cdot)\|^2\Big)\Big|_0^t \leqq \int_0^t \| u(s, \cdot)\|^2\, ds + \int_0^t \Big\|\frac{\partial u}{\partial t}(s, \cdot)\Big\|^2\, ds,$$

we get that for any t

$$|||u(t, \cdot)|||_1^2 \leqq K_3\Big(\int_0^t |||u(s, \cdot)|||_1^2\, dt + \int_0^t \|(\Box u)(s, \cdot)\|^2 ds + |||u(0, \cdot)|||_1^2\Big).$$

From this it follows

$$|||u(t, \cdot)|||_1^2 \leqq K_3 e^{K_3 t}\Big(\int_0^t \|(\Box u)(s, \cdot)\|^2 ds + |||u(0, \cdot)|||_1^2\Big). \qquad \text{q. e. d.}$$

Now we consider the case when the boundary condition on B_2 is Dirichlet one, that is, $\mathcal{B}_2 u = u = 0$ on B_2.

LEMMA 3(a). *The necessary and sufficient condition that the hypotheses in Lemma 2 are fulfilled is that all of the following relations hold:*

$$A_0 > 0,$$

$$A_1^2 + A_2^2 < A_0^2,$$

$$2c A_0 + (c^2+1) A_1 \geqq 0,$$

$$(b^2 - c^2 - 1) A_1^2 - b^2 A_0^2 + 2b A_0 A_2 - c^2 A_2^2 - 2c A_0 A_1 + 2b A_1 A_2 \geqq 0$$

and $$A_2 \geqq 0.$$

PROOF. Let $\frac{\partial u}{\partial x}(t, 0, y) = b\frac{\partial u}{\partial y}(t, 0, y) + c\frac{\partial u}{\partial t}(t, 0, y)$ and $u(t, x, 0) = 0$ in Lemma 1.

Then

$$G_2(t) = Re\Big\{\Big\langle (2c A_0 + (c^2+1) A_1) \frac{\partial u}{\partial t}(t, 0, y), \frac{\partial u}{\partial t}(t, 0, y)\Big\rangle_{B_1}$$

$$+ 2\Big\langle \frac{\partial u}{\partial t}(t, 0, y), (b A_0 + bc A_1 + c A_2) \frac{\partial u}{\partial y}(t, 0, y)\Big\rangle_{B_1}$$

$$+ \Big\langle ((b^2-1) A_1 + 2b A_2) \frac{\partial u}{\partial y}(t, 0, y), \frac{\partial u}{\partial y}(t, 0, y)\Big\rangle_{B_1}\Big\},$$

$$G_3(t) = \Big\langle A_2 \frac{\partial u}{\partial y}(t, x, 0), \frac{\partial u}{\partial y}(t, x, 0)\Big\rangle_{B_2}.$$

From these expressions and the expression of $G_1(t)$, it is easily shown this lemma.

Next we consider the case when the boundary condition on B_2 is Neumann one, that is, $\mathcal{B}_2 u = \frac{\partial u}{\partial y} = 0$ on B_2.

LEMMA 3(b). *The necessary and sufficient condition that the hypotheses in Lemma 2 are fulfilled is that all of the following relations hold:*

$$|A_1| < A_0,$$

$$2c A_0 + (c^2+1) A_1 \geqq 0,$$

$$(b^2 - c^2 - 1) A_1^2 - b^2 A_0^2 - 2c A_0 A_1 \geqq 0$$

and $$A_2 = 0.$$

PROOF. Let $\frac{\partial u}{\partial x}(t, 0, y) = b\frac{\partial u}{\partial y}(t, 0, y) + c\frac{\partial u}{\partial t}(t, 0, y)$ and $\frac{\partial u}{\partial y}(t, x, 0) = 0$ in Lemma 1.

Then $G_2(t)$ is the same that in Lemma 3(a). $G_3(t) = \Big\langle A_2 \frac{\partial u}{\partial t}(t, x, 0), \frac{\partial u}{\partial t}(t, x, 0)\Big\rangle_{B_2}$

$-\left\langle A_2\frac{\partial u}{\partial x}(t, x, 0), \frac{\partial u}{\partial x}(t, x, 0)\right\rangle_{B_2}$. From these expressions and the expression of $G_1(t)$, we can prove this lemma.

Let us prove Theorem by using Lemmas 1–3. At first we consider the case (i). We set

$$Q_{11}=\{(b, c)\in Q_1, b>0\},$$

$$Q_{12}=\{(b, c)\in Q_1, b\leqq 0\}.$$

Then we see that $Q_1=Q_{11}\cup Q_{12}$.

Let $(b, c)\in Q_{11}$. We set $A_0=(c^2+1)$, $A_1=-c$, $A_2=bc$. Then all of the relations in Lemma 3(a) are satisfied. From Lemma 2 we have proved Theorem in this case.

Let $(b, c)\in Q_{12}$. Then we may set $A_0=1+c^2-b^2$, $A_1=-c$, $A_2=0$.

Next we consider the case (ii). Let $(b, c)\in Q_2$. We set $A_0=1+c^2-b^2$, $A_1=-c$, $A_2=0$. Then we can prove Theorem in this case by the same method as above.

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