ON THE ℓ-CLASS FIELD TOWERS OF CYCLIC FIELDS OF DEGREE ℓ

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1. Let ℓ be an odd prime, let $K/\mathbf{Q}$ be a cyclic extension of degree ℓ, and let $p_1, \ldots, p_t$ be the primes ramified in K. Assume ℓ is not ramified in K, Then $p_i \equiv 1 \bmod \ell$ for $i=1, \ldots, t$. Let M_K denote the ℓ-Sylow subgroup of the ideal class group of K and put $r = rank\,(M_K)$.

As is well-known, if $r \geqq 2+2\sqrt{\ell}$, then the ℓ-class field tower of K is infinite. Moreover, we know from a result of Y. Furuta [2] that the ℓ-class field tower of K is infinite on the condition that $t \geqq 8$. On the other hand, if $t=1$ or $r=1$, then the ℓ-class field tower of K is finite.

In the previous paper [4], the author studied in the case where $t=2$ and proved the following.

THEOREM A, *Let $\ell \geqq 13$. Then there exist infinitely many couples of primes (p_1, p_2) with the following conditions:*

(i) $p_i \equiv 1 \bmod \ell$ *for* $i=1, 2$.
p_1 *is an* ℓ*-th power residue mod* p_2.
p_2 *is an* ℓ*-th power residue mod* p_1.

(ii) *If* $K/\mathbf{Q}$ *is a cyclic extension of degree* ℓ *with only* p_1, p_2 *ramified, then the* ℓ*-class field tower of* K *is infinite.*

THEOREM B. *Let p_1 be an odd prime with $p_1 \equiv 1 \bmod \ell$. Let $k_1/\mathbf{Q}$ be the unique cyclic extension of degree ℓ with only p_1 ramified. Assume $4(2+\ell) \leqq h(k_1)$, where $h(k_1)$ is the class number of k_1. Then there exist infinitely many primes p_2 with the following conditions:*

(i) $p_2 \equiv 1 \bmod \ell$ *and* p_2 *is not an* ℓ*-th power residue mod* p_1.

(ii) *If* $K/\mathbf{Q}$ *is a cyclic extension of degree* ℓ *with only* p_1, p_2 *ramified, then the* ℓ*-class field tower of* K *is finite but the class field tower of* K *is infinite.*

The above results are concerned with the number t of primes ramified in K. Corresponding to them we are able to prove theorems concerned with the ℓ*-rank* r of the ideal class group of K. In fact, in this note we consider the case where $r=2$ and prove the following.

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THEOREM 1. *Let $\ell \geqq 13$. Then there exist infinitely many triples of primes (p_1, p_2, p_3) with the following conditions:*

(i) $p_i \equiv 1 \mod \ell$ *for* $i=1, 2, 3$.

(ii) *If $K/\mathbf{Q}$ is a cyclic extension of degree ℓ with only p_1, p_2, p_3 ramified, then*

$$M_K \approx \mathbf{Z}/\ell\mathbf{Z} \oplus \mathbf{Z}/\ell\mathbf{Z},$$

and the ℓ-class field tower of K is infinite.

THEOREM 2. *Let p_1 be an odd prime with $p_1 \equiv 1 \mod \ell$. Let $k_1/\mathbf{Q}$ be the unique cyclic extension of degree ℓ with only p_1 ramified. Assume $4(2+\ell) \leqq h(k_1)$. Then there exist infinitely many couples of primes (p_2, p_3) with the following conditions:*

(i) $p_i \equiv 1 \mod \ell$ *for* $i=2, 3$.

(ii) *If $K/\mathbf{Q}$ is a cyclic extension of degree ℓ with only p_1, p_2, p_3 ramified, then*

$$M_K \approx \mathbf{Z}/\ell\mathbf{Z} \oplus \mathbf{Z}/\ell\mathbf{Z},$$

and the ℓ-class field tower of K is finite but the class field tower of K is infinite.

2. PROOF of Theorem 1, Let p_1, p_2 be primes satisfying the conditions in Theorem A. Then we can choose a prime p_3 with $p_3 \equiv 1$ *mod* ℓ, such that p_1 and p_2 are not ℓ-th power residues *mod* p_3. (See, for instance, [3], II, Lemma 1.) Let $K/\mathbf{Q}$ be a cyclic extension of degree ℓ with only p_1, p_2, p_3 ramified. For a generator σ of Gal $(K/\mathbf{Q})$, let

$$\left(\left(\frac{p_i : K/\mathbf{Q}}{p_j}\right)\right) = (\sigma^{a_{ij}}),\ a_{ij} \in \mathbf{Z}/\ell\mathbf{Z},$$

where $\left(\frac{p_i : K/\mathbf{Q}}{p_j}\right)$ denotes the norm residue symbol locally at p_j of $K/\mathbf{Q}$. Since $\left(\frac{p_i : K/\mathbf{Q}}{p_3}\right) \neq 1$ for $i=1, 2$, it follows that

$$\mathit{rank}\ (a_{ij}) = \mathit{rank} \begin{pmatrix} *, & 0, & * \\ 0, & *, & * \\ ?, & ?, & ? \end{pmatrix} = 2,$$

where $*$ means a non-zero element. Hence by [3], I, Theorem 2 we have $M_K \approx \mathbf{Z}/\ell\mathbf{Z} \oplus \mathbf{Z}/\ell\mathbf{Z}$. Let $k_i/\mathbf{Q}$ be the unique cyclic extension of degree ℓ with only p_i ramified. By Theorem A the ℓ-class field tower of k_1k_2 is infinite, hence that of $k_1k_2k_3$ is also infinite. Since $k_1k_2k_3/K$ is an unramified abelian ℓ-extension, the ℓ-class field tower of K is infinite. This completes the proof.

PROOF of Theorem 2. Let p_1, p_2 be primes satisfying the conditions in Theorem B. Then we can choose a prime p_3 with the following conditions:

(i) $p_3 \equiv 1$ *mod* ℓ.

(ii) p_1 is not an ℓ-th power residue *mod* p_3.

(iii) p_3 is an ℓ-th power residue *mod* p_1 but not an ℓ-th power residue *mod* p_2.

(See, for instance, [3], II, Lemma 1.) Let $L/\mathbf{Q}$ be the elementary abelian extension of degree ℓ^3 with only p_1, p_2, p_3 ramified, i.e., $L=k_1k_2k_3$. Then by Fröhlich's criterion [1], Theorem 3 (or by [3], II, Theorem 2) we see that $\ell \nmid h(L)$, hence the ℓ-class field tower of

L and that of K are finite. On the other hand, the class field tower of k_1k_2 is infinite by Theorem B. Thus the class field tower of K is also infinite. It is easy to see that $\ell \nmid h(L)$ implies $M_K \approx \mathbf{Z}/\ell\mathbf{Z} \oplus \mathbf{Z}/\ell\mathbf{Z}$. This completes the proof.

Rererences

[1] A. Fröhlich *On the absolute class-group of abelian field,* J. London Math. Soc., 29 (1954), 211–217.
[2] Y. Furuta *On class field towers and the rank of ideal class group,* Nagoya Math. J. 48 (1972), 147–157.
[3] T. Takeuchi *On the structure of p-class groups of certain number fields I, II,* Sci. Rep. Niigata Univ. Ser. A 14 (1977), 25–33; ibid. 15 (1978), 35–42.
[4] T. Takeuchi *Notes on the class field towers of cyclic fields of degree ℓ*, Tōhoku Math. J. 31 (1979), 301–307.

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