On the compact connected transformation group of n-sphere with codimension two principal orbit

By

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Introduction

In this paper, we shall consider compact connected differentiable transformation group on spheres with codimension two principal orbit and two isolated singular orbits.

We shall prove the following

THEOREM. *Let* φ: $G\times S^n \to S^n$ $(n>5)$ *be differentiable action of a compact connected Lie group G on n-dimensional sphere* S^n *with codimension two principal orbit and two isolated singular orbits. Then* φ *has the same orbit structure as one of the following linear actions unless* $n=7, 11, 15, 23, 31, 39$.

(1) *A compact connected Lie group G and a representation* $\varphi=\varphi_1\oplus\theta^1$, *where* φ_1 *is an orthogonal representation with codimension two principal orbit.*

(2) *The group* $SU(k)\times U(1)\times U(1)$ *(or* $SU(k)\times U(1)$*) and the representation* $\varphi=[\mu_k\otimes c(\mu_2^*|U(1)\times U(1)]$ *(or the restriction, respectively).*

(3) *The group* $Sp(k)\times Sp(1)\times Sp(1)$ *(or* $Sp(k)\times Sp(1)$, $Sp(k)\times T^1\times Sp(1)$ *and the representation,* $\varphi=\nu_k\otimes_H(\nu_2^*|Sp(1))\times Sp(1))$ *(or the restriction, respectively). Here "the same orbit structure" means the same orbit types and the equivalent slice representation of the corresponding orbit, and* μ_k *and* ν_k *denote the standard representation of* $SU(k)$ *and* $Sp(k)$ *respectively.*

In this paper we shall consider only differentiable actions and use the following notations:

Q, R, C, H: the field of rational, real, complex or quoternion numbers, respectively
A_n, B_n, C_n, D_n: the classical simple Lie group of rank n
G_2, F_4, E_6, E_7, E_8 the exceptional simple Lie group
$G\sim G'$: two Lie groups G and G' are locally isomorphic
G°: the identity component of G
$K\circ L$: the essentially direct product of K and L
Ad: the adjoint representation

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ρ_k, μ_k, ν_k: the standard representation of $SO(k)$, $SU(k)$, $Sp(k)$ respectively.

1. Examples

In this section we shall give examples of orthogonal transformation groups of $(4k-1)$-sphere and $(8k-1)$-sphere with codimension two principal orbits. This section is due to [5].

Let $M(m, n:F)$ denote the set of all $m\times n$ matrices over F(F denotes C or H). Put $\langle X, Y\rangle=\mathrm{Trace}\, X^*Y$ for $X, Y\in M(m, n:F)$, where X^* denotes the conjugate transpose of X. For $X\in M(m, n:F)$, let *rank* X or *rkX* be the maximun number of linearly independent column vectors of X.

Examples A. Let $G_1=SU(k)\times T^1\times T^1$ and $G_2=SU(k)\times T^1$. Define actions $\varphi_k^{(1)}$ and $\varphi_k^{(2)}$ of G_1 and G_2 on $S^{4k-1}\subset M(k, 2:C)$ by the formulas:

$$\varphi_k^{(1)}((A, z_1, z_2), (X_1, X_2))=(AX_1\bar{z}_1, AX_2\bar{z}_2)$$

$\varphi_k^{(2)}((A, z), (X_1, X_2))=(AX_1\bar{z}, AX_2)$, where $A\in SU(k)$, $z, z_i\in T^1$, $(X_1, X_2)\in M(k, 2:C)$. Straightforward computations show the following;

(1) For $X=(X_1, X_2)$ with $rkX=2$ and $\langle X_1, X_2\rangle\neq 0$

$$(G_1)x=\left\{\left(\begin{pmatrix} z & & 0\\ & z & \\ 0 & & *\end{pmatrix}, z, z\right): z\in T^1\right\}$$

$$(G_2)x=\left\{\left(\begin{pmatrix} 1 & & 0\\ & 1 & \\ 0 & & *\end{pmatrix}, 1\right)\right\}$$

$$G_i(X)=SU(k)/SU(k-2)\times S^1 \qquad i=1, 2.$$

(2) For $X=(X_1, X_2)$ with $rk\,X=2$ and $\langle X_1, X_2\rangle\neq 0$

$$(G_1)x=\left\{\left(\begin{pmatrix} z_1 & & 0\\ & z_2 & \\ 0 & & *\end{pmatrix}, z_1, z_2\right): z_i\in T^1\right\}$$

$$(G_2)x=\left\{\left(\begin{pmatrix} z_1 & & 0\\ & 1 & \\ 0 & & *\end{pmatrix}, z_1\right): z_1\in T^1\right\}$$

$$G_i(X)=SU(k)/SU(k-2) \qquad i=1, 2.$$

(3) For $X=(X_1, X_2)$ with rk $X=1$ and $\langle X_1, X_2\rangle\neq 0$

$$(G_1)x=\left\{\left(\begin{pmatrix} z & 0\\ 0 & *\end{pmatrix}, z, z\right): z\in T^1\right\}$$

$$(G_2)x=\left\{\left(\begin{pmatrix} z & 0\\ 0 & *\end{pmatrix}, 1\right): z\in T^1\right\}$$

$$G_i(X)=S^{2k-1}\times S^1 \qquad i=1, 2.$$

(4) For $X=(X_1, X_2)$ with rk $X=1$ and $<X_1, X_2>=0$

$$(G_1)x=\left\{\left(\begin{bmatrix} z_1 & \\ & * \end{bmatrix}, z_1, z_2\right): z_i\in T^1\right\} \qquad \text{for } X_1\neq 0$$

$$(G_1)x=\left\{\left(\begin{bmatrix} z_2 & \\ & * \end{bmatrix}, z_1, z_2\right): z_i\in T^1\right\} \qquad \text{for } X_2\neq 0$$

$$(G_2)x=\left\{\left(\begin{bmatrix} 1 & \\ & * \end{bmatrix}, z\right): z\in T^1\right\} \qquad \text{for } X_1\neq 0$$

$$(G_2)x=\left\{\left(\begin{bmatrix} z & \\ & * \end{bmatrix}, z\right): z\in T^1\right\} \qquad \text{for } X_2\neq 0$$

$$G_i(X)=S^{2k-1} \qquad i=1, 2$$

Examples B. Let $G_1=Sp(k)\times Sp(1)\times Sp(1)$, $G_2=Sp(k)\times T^1\times Sp(1)$ and $G_3=Sp(k)\times Sp(1)$. Define actions $\varphi_k^{(1)}$, $\varphi_k^{(2)}$ and $\varphi_k^{(3)}$ of G_1, G_2 and G_3 on S^{8k-1} by the formulas:

$$\varphi_k^{(1)}(A, q_1, q_2), (X_1, X_2)=(AX_1\bar{q_1}, AX_2\bar{q_2})$$

$$\varphi_k^{(2)}(A, z, q), (X_1, X_2))=(AX_1\bar{z}, AX_2\bar{q})$$

$\varphi_k^{(3)}(A, q), (X_1, X_2))=(AX_1\bar{q}, AX_2)$, where $A\in Sp(k)$, $z\in T^1$, $q_i\in Sp(1)$ $(X_1, X_2)\in M(k, 2:H)$. Straightforward computations show the following;

(1) For $X=(X_1, X_2)$ with rk $X=2$ and $<X_1, X_2>\neq 0$

$$(G_1)x=\left\{\left(\begin{pmatrix} q & & \\ & q & \\ & & * \end{pmatrix}, q, q\right): q\in Sp(1)\right\}$$

$$(G_2)x=\left\{\left(\begin{pmatrix} z & & \\ & z & \\ & & * \end{pmatrix}, z, z\right): z\in T^1\right\}$$

$$(G_3)x=\left\{\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & * \end{pmatrix}, 1\right)\right\}$$

$$G_i(X)=Sp(k)/Sp(k-2)\times S^3 \qquad 0=1, 2$$

(2) For $X=(X_1, X_2)$ with rk $X=2$ and $<X_1, X_2>=0$

$$(G_1)x=\left\{\left(\begin{pmatrix} q_1 & & \\ & q_2 & \\ & & * \end{pmatrix}, q_1, q_2\right): q_i\in Sp(1)\right\}$$

$$(G_2)x=\left\{\left(\begin{pmatrix} z & & \\ & q & \\ & & * \end{pmatrix}, z\ q\right): z\in T^1,\ q\in Sp(1)\right\}$$

$$(G_3)x=\left\{\left(\begin{pmatrix} q & & \\ & 1 & \\ & & * \end{pmatrix}, q\right): q\in Sp(1)\right\}$$

$$G_i(X)=Sp(k)/Sp(k-2) \qquad i=1, 2$$

(3) For $X=(X_1, X_2)$ with rk $X=1$ and $\langle X_1, X_2\rangle \neq 0$

$$(G_1)x=\left\{\left(\begin{bmatrix} q & \\ & * \end{bmatrix}, q, q\right): q\in Sp(1)\right\}$$

$$(G_2)x=\left\{\left(\begin{bmatrix} z & \\ & * \end{bmatrix}, z, z\right): z\in T^1\right\}$$

$$(G_3)x=\left\{\left(\begin{bmatrix} 1 & \\ & * \end{bmatrix}, 1\right)\right\}$$

$$G_i(X)=S^{4k-1}\times S^3 \qquad i=1, 2$$

(4) For $X=(X_1, X_2)$ with rk $X=1$ and $\langle X_1, X_2\rangle=0$

$$(G_1)x=\left\{\left(\begin{bmatrix} q_1 & \\ & * \end{bmatrix}, q_1, q_2\right): q_i\in Sp(1)\right\} \qquad \text{for } X_1\neq 0$$

$$(G_1)x=\left\{\left(\begin{bmatrix} q_2 & \\ & * \end{bmatrix}, q_1, q_2\right): q_i\in Sp(1)\right\} \qquad \text{for } X_2\neq 0$$

$$(G_2)x=\left\{\left(\begin{bmatrix} z & \\ & * \end{bmatrix}, z, q\right): z\in T^1,\ q\in Sp(1)\right\} \qquad \text{for } X_1\neq 0$$

$$(G_2)x=\left\{\left(\begin{bmatrix} q & \\ & * \end{bmatrix}, z, q\right): z\in T^1,\ q\in Sp(1)\right\} \qquad \text{for } X_2\neq 0$$

$$(G_3)x=\left\{\left(\begin{bmatrix} q & \\ & * \end{bmatrix}, q\right): q\in Sp(1)\right\} \qquad \text{for } X_1\neq 0$$

$$(G_3)x=\left\{\left(\begin{bmatrix} 1 & \\ & * \end{bmatrix}, q\right): q\in Sp(1)\right\} \qquad \text{for } X_2\neq 0$$

$$G_i(X)=S^{4k-1}. \qquad i=1, 2$$

Note that in the above presentation of isotropy subgroup the notation "=" means "up to conjugacy."

2. Statement of results

In this section we shall prove the main theorem modulo some propositions which are proved in the subsequent sections. In the following let G be a compact connected Lie group and G act on n-sphere S^n almost effectively with codimension two principal orbit G/H, two isolated singular orbits G/K_1, G/K_2 and hence two non-isolated singular orbits G/L_1, G/L_2. Put $k_i=$codim G/K_i and $l_i=$codim G/L_i. We may assume that $H\subset L_1, L_2$, $L_1, L_2\subset K_1$ and $L_1, L_2\subset K_2$.

We have the following

PROPOSITION 1. *G/K_i is simply connected for $i=1, 2$ and hence K_i is connected for $i=1$, 2.*

PROPOSITION 2. *If $G/K_i\neq pt.$, then we have*

$$P(G/K_i)=\begin{cases}(1+t^{k_{3-i}-1})\sum_{i=0}^{N} t^{i(k_1+k_2-2)} \text{ if } k_1\neq k_2, \text{ where} \\ \qquad\qquad n-1=(N+1)(k_1+k_2-2) \\ \sum_{i=0}^{N} t^{i(k-1)} \text{ if } k_1=k_2=k, \text{ where } n-1=(N+1)(k-1).\end{cases}$$

Here $P(X)$ denotes the Poincaré polynomial of a space X.

REMARK 1. It is easy to see that G/K_1 is a point if and only if G/K_2 is a point.

Consider the action of K_i on S^{k_i-1} induced by the slice represetation $K_i \longrightarrow SO(k_i)$. This action has codimension one principal orbit K_i/H and two singular orbits K_i/L_1 and K_i/L_2. Put $p=\dim K_i/H-\dim K_i/L_1=l_1-2$ and $q=\dim K_i/H-\dim K_i/L_2=l_2-2$.

It follows from results in [4] that there are following cases;

Case 1. p, q; odd$>$1.

$$K_i/L_j=S^{k_i-l_j} \text{ and } K_i/H=K_i/L_1\times K_i/L_2$$

Case 2. p; odd$>$1, q; even.

1) $K_i/L_j=S^{k_i-l_j}$ and $K_i/H=K_i/L_1\times K_i/L_2$
2) $k_i-1=2(p+q)+1$
   $P(K_i/L_1)=(1+t^q)(1+t^{p+q})$, $P(K_i/L_2)=(1+t^p)(1+t^{p+q})$
   $P(K_i/H)=(1+t^p)(1+t^q)(1+t^{p+q})$.

Case 3. $p=1$, q; even.

1) $K_i/L_j=S^{k_i-l_j}$ and $K_i/H=S^1\times S^q$
2) $k_i-1=2q+3$
   $P(K_i/L_1)=(1+t^q)(1+t^{q+1})$, $P(K_i/L_2)=(1+t)(1+t^{q+1})$
   $P(K_i/H)=(1+t)(1+t^q)(1+t^{q+1})$

Case 4. $p=1$, q; odd.

1) $K_i/L_j=S^{k_i-l_j}$ and $K_i/H=S^1\times S^q$
2) $k_i-1=2q+3$
   $P(K_i/L_1)=1+t^{2q+1}$, $P(K_i/L_2)=1+t$
   $P(K_i/H)=(1+t)(1+t^{2q+1})$
3) $k_i-1=5$, $p=q=1$
   $P(K_i/L_1)=1+t$, $P(K_i/L_2)=1+t^3$
   $P(K_i/H)=(1+t)(1+t^3)$
4) $k_i-1=4$, $p=q=1$
   $P(K_i/L_j)=1$, $P(K_i/H)=1+t^3$
5) $k_i-1=7$, $p=p=1$
   $P(K_i/L_j)=1+t^3$, $P(K_i/H)=(1+t^3)^2$

Case 5. p, q; even.

1) $K_i/L_j=S^{k_i-l_j}$ and $K_i/H=K_i/L_1\times K_i/L_2$
2) Let K_i'' be the almost effective part of the action of K_i. $K_i''\sim A_2$, B_2, or G_2. $K_i''\cap L_j\sim T^1\times A_1$ and $K_i''\cap H\sim T^1\times T^1$.

3) $K_i/L_j=C_3/C_1\times C_2$, $K_i/,H=C_3/C_1\times C_1\times C_1$.

4) $K_i/L_j=F_4/B_4$, $K_i/H=F_4/D_4$.

REMARK 2. Since K_i acts linearly on S^{k_i-1} with codimention 1 principal orbit, it follows from result in [2] that the action of K_i on S^{k_i-1} is induced by one of the following representations in Table 1 if the slice representation of K_i is maximal i.e. there is no compact connected Lie group K such that K contains K_i as proper subgroup and K admits a representation $K\longrightarrow SO(k_i)$ with codimension two principal orbit.

Assume that the slice repesentation of K_i is not maximal. Let K_i'' be the almost effective part of the action of K_i on S^{k_i-1}. Then there is a compact connected Lie group K which contains K_i'' as proper subgroup and admits a representation $K\longrightarrow SO(k_i)$ which is maximal in the sense above. Let H_i and H be principal isotropy subgroup of the action of K_i'' and K respectively. We can show that $K_i''/H_i=K/H$. In fact, let $H_i=(K'')_x=K_x\cap K_i''$. Since K_i''/H_i is a sumanifold of K/K_x and hence dim $K_i''/H_i=k_i-2=$dim K/K_x, we see that $K_i/H_i=K/K_x$ and K_x is a principal isotropy subgroup. It follows that the maximal compact connected Lie groups for cases on pages from 5 to 6 are given in Table 1.

Table 1

Case	Group	Representation	dim	Principal orb.	Poincaré polyn. of sing. orb.
Case i–1)	$G_1\times G_2$	G_i is transitive on S^{r_i-1}	r_1+r_2	$S^{r_1-1}\times S^{r_2-1}$	$1+t^{r_1-1}$ $1+t^{r_2-1}$
Case 2–2)	$SU(2)\times SU(r)$	$[\mu_2\otimes_C\mu_r]_R$	$4r$	$W_{r,2}\times S^2$	$(1+t^{2r-3})(1+t^{2r-1})$ $(1+t^2)(1+t^{2r-1})$
	$Sp(2)\times Sp(r)$	$\nu_2\otimes_H\nu_r^*$	$8r$	$X_{r,2}\times S^4$	$(1+t^{4r-1})(1+t^{4r-5})$ $(1+t^4)(1+t^{4r-1})$
	$U(1)\times Spin\ (10)$	$[\mu\otimes_C\Delta^{10}{}_+]_R$	32	$G/T^1\times SU(4)$	$(1+t^6)(1+t^{15})$ $(1+t^9)(1+t^{15})$
	$U(5)$	$[\Lambda^2\mu_5]_R$	20	$G/SU(2)^2\times T^1$	$(1+t^4)(1+t^9)$ $(1+t^5)(1+t^9)$
Case 3–2)	$SO(2)\times SO(r)$ r; even	$\rho_2\otimes_R\rho_r$	$2r$	$V_{r,2}\times S^1$	$(1+t^{r-2})(1+t^{r-1})$ $(1+t)(1+t^{r-1})$
Case 4–2) -3)	$SO(2)\times SO(r)$ r; odd	$\rho_2\otimes_R\rho_r$	$2r$	$V_{r,2}\times S^1$	$1+t^{2r-3}$ $1+t$
Case 4–4)	$SO(3)$	$S^2\rho_3-\theta$	5	$SO(3)/Z_2+Z_2$	1

Case 4–5)	$SU(2)\times Sp(1)$	$\mu_2\otimes_H \nu_1{}^*$	8	G/finite gr.	$1+t^3$
Case 5	$SU(3)$	Ad	8	$SU(3)/T^2$	$SU(3)/A_1\times T^1$
Case 5	$Sp(3)$	$\Lambda^2\nu_3-\theta$	14	$Sp(3)/Sp(1)^3$	$Sp(3)/Sp(2)$
Case 5	$Sp(2)$	Ad	10	$Sp(2)/T^2$	$Sp(2)/Sp(1)\times T^1$
Case 5	G_2	Ad	14	G_2/T^2	$G_2/Sp(1)\times T^1$
Case 5	F_4	φ_4	26	$F_4/Spin(8)$	$F_4/Spin(9)$

Note that groups $G_1\times G_2$ in case i–1) are not necessarily maximal

We have the following

PROPOSITION 3. $k_1=k_2$.

PROPOSITION 4. $P(K_1/L_i)=P(K_2/L_i)$ for $i=1, 2$.

PROPOSITION 5. *If n is even, then* $K_1=K_2=G$.

We shall consider the cases in which two isolated singular orbits are not both fixed points and hence n is odd.

Let $G=\bar{G}\times T^a$ be the decomposition of G into product of semi-simple part $\bar{G}$ and torus T^a. We may assume $\bar{G}$ is simply connected. Put $K_i=\bar{K}_i\circ T^b(\bar{K}_i=$the semi-simple part of $K_i)$.

Consider the following commutative diagram;

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \pi_1(\bar{G}\cap K_i)\otimes Q & \longrightarrow & \pi_1(\bar{G})\otimes Q=0 & & \\
 & & \downarrow & & \downarrow & & \\
\pi_2(G/K_i)\otimes Q & \longrightarrow & \pi_1(K_i)\otimes Q & \longrightarrow & \pi_1(G)\otimes Q & \longrightarrow & 0 \\
 & & \downarrow p_\sharp & & \downarrow p_\sharp & & \\
 & & \pi_1(p(K_i))\otimes Q & \longrightarrow & \pi_1(T^a)\otimes Q & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

where p is the projection $G\longrightarrow T^a$. It follows that $a=b$ and $(\bar{G}\cap K_i)^\circ=\bar{K}_i$. Since $\bar{G}/\bar{K}_i\longrightarrow G/K_i$ is a finite covering and $\pi_1(G/K_i)=1$, we have $\bar{G}/\bar{K}_i=G/K_i$ and $\bar{G}\cap K_i=\bar{K}_i$. Thus we have proved the following

PROPOSITION 6. $G/K_i=\bar{G}/\bar{K}_i$.

REMARK 3. If $\pi_2(G/L_i)\otimes Q=\pi_1(G/L_i)\otimes Q=0$, then we see that $G/L_i=\bar{G}/\bar{L}_i$, where $\bar{L}_i$ is the semi-simple part of L_i.

We have the following

PROPOSITION 7. *Let $\overline{G}$ be the product of simple Lie groups G_i; $\overline{G}=G_1\times\dots\times G_t$ and K be a semi-simple subgroup of $\overline{G}$ with rk $K=$ rk $\overline{G}-1$. Then we have the following two possibilities;*

$K=K_1\times\dots\times K_t$, *$K_i$ is a subgroup of G_i*

or

$K=(K_1\times\dots\times K_t)\circ K_o$, *$K_i$ is a subgroup of G_i, rk $K_o=1$ and the number of i such that $p_i(K_o)\neq 1$ is just two and rk K_i+rk $p_i(K_o)=$ rk G_i for any i. Here p_i denotes the homomorphism $p_i: K_o \subsetneq K \subsetneq G \xrightarrow[proj]{} G_i$.*

PROPOSITION 8. *If k is odd, then rk $G=$ rkK_i for $i=1, 2$ and if k is even, then rk $G=$ rk K_i+1 for $i=1, 2$.*

PROPOSITION 9. *Let t and s_i be the number of simple factors of $\overline{G}$ and $\overline{K}_i$ respectively. Then we have*

if $k\geqq 6$, $s_i=t$

if $k=5$, $s_i=t+1$ and

if $k=4$, $s_i=t-1$.

PROPOSITION 10. *Let $\overline{G}=G_1\times\dots\times G_t$ be the decomposition of $\overline{G}$ into the product of simple Lie groups. Then we have*

(1) *If $k\geqq 6$, then $\overline{G}/\overline{K}_i=G_1/K_{i1}\times\dots\times G_t/K_{it}$ or $\overline{G}/\overline{K}_i=(A_1\times\dots\times G_t)/(1\times\dots\times K_{it})\circ K_{io}$ $=G_2/K_{i2}\times\dots\times G_t/K_{it}$, where K_{ij} is simple Lie group.*

(2) *If $k=5$, Then $\overline{G}/\overline{K}_i=G_1/K_{ij}\times\dots\times G_t/K_{it}$, where rk $G_i=$ rk K_{ij}.*

Let U_i be the identity component of the ineffective kernel of the action of K_i on S^{k-1}; $K_i=K_i''\circ U_i$. Assume that there is no direct product summand G_j/K_{ij}, where K_{ij} is a factor of K_i'', in the product decomposition $G_1/K_{i1}\times\dots\times G_t/K_{it}$ (see Proposition 10) for both $i=1, 2$. Then there is a simple factor of G, say G_1, which contains semi-simple factor of U_1 and U_2. It is clear that rk $G_1\geqq 2$.

We have the following

PROPOSITION 11. *Assume that $U_1\neq 1$ and $U_2\neq 1$ and there is a simple factor G_1 of G such that $G_1\cap K_1$ and $G_2\cap K_2$ are semi-simple factors of U_1 and U_2, respectively. Then the action of the restriction has a unique orbit type.*

PROPOSITION 12. *Let G be a simply connected simple Lie group of rank $\geqq 2$ and H a subgroup of G such that rk $H\geqq$ rk $G-1$. If n is greater than 5 and dim $G/H\leqq n-2$, then G cannot act on S^n with a unique orbit type G/H.*

REMARK 4. It follows from the above arguments that we may consider only the case in which there is at least one non-trivial direct product summand G_j/K_{ij} of G/K_i where K_{ij} is a factor of K_i'' for at least one i.

For convience, we shall list the Poincaré polynomials of some homogeneous spaces.

Table 2

K	G	$P(G/K)$	K	G	$P(G/K)$
A_7	E_7	$1+t^6+\dots$	B_r	B_{r+1}	$1+t^{4r+3}$
A_8	E_8	$1+t^6+\dots$	B_r	D_{r+1}	$1+t^{2r+1}$
A_2	G_2	$1+t^6$	C_r	C_{r+1}	$1+t^{4r+3}$
A_r	A_{r+1}	$1+t^{2r+1}$	C_3	F_4	$1+t^7+\dots$
A_r	C_{r+1}	$1+t^6+\dots$	D_r	B_r	$1+t^{2r}$
A_r	B_{r+1}	$1+t^6+\dots$	D_8	E_8	$1+t^{12}+t^{20}+\dots$
A_r	D_{r+1}	$1+t^6+\dots$	D_4	F_4	$(1+t^8)(1+t^8+t^{16})$
A_5	E_6	$1+t^6+\dots$	D_r	B_{r+1}	$1+t^{4r-1}+t^{4r+3}+t^{8r+2}$
A_6	E_7	$1+t^6+\dots$	D_5	E_6	$1+t^8+t^{16}+t^{17}+\dots$
A_7	E_8	$1+t^6+\dots$	D_6	E_7	$1+t^8+t^{16}+\dots$
A_3	F_4	$1+t^6+\dots$	D_7	E_8	$1+t^8+\dots$
A_1	G_2	$1+t^6+\dots$	G_2	B_3	$1+t^7$
B_4	F_4	$1+t^8+t^{16}$	D_r	D_{r+1}	$1+t^{2r}+t^{2r+1}+t^{4r+1}$
E_6	E_7	$1+t^{10}+\dots$	E_7	E_8	$1+t^{12}+\dots$

We shall consider each case on pages from 5 to 6 separately.

At first we shall prove that the cases 2, 3, 4–2), 3), 4), 5) and 5 do not occur. In the following let K_i'' be the almost effective part of the action of K_i on S^{k-1} and U_i the identity component of the ineffective kernel of K_i on S^{k-1}.

Case 2. $l_1 \geq 5$; odd, $l_2 \geq 4$; even.

1) Put $k-l_1=2r_1$ and $k-l_2=2r_2+1$. Note that $k=l_1+l_2-2$. Since k is odd, we see that rk $G=$ rk K_1 and $G/K_1=\bar{G}/\bar{K}_1=G_1/K_{11}\times\dots\times G_t/K_{1t}$, where rk $G_i=$ rk K_{1i}. It follows from remark 4 that we may assume K_{11} and K_{12} are factors of K_1'' and $G_1/K_{11}\neq$pt or $G_2/K_{12}\neq$pt. Moreover we may assume that K_{11} and K_{12} act on S^{2r_1} and S^{2r_2+1} transitively respectively. Possible type of K_{11} is B_{r_1}, $C_2(r_1=2)$, $G_2(r_1=3)$ and possible type of K_{12} is A_{r_2}, D_{r_2+1}, $C_{r_2+1/2}$, $C_{r_2+1/2}\times C_1$, $B_3(r_2=3)$ or $B_4(r_2=7)$.

Assume $G_1/K_{11}=$pt.

$K_{12}\sim A_{r_2}$; Since rk $G_2=$rk K_{12}, G_2 is $G_2(r_2=2)$, $E_7(r_2=7)$ or $E_8(r_2=8)$. In this case we see that $P(G/K_1)=(1+t^6+\dots)P$. Since $k=l_1+l_2-2=2(r_1+r_2)+3$, this is impossible.

$K_{12}\sim D_{r_2+1}$; In this case G_2 is B_{r_2+1}, $E_8(r_2=7)$, $F_4(r_2=3)$. We see that $P(G/K_1)=(1+t^{2r_2+2})P$, $(1+t^{12}+t^{20}+\dots)P$ or $(1+t^8+t^{16})(1+t^8)P$. Since $k=2(r_1+r_2)+3$, these cases are all impossible.

$K_{12}\sim C_r$ or $C_r\times C_1$; This case does not occur, because there is no simple Lie group of rank r other than C_r which contains as proper subgroup.

$K_{12}\sim B_3$; This case does not occur, because B_3 is the only Lie group of rank 3 which contains B_3.

$K_{12}\sim B_4$; Clearly G_2 must be F_4. Hence we have $P(G/K_1)=(1+t^8+t^{16})P$, which is not equal to $\sum_{i=0}^{N} t^{i(k-1)}$, because $k=2(r_1+r_2)+3>9$.

Thus we have shown that $G\neq K_{11}$. Recall that possible type of K_{11} is B_{r_1}, $C_2(r_1=2)$ or $G_2(r_1=3)$.

$K_{11}\sim B_{r_1}$; In this case G_1 is $F_4(r_1=4)$. Hence we have $P(G/K_1)=(1+t^8+t^{16})P$, which is not oqual to $\sum_{i=0}^{N} t^{i(k-1)}$.

$K_{11}\sim C_2, G_2$; This case does not occur, because there is no simple Lie group of rank 2 which contains C_2 or G_2 as proper subgroup.

2) $k-1=2(p+q)+1=2(l_1+l_2-4)+1$.

Since k is greater than 6, we see that $\bar{G}/\bar{K}_1=G_1/K_{11}\times\ldots\times G_t/K_{1t}$ or $G_2/K_{12}\times\ldots\times G_t/K_{1t}$, where K_{1j} is a simple Lie group. Let K_{11} be a simple factor of K_1'' such that $G_1/K_{11}\neq$pt. Let K be the maximal compact connected Lie group which contains K_1'' and admits a representation $K\longrightarrow SO(k)$ with the same principal orbit as K_1''. It follows from Table 1 that possibilities of K are followings.

① $K\sim SU(2)\times SU(r)$;

In this case we see that $k=4r$. Note that K_{11} is locally isomorphic to a subgroup of $SU(2)$ or $SU(r)$.

$K_{11}\sim A_1$: Possibility of G_1 is A_2, B_2 or G_2, none of which satifies $P(G/K_1)=\sum_{i=0}^{N} t^{i(k-1)}$, since $k>8$.

$K_{11}\sim A_s(s\leqq r)$: Possibility of G_1 is A_{s+1}, B_{s+1}, C_{s+1}, D_{s+1}, E_{s+1} $(s=5, 6, 7)$, $E_s(s=7, 8)$ or $G_2(s=2)$. For any G_1, it follows from Table 2 that $P(G/K_1)\neq\sum_{i=0}^{N} t^{i(k-1)}$.

$K_{11}\sim B_s(r\geqq 2s+1)$: G_1 is one of B_{s+1}, $F_4(s=3)$ or $F_4(s=4)$. It is easy to see that $P(G/K_1)\neq\sum_{i=0}^{N} t^{i(k-1)}$.

By the same arguments it can be shown that cases of $K_{11}\sim C_s$, D_s and E_s do not occur.

② $K\sim Sp(2)\times Sp(r)$;

In this case we see that $k=8r$. By the same arguments as the case of ①, we can show that this case does not occur.

③ $K\sim Spin(10)$;

In this case we see that $k=32$. Since K_{11} is locally isomorphic to a subgroup of $Spin(10)$, K_{11} is type of $D_r(4\leqq r\leqq 5)$, $A_r(1\leqq r\leqq 4)$, $B_r(2\leqq r\leqq 4)$ or G_2. It is not difficult to see that there is no simple Lie group G_1 such that $P(G_1/K_1)\neq\sum_{i=0}^{N} t^{i(k-1)}$.

④ $K\sim SU(5)$;

In this case we see that $k=20$ and K_{11} is locally isomorphic to a subgroup of $SU(5)$. It is easy to see that this case does not occur.

Case 3. $l_1=3$, $l_2\geqq 4$; even.

1) $k=l_1+l_2-2$; *odd.*

Let G_1/K_{11} be a positive dimensional direct product summand of $\bar{G}/\bar{K}_1$ where K_{11} is a simple factor of K_1''. Since k is odd, we see that rk $G_1=$rk K_{11}. Note that if k is 5, K_{11} is not necessary simple. Since K_1'' acts on $S^1\times S^{2r}$ $(k-l_1=2r)$ transitively, we see that K_{11} is locally isomorphic to a subgroup of $SO(2r+1)$ and acts on S^{2r} transitively. Hence K_{11} is type of $A_1(r=1)$ B_2, $\boldsymbol{G}_2(r=3)$ or $C_3(r=2)$. It is not difficult to see that there is no simple Lie group G_1 such that rk $G_1=$rk K_{11} and $P(\bar{G}/\bar{K}_1)=\sum_{i=0}^{N} t^{i(k-1)}$.

2) $k-1=2q+3=2l_2-1$.

Let K be the maximal compact connected Lie group which contains K_1'' and admits a representation $K\longrightarrow SO(k)$ with the same principal orbit as K_1''. It follows from Table 1 that K is locally isomorphic to $SO(r)$ (r; even). Thus K_1'' is locally isomorphic to a subgroup of $SO(r)$. It is easy to see that r is l_2. Let K_{11} be a simple factor of K_1'' such that G_1/K_{11} is a non-trivial direct product summand of G/K_1.

$K_{11}\sim A_s(r\geqq 2s)$; In this case possible type of G_1 is A_{s+1}, B_{s+1}, C_{s+1}, D_{s+1}, E_{s+1} $(s=5, 6, 7)$, $F_4(s=3)$, $\boldsymbol{G}_2(s=1, 2)$, $E_s(s=7, 8)$. It is easy to see that $P(\bar{G}/\bar{K}_1)\neq\sum_{i=0}^{N} t^{i(k-1)}$.

$K_{11}\sim B_s(r\geqq 2s+1)$; In this case possible type of G_1 is B_{s+1}, D_{s+1} or $F_4(s=3, 4)$. If $G_1=B_{s+1}$, then $P(G/K_1)=1+t^{4s+1}+\ldots$. Since $k-1=2r-1>4s+1$, $P(G/K_1)\neq\sum_{i=0}^{N} t^{i(k-1)}$. It is easy to see that cases of $K_{11}\sim D_{s+1}$ and F_4 do not occur.

$K_{11}\sim C_s(r\geqq 4s)$; In this case possible type of G_1 is C_{s+1} or $F_4(s=3)$. It is easy to see that C_{s+1} and F_4 are inadequate.

We shall omit the proof of the fact that cases of $K_{11}\sim D_s$, $\boldsymbol{G}_2$, F_4, E_s $(s=6, 7, 8)$ do not occur, since they are tedius, but are not difficult.

Case 4. $l_1=3$, $l_2\geqq 3$; odd.

2) In this case we see that $k=2l_2$. Let K_{11} be a simple factor of K_1'' such that $G_1/K_{11}\neq$pt, where G_1 is a simple factor of G. It follows from Table 1 that K_{11} is locally isomorphic to a subgroup of $SO(r)$ $(r=l_2)$. Hence K_{11} is type of $A_s(r\geqq 2s+1)$, $B_s(r\geqq 2s+1)$, $C_s(r\geqq 4s)$, $D_s(r\geqq 2s)$, $\boldsymbol{G}_2(r\geqq 7)$, $F_4(r\geqq 26)$, $E_6(r\geqq 27)$, $E_7(r\geqq 56)$ or $E_8(r\geqq 248)$.

$K_{11}\sim A_s$; Note that $k=2r$. If $G_1=A_{s+1}$, then we see that $P(G/K_1)=1+t^{2s+3}+\ldots$, which is not equal to $\sum_{i=0}^{N} t^{i(k-1)}$, because $r\geqq 2s+1$. If G_1 is a simple Lie group which contains A_s, of rank s or $s+1$ and is not A_{s+1}, then we see that $P(G/K_1)=1+t^6+\ldots$, which is impossible.

By the same arguments it can be shown that other case does not occur.

3) $k=6$, $l_1=l_2=3$.

Let K_{11} be as above. Then it follows from Table 1 that $K_{11}\sim A_1$ and hence $G_1=A_2$

or G_2. It is easy to see that $G_1=A_2$. Thus we see that $\bar{G}/\bar{K}_1=A_3/A_1$ or $A_2/A_1\times G'/U_1$. Similarly we see that $\bar{G}/\bar{K}_2=A_2/A_1$ or $A_2/A_1\times G''/U_2$. If one of G'/U_1 and G''/U_2 is a point, then $n=11$. If $G'/U_1\neq$pt and $G''/U_2\neq$pt, then it follows from Propositions 11 and 12 that $G'=G''=A_2$, $U_1=U_2=A_1$, which implies that $P(G/K_1)=1+2t^5+\dots$. This is impossible.

4) $k=5$, $l_1=l_2=3$.

Let K_{11} be as above. Since k is odd, we see that rk $G_1=$rk K_{11}. It is easy to see that this case does not occur.

5) $k=8$, $l_1=l_2=3$·

Let K_{11} be as above. It follows from Table 1 that K_{11} is type of A_1.. Then G_1 must be B_2. By the same arguments as in 3), it can be shown that $n=15$.

Case 5. $l_1, l_2\geqq 4$; *even.*

1) Put $2r_i=k-l_i$. Then we have $k=2(r_1+r_2)+2$ and $k\geqq 6$. Since K_1'' acts transitively on $S^{k-l_1}\times S^{k-l_2}$, $K_1''=K_{11}\times K_{12}$ and K_{1j} must be one of A_1, B_r, C_2 or G_2. We assume that G_1/K_{11} is not point.

$K_{11}\sim A_1$; G_1 must be of rank 2 and one of A_2, B_2 or G_2. It is clear that if $G_1=G_2$, then $P(G/K_1)\neq\sum_{i=0}^{N}t^{i(k-1)}$.

Assume that $G_1=A_2$. Then k must be 6 and hence $r_1=r_2=1$. Thus we see that $K_{11}=K_{12}=A_1$. If $K_1''=K_1$, then $G/K_1=\bar{G}/\bar{K}_1=G_1/A_1$ and hence $n=11$. Thus we have $U_1\neq 1$ and $U_2\neq 1$ and $G/K_1=A_2/A_1\times G'/U_1$, $G/K_2=A_2/A_1\times G''/U_2$, where $G'/U_1\neq$pt, $G''/U_2\neq$pt. It follows from Propositions 11 and 12 that $G'=A_2$, $G''=A_2$ and $U_1=U_2=A_1$, which is impossible. Next assume that $G_1=B_2$. Then k must be 7 and hence $r_1=1$ and $r_2=2$. Since $r_2=2$, we see that $K_{12}\sim B_2$. If $G_2/K_{12}\neq$pt, G_2 must be B_3, C_3 or D_3 and hence $P(G/K_1)=1+t^7+t^{11}+\dots$ or $1+2t^7+\dots$, which is impossible. Thus we have shown that $G/K_1=B_2/B_1$, which implies that $n=15$, or $U_1\neq 1$. Thus we see that $U_1\neq 1$ and $U_2\neq 1$. By the same arguments as in case of $G_1=A_2$ we can show that this does not occur.

$K_{11}\sim B_r$; If G_1 is of rank r, then $r=4$ and $G_1=F_4$. We have $P(G/K_1)=1+t^8+t^{16}+\dots$. Since $k=2(r_1+r_2+1)\geqq 10$, $P(G/K_1)$ is not equal to $\sum_{i=0}^{N}t^{i(k-1)}$. Next we assume that rk $G_1=r+1$. Then $G_1=B_{r+1}$ or D_{r+1}. It is not difficult to see that $P(G/K_1)\neq\sum_{i=0}^{N}t^{i(k-1)}$.

$K_{11}\sim C_2$; Since there is no simple Lie group except C_3 which contains C_2 as proper subgoup and of rank 3, G_1 is C_3. We have then $P(G/K_1)=1+t^{11}+\dots$. Since $k=2(r_1+r_2+1)=6+2r_2$, r_2 must be 3 and $G_2=K_{12}=C_1$ or $C_1\times C_1$, which means that $n=23$.

$K_{11}\sim G_2$; It is easy to see that this case does not occur.

2) Assume $K_1''\sim A_2$. Then $k=8$ and $l_1=l_2=4$. G_1 must be G_2, A_3, C_3 or B_3. It is easy to see that $G_1=A_3$. If $U_1=1$, then $n=15$. If $U_1\neq 1$ and $U_2\neq 1$, then we see that $G/K_1=S^7\times G'/U_1$ and $G/K_2=S^7\times G''/U_2$. By the same arguments as the case 1) we can show that this case does not occur. Assume $K_1''\sim C_2$. Then $k=10$ and G_1 must be C_3, which is not adequate, because $P(G/K_1)=1+t^{11}+\dots\neq 1+t^{k-1}+\dots$ Assume $K_1''\sim G_2$·

Then $k=12$ and $G_1=B_3$, which is easily seen to be inadequate. Since the proof of the fact that the cases 3) and 4) do not occur is completely analoguos to the above cases, we shall omit its proof. Thus we have proved that the cases 2, 3, 4–2), 3) 4) 5) and 5 do not occur.

Next we shall consider the cases 1 and 4–1). We use the the same notations as above; U_i denotes the identity component of the ineffective kernel of the action of K_i on S^{k-1} and K_i'' the almost effective part of the action of K_i on S^{k-1}.

Case 1. $l_1 \geqq 5$; odd, $l_2 \geqq 5$; odd.

At first we shall show that the semi-simple part $\bar{G}$ of G acts on S^n with same orbits as G and isotropy subgroups of the action $\bar{G}$ are all semi-simple, in other words, $G/K_i=\bar{G}/\bar{K}_i$, $G/L_j=\bar{G}/\bar{L}_j$ and $G/H=\bar{G}/\bar{H}$, where $\bar{K}_i$, $\bar{L}_j$ and $\bar{H}$ are semi-simple part of K_i, L_j and H, respectively. In section 3 we shall prove that $P(G/L_j)=P(G/K_i)P(K_i/L_j)$. It follows that $\pi_2(G/L_j)\otimes Q=\pi_1(G/L_j)\otimes Q=0$. As noted in Remark 3, we see that $G/L_j=\bar{G}/\bar{L}_j$. From the homotopy exact sequence of the fibre bundle: $L_j/H\longrightarrow G/H\longrightarrow G/L_j$, it follows that $\pi_2(G/H)\otimes Q=\pi_1(G/H)\otimes Q=0$, which implies that $G/H=\bar{G}/\bar{H}$. From the following commutative diagram, where the horizontal sequence is the canonical fibre bundle;

$$\begin{array}{ccccc} \bar{K}_i/\bar{L}_j & \longrightarrow & \bar{G}/\bar{L}_j & \longrightarrow & \bar{G}/\bar{K}_i \\ \downarrow & & \downarrow & & \downarrow \\ K_i/L_j & \longrightarrow & G/L_j & \longrightarrow & G/K_i, \end{array}$$

it follows that $\bar{K}_i/\bar{L}_j=K_i/L_j$. Similarly we see that $K_i/H=\bar{K}_i/\bar{H}$ and $L_j/H=\bar{L}_j/\bar{H}$. Thus we have proved our assertion.

We consider firstly the case in which $U_1=1$. Put $k-l_i=2r_i-1$. We see that $k=l_1+l_2-2=2(r_1+r_2)$. It follows from Table 1 that $\bar{K}_1=K_{11}\times K_{12}$, where K_{1j} acts on S^{2r_j-1} transitively. It follows from Propositions 9 and 10 that $\bar{G}/\bar{K}_1=G_1/K_{11}\times G_2/K_{12}$ in which we may assume that rk $G_1=$rk $K_{11}+1$ and rk $G_2=$rk K_{12}. Possible type of K_{1j} is A_r, C_r, $C_r\times C_1$, D_r, B_3 or B_4.

$K_{11}\sim A_r$: In this case we see that $r=r_1-1$ and possible type of G_1 is A_{r+1}, B_{r+1}, C_{r+1}, D_{r+1}, $E_{r+1}(r=5, 6, 7)$, $F_4(r=3)$ or $G_2(r=1)$. If $G_1=A_{r+1}$, then we have $P(\bar{G}/\bar{K}_1)=1+t^{2r+3}+\dots$. Since $k-l_1=2r_1-1$, we have $k-1=l_1+2r_1-2>2r+3$, which implies that $P(\bar{G}/\bar{K}_1)\neq \sum_{i=0}^{N} t^{i(k-1)}$. If $G_1\neq A_{r+1}$, then it follows from Table 2 that $P(\bar{G}/\bar{K}_1)=1+t^6+\dots$, which is not equal to $\sum_{i=0}^{N} t^{i(k-1)}$.

$K_{11}\sim D_r$: In this case we see that $r=r_1$ and possible type of G_1 is B_{r+1}, D_{r+1}, $E_{r+1}(r=5, 6, 7)$. If $G_1=B_{r+1}$, then $P(\bar{G}/\bar{K}_1)=1+t^{4r-1}+t^{4r+3}+\dots$, which is not equal to $\sum_{i=0}^{N} t^{i(k-1)}$. It is easy to see that the cases of $G_1=D_{r+1}$, E_{r+1} do not occur.

$K_{11}\sim B_3$: In this case we see that $r_1=4$ and possible type of G_1 is B_4, D_4 or F_4. It is easy to see that the cases of $G_1=F_4$ and D_4 do not occur. Assume $G_1=B_4$. Then we have $P(\bar{G}/\bar{K}_1)=1+t^{15}+\dots$ It is not difficult to see that $P(\bar{G}/\bar{K}_1)=\sum_{i=0}^{N} t^{i(k-1)}$ if and only if $r_2=4$

and $K_{12}=G_2, B_2, B_3$ or A_3. Since $\dim=\overline{G}/\overline{K}_1=\dim G_1/K_{11}+\dim G_2/K_{12}=15$ and $k=16$, we see that $n=31$.

$K_{11}\sim B_4$: In this case we see that $r_1=8$ and possible type of G_1 is B_5 or D_5. If $G_1=D_5$, then we have $P(\overline{G}/\overline{K}_1)=1+t^9+\dots$. Since $k-1>15$, this is impossible. Assume $G_1=B_5$. Then we have $P(\overline{G}/\overline{K}_1)=1+t^{19}+\dots$. Since K_{12} is transitive on S^{2r_2-1}, K_{12} is one of A_{r_2}, D_{r_2}, $C_{r_2/2}$, $C_{r_2/2}\times C_1$, B_3 or B_4. By the same arguments as case of $K_{11}\sim C_r$, it can be shown that $P(\overline{G}/\overline{K}_1)=\sum_{i=0}^{N} t^{i(k-1)}$ if and only if $G_2=K_{12}$ and $r_2=2$, which implies that $G_2=K_{12}=C_1$ or $C_1\times C_1$. Clearly n is 39.

$K_{11}\sim C_r$: In this case we see that $r_1=2r$. It is obvious that G_1 is C_{r+1} or F_4 $(r=3)$. It is easy to see that G_1 is not F_4. Then we see that $P(\overline{G}/\overline{K}_1)=(1+t^{4r+3})P(G_2/K_{12})$. Note that $k=4r+2r_2$. Possible type of K_{12} is $C_1\times C_1$, A_s, D_s, C_s, $C_s\times C_1$, B_3 or B_4. Since $\mathrm{rk}\,G_2=\mathrm{rk}\,K_{12}$, possible type of G_2 is $C_1\times C_1$ (for $K_{12}\sim C_1\times C_1$), A_s, E_7, E_8, G_2 (for $K_{12}\sim A_s$), D_s, B_s, E_8, F_4 (for $K_{12}\sim D_s$), C_s (for $K_{12}\sim C_s$), $C_s\times C_1$ (for $K_{12}\sim C_s\times C_1$), B_3 (for $K_{12}\sim B_3$), B_4, G_4(for $K_{12}\sim B_4$). It is not difficult to see that the cases of $K_{12}\sim D_s$, C_s, $C_s\times C_1$. B_3 and B_4 do not occur. If $K_{12}\sim C_1\times C_1$ or A_s, then G_2 must be $C_1\times C_1$ and A_s respectively and $2r_2-1=3$. Thus we have proved that if $K_{11}\sim C_r$, then $G_1=C_{r+1}$, $K_{12}=G_2=A_1$ or $C_1\times C_1$ and $r_2=2$.

$K_{11}\sim C_r\times C_1$: In this case we see that $r_1=2r$. By the same arguments as above we see that possible type of pair $(\overline{G},\overline{K}_1)$ is $(C_{r+1}\times C_1\times C_1, C_r\times C_1\times C_1)$ or $(C_{r+1}\times C_1\times C_1\times C_1, C_r\times C_1\times C_1\times C_1)$.

Now we shall determine G, K_i, L_j, and H. We need the following

PROPOSITION 13. *K_1 and K_2 are not conjugate.*

Let W_i be the identity component of the ineffective kernel of the natural action of G on G/K_i; $G=G^{(i)}\times W_i$ and $K_i=K_i'\circ W_i$. We have shown that $G/K_i=G^{(i)}/K_i'=C_{r+1}/C_r$ and hence $G^{(i)}=C_{r+1}$, $C_{r+1}\times T^1$ or $C_{r+1}\times C_1$.

$G^{(i)}=C_{r+1}$: In this case we see that $K'_1=C_r$, $W_1=C_1$, $C_1\times T^1$ or $C_1\times C_1$ and W_1 acts transitively on S^3. If $W_1=C_1$, then we have $G=C_{r+1}\times C_1$ and $K_1=C_r\times C_1$. In this case we see that

$$K_1=\left\{\left(\begin{bmatrix}1 & 0\\ 0 & *\end{bmatrix}, q_1\right): q_1\in C_1\right\}$$

It follows from Proposition 13 that

$$K_2=\left\{\left(\begin{bmatrix}q & 0\\ 0 & *\end{bmatrix}, q\right);\ q\in C_1\right\}$$

Since $L_1\sim C_{r-1}\times C_1$, L_1 is assumed to be a subgroup of K_1 and is isomorphic to a subgroup of K_2, we see

$$L_1=\left\{\left(\begin{pmatrix}1 & & \\ & q & \\ & & *\end{pmatrix}, q\right);\ q\in C_1\right\}.$$

Similarly we have

$$L_2=\left\{\left(\begin{bmatrix}1 & \\ & *\end{bmatrix}, 1\right)\right\}$$

and

$$H=\left\{\left(\begin{pmatrix}1 & & \\ & 1 & \\ & & *\end{pmatrix}, 1\right)\right\}.$$

This orbit structure coincides with that of example $\phi^{(3)}_{r+1}$. If $W_1=C_1\times T^1$, then we have $G=C_{r+1}\times T^1\times C_1$ and $K_1=C_r\times T^1\times C_1$. In this case we see that

$$K_1=\left\{\left(\begin{bmatrix}1 & \\ & *\end{bmatrix} z,\ q\right);\ z\in T^1,\ q\in C_1\right\}.$$

It follows from Proposition 13 that possible embedding of K_2 is

$$\left\{\left(\begin{bmatrix}z & \\ & *\end{bmatrix}, z, q\right);\ z\in T^1,\ q\in C_1\right\} \text{ or } \left\{\left(\begin{bmatrix}q & \\ & *\end{bmatrix}, z, q\right)\right\}.$$

Since $L_1\sim C_{r-1}\times T^1\times C_1$, L_1 is assumed to be a subgroup of K_1 and conjugate to a subgroup of K_2, possible embedding of L_1 is

$\left\{\left(\begin{pmatrix}1 & & \\ & z & \\ & & *\end{pmatrix}, z, q\right)\right\}$ or $\left\{\left(\begin{pmatrix}1 & & \\ & q & \\ & & *\end{pmatrix}, z, q\right)\right\}$. Assume $L_1=\left\{\left(\begin{pmatrix}1 & & \\ & z & \\ & & *\end{pmatrix}, z, q\right)\right\}$.

Then we see that $gL_1g^{-1}\subset K_2$ for some $g=(A, z_0, g_0)$. Hence there are $z'\in T^1$ or $q'\in C_1$ such that $z_0z'=zz_0$ or $z_0q=q'z_0$ which implies $z'=1$ or $q'=1$. This means that L_1 is not type of $C_{r-1}\times T^1\times C_1$. Thus the case of $W_1=C_1\times T^1$ does not occur.

By the same arguments we can show that the case of $W_1=C_1\times C_1$ does not occur.

$G^{(1)}=C_{r+1}\times T^1$: In this case we see that $K'_1=C_r\times T^1$, $W_1=C_1$, $C_1\times T^1$ or $C_1\times C_1$. If $W_1=C_1$, then we have $G=C_{r+1}\times T^1\times C_1$ and $K_1=C_r\times T^1\times C_1$. In this case we see that

$$K_1=\left\{\left(\begin{bmatrix}z & \\ & *\end{bmatrix}, z, q\right);\ z\in T^1,\ q\in C_1\right\}.$$

It is not difficult to see that

$$K_2=\left\{\left(\begin{bmatrix}q & \\ & *\end{bmatrix}, z, q\right);\ z\in T^1,\ q\in C^1\right\}$$

and

$$L_1=\left\{\left(\begin{pmatrix}z & & \\ & q & \\ & & *\end{pmatrix}, z, q\right)\right\}$$

$$L_2=\left\{\left(\begin{bmatrix}q & \\ & *\end{bmatrix}, z, z\right)\right\}$$

$$H=\left\{\left(\begin{pmatrix}z & & \\ & z & \\ & & *\end{pmatrix}, z, z\right)\right\}.$$

This orbit structure coincides with that of example $\psi^{(2)}_{r+1}$.

$G^{(1)}=C_{r+1}\times C_1$: By the same arguments as case of $G^{(1)}=C_{r+1}\times T^1$ we can show that this case has the same orbit structure as example $\psi^{(1)}_{r+1}$.

Next we consider the case in which $U_i\neq 1$ for $i=1, 2$. If $\bar{K}_i\cap U_i=\bar{U}_i=1$ for at least one i, then the same argument as in case $U_i=1$ show that $(\bar{G}, \bar{K}_i)$ is one of $(C_{r+1}\times C_1, C_r\times C_1)$, $(C_{r+1}\times C_1\times T^1, C_r\times C_1\times T^1)$ or $(C_{r+1}\times C_1\times C_1, C_r\times C_1\times C_1)$. Assume $\bar{U}_i\neq 1$ for $i=1, 2$. Put $\bar{G}=\bar{G}^{(i)}\times \bar{W}_i$ and $\bar{K}=\bar{K}_i'\circ\bar{W}_i=\bar{K}_i''\circ\bar{U}_i$. It follows from Remark 4 that we may assume that there are simple factor K_{11} and G_1 of $\bar{K}_1''$ and $\bar{G}$ respectively such that $G_1/K_{11}$is a non-trivial direct summand of G/K_1. Clearly K_{11} is type of $A_r\,(r\geqslant 1)$, $C_r\,(r\geqslant 2)$, $D_r\,(r\geqslant 4)$, B_3 or B_4.

The case rk $K_{11}=$rk G_1.

In this case it is easy to see that there are no K_{11} and G_1 satisfying $P(G/K_1)=\sum_{i=0}^{N} t^{i(k-1)}$.

The case rk $K_{11}=$rk G_1-1.

It follows from Table 1 that $\bar{K}_i''=K_{i1}\times K_{i2}$, K_{ij} acts transitively on S^{2r_i-1}.

$K_{11}\sim A_r$: In this case we see that $r=r_1-1$. Possible type of G_1 is A_{r+1}, B_{r+1}, C_{r+1}, D_{r+1}, $E_{r+1}\,(r=5, 6, 7)$, $F_4(r=3)$ or $G_2(r=1)$. If $G_1=A_{r+1}$, then we have $P(G/K_1)=1+t^{2r+3}+\dots$, which is not equal to $\sum_{i=0}^{N} t^{i(k-1)}$, because $k-1=2(r_1+r_2)-1>2r+3$. If G_1 is exceptional, then we have $P(G/K_1)1+t^6+\dots$, which is impossible. Assume $G_1=B_{r+1}$. Then we have $P(G/K_1)=1+t^6+\dots\,(r\geqslant 2)$ or $1+t^7+\dots\,(r=1)$. Thus $r=1$ and hence $r_2=2$. Since K_{12} acts transitively on S^3, K_{12} is type of C_1 or $C_1\times C_1$. Put $G/K_1=\bar{G}/\bar{K}_1=G_1/K_{11}\times G_1'/K_{12}\times G_1''/\bar{U}_1$. Since rk $G_1=$rk $K_{11}+1$, we see that rk $G_1'=$rk K_{12} and rk $G_1''=$rk $\bar{U}_1$, which implies $G_1'=K_{12}$. If $G_1''/\bar{U}_1=$pt, then we have $n=15$. Since $r=1$ and $r_2=2$, we see that $\bar{K}_2\sim C_1\times C_1\times\bar{U}_2$ or $C_1\times C_1\times C_1\times\bar{U}_2$. Since it follows from Propositions 11 and 12 that $\bar{U}\cap G_2''=1_2$, we have $\bar{G}/\bar{K}_2=G_1/\bar{U}_2\times(G'_1\times G_1''/K_{12}\times K_{22})$. If rk $(G_1'\times G_2'')=$rk $(K_{21}\times K_{22})$, then have $G_1'\times G_1''=K_{21}\times K_{22}$ and hence $G/K_2=G_1/U_2$, which leads a contradiction, because of Propositions 11 and 12. Thus we have rk $G_1=$ rk $\bar{U}_2$, which means $G_1=\bar{U}_2$, because $G_1=B_2$ contains no proper simple subgroup of rank 2. This is a contradiction. It is easy to see that $G_1=C_{r+1}$, D_{r+1} is inadequate.

$K_{11}\sim D_r$: It is easy to see that this case does not occur.

$K_{11}\sim C_r$: In this case we see that $r_1=2r$. Possible type of G_1 is $B_3(r=2)$, C_{r+1} or $F_4(r=3)$. If $G_1=B_3$ or F_4, then we have $P(G/K_1)=1+t^7+\dots$, which is impossible, because $k-1>7$. Assume $G_1=C_{r+1}$. Then we have $P(G/K_1)=1+t^{4r+3}+\dots$ and hence $r_2=2$. Hence K_{12} is type of C_1 or $C_1\times C_1$. Thus we have $\bar{G}=C_{r+1}\times G_1'\times G''_1$, $\bar{K}_1=C_r K_{12}\times\bar{U}_1$ and $\bar{G}/\bar{K}_1=C_{r+1}/C_r\times G_1''/\bar{U}_1$, where rk $G_1''=$rk $\bar{U}_1$. We may assume $\bar{K}_2\sim C_r\times K_{22}\times U_2$, $K_{22}\sim C_1$ or $C_1\times C_1$. Since $\bar{U}_2\cap G_1''=1$, we have $\bar{G}/\bar{K}_2=C_{r+1}/\bar{U}_2\times(G_1'\times G_1'')/(K_{21}\times K_{22})$ or $(C_{r+1}\times G_1')/\bar{U}_2\times G_1''/(K_{21}\times K_{22})$. If rk $(K_{21}\times K_{22})=$rk $(G_1'\times G_1'')$ or rk G_1'', then we see that the second summand of G/K_2 is a point, which leads a contradiction, because of Propositions 11 and 12. Thus we have rk $\bar{U}_2=$rk C_{r+1}, which means $\bar{U}_2=C_{r+1}$. This is a contradiction.

$K_{11}\sim B_3, B_4$: By the same arguments as case of $K_{11}\sim C_r$, it can be shown that this case does not occur.

Thus we have shown that $\bar{U}_i=1$ for at least one i and hence $G/K_i=\bar{G}/\bar{K}_i=C_{r+1}/C_r$. Assume $\bar{U}_1=1$. Thus U_1 is a torus. But this is impossible, because $U_1\lhd K_1$ and $G^{(1)}$ acts on $G/K_1=S^{4r+3}$ almost effectively. Thus we have proved that $U_1=U_2=1$ and hence the action of G has the same orbit structure as one of examples $\varphi_{r+1}^{(1)}$, $\varphi_{r+1}^{(2)}$ and $\varphi_{r+1}^{(3)}$.

Case 4–1). $l_1=3$, $l_2\geqslant 3$; odd.

At first we shall consider the case in which U_1 or U_2 is trivial. We see that $k=l_1+l_2-2$ and $K_1\sim SO(2)\times K_{11}$, where K_{11} acts transitively on S^q. Assume $q=1$ and hence $k=4$. We have $\bar{K}_1=1$. Thus we see that $G/K_1=\bar{G}/\bar{K}_1=S^3$, which means $n=7$. Thus we may assume $q\geqslant 3$. Since K_{11} is transitive on S^q, possible type of K_{11} is A_r, C_r, $C_r\times C_1$, D_r, B_3 or B_4. Note that $\bar{K}_1=K_{11}$.

$K_{11}\sim A_r$: In this case we see that $k=2r+4$. It is easy to see that $\bar{G}_1=A_{r+1}$.

It is not difficult to see that the cases of $K_{11}\sim C_r$, $C_r\times C_1$, D_r, B_3 and B_4 don't occur.

Now we shall determine G, K_i, L_j and H. As before we put $G=G^{(i)}\times W_i$, $K_i=K_i'\circ W_i$. As shown above, possible pair of $(G^{(i)}, K'_i)$ is (A_{r+1}, A_r), or $(A_{r+1}\times T^1, A_r\times T^1)$. Since W_i is transitive on S^1, we see that $W_i=T^1$. Thus possible type of (G, K_i) is $(A_{r+1}\times T^1, A_r\times T^1)$ or $(A_{r+1}\times T^1\times T^1, A_r\times T^1\times T^1)$.

Case of $(G, K_i)=(A_{r+1}\times T^1, A_r\times T^1)$.

In this case we see that

$$K_1=\left\{\left(\begin{bmatrix}1 & \\ & *\end{bmatrix}, z\right); z\in T^1\right\}.$$

It follows from Proposition 13 that we see

$$K_2=\left\{\left(\begin{bmatrix}z & \\ & *\end{bmatrix}, z\right); z\in T^1\right\}.$$

If we choose L_1 such that $L_1\subset K_1$, then we have

$$L_1=\left\{\left(\begin{bmatrix}1 & & \\ & z & \\ & & *\end{bmatrix}, z\right); z\in T^1\right\}$$

and

$$L_2=\left\{\left(\begin{bmatrix}1 & \\ & *\end{bmatrix}, 1\right)\right\}$$

$$H=\left\{\left(\begin{bmatrix}1 & & \\ & 1 & \\ & & *\end{bmatrix}, 1\right)\right\}.$$

This orbit structure coincides with that of example $\varphi_{r+1}^{(2)}$.

Case of $(G, K_i)=(A_{r+1}\times T^1\times T^1, A_r\times T^1\times T^1)$.

By the same arguments as above it can be shown that

$$K_1=\left\{\left(\begin{bmatrix}z & \\ & *\end{bmatrix}, z, z_2\right); z, z_2\in T^1\right\}$$

$$K_2=\left\{\left(\begin{bmatrix} z & \\ & * \end{bmatrix}, z_1, z\right); z, z_1\in T^1\right\}$$

$$L_1=\left\{\left(\begin{pmatrix} z_1 & & \\ & z_2 & \\ & & * \end{pmatrix}, z_1, z_2\right); z_i\in T^1\right\}$$

$$L_2=\left\{\left(\begin{bmatrix} z & \\ & * \end{bmatrix}, z, z\right); z\in T^1\right\}$$

and

$$H=\left\{\left(\begin{pmatrix} z & & \\ & z & \\ & & * \end{pmatrix}, z, z\right); z\in T^1\right\}.$$

This coincides with the orbit structure of example $\varphi^{(1)}_{r+1}$.

By the same arguments as case 1 we can show that the case in which $U_i\neq 1$ for $i=1$, 2 does not occur. Thus we have shown that in the case 4–1) the orbit structure of the action has the same orbit structure as one of the examples $\varphi^{(1)}_{r+1}$, and $\varphi^{(2)}_{r+1}$.

3. Proof of Propositions 1, 2, 3, 4 and 5

In this section we shall prove Proposition 1, 2, 3, 4 and 5 in section 2. Recall the notations; G is a compact connected Lie group which acts on S^n almost effectively with codimension two principal orbit G/H, two isolated singular orbits G/K_1, G/K_2 and two non-isolated singular orbits G/L_1, G/L_2. k_i and l_i denote codim G/K_i and codim G/L_i respectively.

It is well known that the orbit space is 2–dimensional disk and dim G/K_i is strictly smaller than dim G/L_j ([1], chap. IV, section 8). It is easy to see that S^n is equivariantly diffeomorphic to a G–manifold $M_1\underset{f}{\cup} M_2$, where M_i is a G–equivariant k_i– disk bundle over G/K_i and $f:bM_1\longrightarrow bM_2$ is an equivariant diffeomorphism (bM_i is the boundary of M_i). Since $3<k_i$, we see that the simply connectedness of S^n implies that G/K_i is simply connected for $i=1.\ 2$. Thus we have proved proposition 1.

We identify bM_1 and bM_2 by f and put $M_0=bM_1=bM_2$. From Mayer-Vietoris exact sequence, it follows that $H^i(M_0; Q)$ is isomorphic to $H^i(G/K_1; Q)\oplus H^i(G/K_2; Q)$ for $0<i<n-1$. In particular, the projection $p_i:M_0\longrightarrow G/K_i$ induces isomorphism $p_i^*:H^j(G/K_i:Q)\longrightarrow H^j(M_0: Q)$. Hence we have $M_0\underset{Q}{\sim} G/K_i\times S^{k_i-1}$, where $X\underset{Q}{\sim} Y$ means that spaces X and Y have the same graded cohomology modules.

We have already noted that

(1) $H^i(M_0; Q)\simeq H^i(M_1; Q)\oplus H^i(M_2; Q)$ for $0<i<n-1$

(2) $M_0\underset{Q}{\sim} G/K_i\times S^{k_i-1}$.

Thus we have

(3) $P(M_0)=P(G/K_1)+P(G/K_2)+t^{n-1}-1$
$\qquad =P(G/K_i)(1+t^{k_i-1})$ for $i=1, 2$

and hence we have

(4) $P(G/K_1)(1-t^{k_1+k_2-2})=(1+t^{k_2-1})(1-t^{n-1})$

and

(5) $P(G/K_2)(1-t^{k_1+k_2-2})=(1+t^{k_1-1})(1-t^{n-1})$.

Multiply both hand sides of (4) by $\sum_{i=0}^{N} t^{i(k_1+k_2-2)}$. Then we have $n-1\equiv 0$ or k_2-1 mod (k_1+k_2-2) and $n+k_2-2\equiv 0$ or k_2-1 mod (k_1+k_2-2), because every term of the left hand side has positive coefficient mod $t^{(N+1)(k_1+k_2-2)}$. Similarly we have $n-1\equiv 0$ or k_1-1 mod (k_1+k_2-2) and $n+k_2-2\equiv 0$ or k_1-1 mod (k_1+k_2-2). Assume $k_1\neq k_2$. We may assume $k_1<k_2$. If $n-1\equiv k_2-1$ and $n+k_2-2\equiv 0$ mod (k_1+k_2-2), then we have $2(k_2-1)=m(k_1+k_2-2)$, which is a contradiction, because $k_1\neq k_2$. If $n-1\equiv 0$ and $n+k_2-2\equiv 0$ mod (k_1+k_2-2), then we have $k_2-1=m(k_1+k_2-2)$, which is a contradiction. By the same arguments we can conclude that $n-1=(N+1)(k+k_2-2)$ for some N. Multiply both hand side of (4) by $\sum_{i=0}^{N} t^{i(k_1+k_2-2)}$, we have

$$P(G/K_1)=(1+t^{k_2-1})\sum_{i=0}^{N} t^{i(k_1+k_2-1)}.$$

Similarly we have

$$P(G/K_2)=(1+t^{k_1-1})\sum_{i=0}^{N} t^{i(k_1+k_2-2)}.$$

Next assume $k=k_1=k_2$. In this case it is easy to see that $n-1\equiv 0$ mod $(k-1)$ and $P(G/K_i)=\sum_{i=0}^{N} t^{ik-1)}$, where $n-1=(N+1)(k-1)$.

Thus we have proved the Proposition 2.

Next we shall prove the Proposition 3. Consider the fibering: $K_i/L_j \longrightarrow G/L_j \longrightarrow G/K_i$. Since $\pi_1(G/K_i)=1$, we have

$$E_2^{p,q}=H^p(G/K_i)\otimes H^q(K_i/L_j)$$

for the spectral sequence of the fibering, where the coefficient of cohomology is Q. In the following the coefficient of cohomology is assumed to be rational numbes, unless it is stated to the contrary. We shall show that $P(G/L_j)=P(G/K_i)\,P(K_i/L_j)$. We have the following

LEMMA. *Let $F\longrightarrow E\longrightarrow B$ be a fibre bundle with $\pi_1(B)=1$. Assume $H^i(B)=0$ for $0<i<k(k\geqq 3)$ and $H^i(F)=0$ for $i>l$ and $l+2<k$. Then we have $P(E)=P(B)P(F)$.*

This lemma is proved by the standard argument of spectral sequence and our asertion is proved as follows. It follows from Proposition 1 that $H^i(G/K_j)=0$, $0<i<k_{3-j}-1$. Since dim $K_i/L_j=k_i-l_j$, we have $H^t(K_i/L_j)=0$ for $t>k_i-l_j$. Since l_j is greater than 3 and $k_i>l_j$, the hypotheses of Lemma hold for the fibre bundle $K_i/L_j\longrightarrow G/L_j\longrightarrow G/K_i$. Thus we have proved that $P(G/L_j)=P(G/K_i)\,P(K_i/L_j)$.

Now we shall prove that $k_1=k_2$. Assume the contrary; $k_1<k_2$. From the equality

$P(G/L_j)=P(G/K_i)P(K_i/L_j)$, it follows that $(1+t^{k_1-1})\sum_{i=0}^{N} t^{i(k_1+k_2-2)} P(K_i/L_j)=(1+t^{k_2-1})$ $\sum_{i=0}^{N} t^{i(k_1+k_2-2)}P(K_1/L_j)$, in other words, $(1+t^{k_1-1})P(K_2/L_j)=(1+t^{k_2-1})P(K_1/L_j)$. Since dim $K_2/L_j=k_2-l_j$ the coefficient of t^{k_1-1} on the left hand side is at least 1. But it is clear that the coeffient of t^{k_1-1} on the right hand side is zero. This proves that $k_1=k_2$. Now Proposition 4 and 5 follows immeadiately from Proposition 3.

4. Proof of Proposition 7, 8, 9, 10, 11, 12 and 13

In this section we shall prove the Proposition 7, 8, 9, 10, 11, 12 and 13. We shall omit the proof of Proposition 7, since it is proved in [3].

Proof of Proposition 8.

If k is odd, then the Euler characteristic $X(G/K_i)$ is non-zero and hence we have that rk $G=rk$ K_i. Assume k is even. Then we see that $X(G/K_i)=0$ and hence rk $K_i<$rk G. Let T be a maximal torus of G and consider the action of T on S^n obtained by the restriction. Since rk $K_i<$rk G, the fixed point set $F(T, S^n)$ is empty. But it follows from the Borel formula that there is a codimension 1 subtorus S of T such that $F(S, S^n)$ is not empty, which implies that rk $K_i=$rk $G-1$. This proves the Proposition 8.

Proof of Proposition 9.

Consider the homotopy exact sequence;

$\pi_4(G)\otimes Q\longrightarrow\pi_4(G/K_i)\otimes Q\longrightarrow\pi_3(K_i)\otimes Q\longrightarrow\pi_3(G)\otimes Q\longrightarrow\pi_3(G/K_i)\otimes Q\longrightarrow 0.$

If $k\geqq 6$, then we have that $\pi_3(K_i)\otimes Q\cong\pi_3(G)\otimes Q$, which implies that $s_i=t$. If $k=5$, then we have the exact sequence;

$0\longrightarrow Q\longrightarrow\pi_3(K_i)\otimes Q\longrightarrow\pi_3(G)\otimes Q\longrightarrow 0$, which implies that $s_i=t+1$. Assume $k=4$. Consider the spectral sequence of the fibre bundle; $\bar{K}_i\longrightarrow\bar{G}\longrightarrow\bar{G}/\bar{K}_i$. We have $H^3(G)=tQ$, $H^3(\bar{K}_i)=s_iQ$ and $H^3(\bar{G}/\bar{K}_i)=Q$. It follows that $E_2^{3,0}=E_\infty^{3,0}=Q$ and $E_2^{0,3}=E_\infty^{3,0}=sQ$. It follows that $\bar{H}^3(G)=(s+1)Q$ and hence $t=s_i+1$. This completes the proof of Proposition 9.

Proof of Proposition 10.

If $k=5$, then rk $G=$ rk K_i and hence we have $\bar{G}/\bar{K}_i=G_1/K_{i1}\times G_2/K_{i2}\times\ldots\times G_t/K_{it}$, where rk $G_j=$rk K_{ij}.

Assume $k\geqq 6$. Let $\bar{G}=G_1\times\ldots\times G_t$ be the decomposition into product of simple Lie groups. It follows from Proposition 7 that $\bar{K}_i=K_{i1}\times\ldots\times K_{it}$ or $\bar{K}_i=(K_{i1}\times\ldots\times K_{it})\circ K_{i0}$, where $K_{ij}\subset G_j$ and K_{i0} is of rank 1. From Proposition 9 it follows that K_{ij} is simple for the first case and some of K_{ij} is trivial for the second case. In the first case, it is clear that $\bar{G}/\bar{K}_i=G_1/K_{i1}\times\ldots\times G_t/K_{it}$. Consider the second case. We may assume $p_1(K_{i0})\neq 1$ and $p_2(K_{i0})\neq 1$ and $p_j(K_{i0})=1$ for $j\geqq 3$, where p_j denotes the projection $\bar{G}\longrightarrow G_j$. Then we have

$$\bar{G}/K_i=(G_1\times G_2)/((K_{i1}\times K_{i2})\circ K_{i0})\times G_3/K_{i3}\times\ldots\times G_t/K_{it},$$

where rk G_j=rk K_{ij} for $j \geqq 3$. Since the number of simple factors of $(K_{i1} \times K_{i2}) \circ K_{i0}$ is 2, we may assume $K_{i1}=1$ and hence rk $G_1=1$. It is easy to see that the natural action of G_2 on $(G_1 \times G_2)/(K_{i0} \circ K_{i2})$ induced by the left translation is transitive and hence $(G_1 \times G_2)/(K_{i0} \circ K_{i2}) = G_2/K_{i2}$. This completes the proof of Proposition 10.

Proof of Proposition 12.

Assume the contrary. Then it is well known that $S^n = G/H \times_{\Gamma_H} F(H, S^n)$, where $\Gamma_H = N(H, G)/H$ ($N(H, G)$=the normalizer of H in G). Since rk $H \geqq$ rk $G-1$, Γ_H is finite group or Lie group of rank 1. If Γ_H is finite, we have $S^n = G/H \times F$, where F is a component of $F(H, S^n)$, because S^n is simply connected. Since dim $G/H < n$, dim $F < n$, this is clearly a contradiction. Assume rk $\Gamma_H=1$. Then we have $S^n = G/H \times_{\Gamma_H^0} F$, where F is a component of $F(H, S^n)$ and Γ_H^0 is the identity component of Γ_H. It is clear that the Euler class of the fibre bundle; $\Gamma_H^0 \longrightarrow G/H \times F \longrightarrow S^n$ is zero and hence we see that $G/H \times F \underset{Q}{\sim} S^n \times \Gamma_H^0$, which implies $H^i(G/H \times F)=0$ for $4 < i < n-1$. Since dim $G/H \leqq n-2$, we have dim $G/H \leq 3$ and dim $F \leq 3$, which implies n=dim $(G/H \times F)$-dim $\Gamma_H \leqq 6-1=5$. This proves the Proposition 12.

Proof of Proposition 11.

Recall the notations; W_i = the identity component of ineffective kernel of the action of $\bar{G}$ on G/K_i, U_i=the identity component of ineffective kernel of the action of K_i on S^{k-1} and $U_i' = U_i \cap \bar{K}_i$.

We see that $G_1 \cap K_i = G_1 \cap \bar{K}_i \subset U_i \subset H$ and hence we have $G_1 \cap H \subset G_1 \cap L_j \subset G_1 \cap K_i \subset G_1 \cap H$, which implies that $G_1 \cap H = G_1 \subset L_j = G_1 \cap K_i$. For any point $x \in S^n$, we have $G_x = gK_ig^{-1}$, gL_jg^{-1} or gHg^{-1} for some $g \in G$. If $G_x = gK_ig^{-1}$, then we have that $G_1 \subset G_x = gG_1 \cdot g^{-1} \cap gK_ig^{-1} = g(G_1 \cap K_i)g^{-1} = g(G_1 \cap H)g^{-1}$. By the same arguments we have that for every $x \in S^n$, $G_1 \cap G_x$ is conjugate to $G_1 \cap H$. This proves the Proposition 11.

Proof of Proposition 13.

Assume the contrary. Then we may assume $K_1 = K_2$. Let T be a maximal torus of K_1. Then $F(T, G/K_1)$ (=the fixed point set of T)$\neq \phi$. It is clear that $F(T, S^n) = F(T, G/K_1) \cup F(T, G/K_2)$ (disjoint union), which contradicts the connectedness of $F(T, S^n)$. This completes the proof of Proposition 13.

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