3-dimensional homogeneous Riemannian manifolds II

By
Kouei SEKIGAWA*

(Received November 30, 1977)

0. Introduction

In the previous paper, [1], we have given a list of Lie algebras of Lie groups of full isometries acting transitively and effectively on 3-dimensional connected homogeneous Riemannian manifolds. In this paper, we shall give a list of all 3-dimensional connected homogeneous Riemannian manifolds. The arguments in this paper is the continuation of the ones in [1]. To avoid repitition, we shall adopt the same notations and terminologies as [1]. In our arguments, the following result plays an important role.

THEOREM A (J. A. Wolf [3]) *Let $\widetilde{M}$ and M be Riemannian manifolds and let $\widetilde{M}=M/\Gamma$, where Γ is a group of isometries of $\widetilde{M}$ acting freely and properly discountinously. Let G be the centralizer of Γ in the group $I(\widetilde{M})$ of all isometries on $\widetilde{M}$. Then, M is homogeneously if and only if G is transitive on $\widetilde{M}$. And if M is homogeneous, then every element of Γ is a Clifford translation.*

Let M be a 3-dimensional connected homogeneous Riemannian manifold and $\widetilde{M}$ be its Riemannian universal covering manifold.

1. Cases I, II, III

First, we consider the case I. In this case, M is isometric with a 3-dimensional sphere with a Riemannian metric of a positive constant curvature. Then, according to J. A. Wolf, [3], M is of the form, $\boldsymbol{S}^3/\Gamma$, where Γ is any one of the following groups:

$$(1)\ \{1\},\quad (2)\ \boldsymbol{Z}_m,\quad (3)\ \boldsymbol{D}^*_m,\quad (4)\ \boldsymbol{T}^*,\quad (5)\ \boldsymbol{O}^*,\quad (6)\ \boldsymbol{I}^*,$$

here $\boldsymbol{D}^*_m$, $\boldsymbol{T}^*$, $\boldsymbol{O}^*$, and $\boldsymbol{I}^*$ denote the binary dihedral, binary tetrahedral, binary octahedral and binary icosahedral groups as usual, and m is any positive integer.

Secondary, we consider the case II. In this case, $\widetilde{M}$ is isometric with a 3-dimensional Euclidean space $\boldsymbol{E}^3$. Then, according to J. A. Wolf, [3], M is of the form, $\boldsymbol{E}^3/\Gamma$, where Γ is any one of the following groups:

$$(1)\ \{1\},\quad (2)\ \boldsymbol{Z},\quad (3)\ \boldsymbol{Z}\times\boldsymbol{Z},\quad (4)\ \boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}.$$

Lastly, we consider the case III. In this case, $\widetilde{M}$ is isometric with a 3-dimensional Hyperbolic space $\boldsymbol{H}^3$. Then, according to J. A. Wolf, [3], M is $\boldsymbol{H}^3$ alone.

* Niigata University

2. Case

First, we consider the case IV-(i). Then, $\widetilde{M}$ is isometric with $\boldsymbol{S}^2\times\boldsymbol{E}^1$. And $I_0(\widetilde{M})$ is isomorphic with $SO(3)\times\boldsymbol{R}^1$. In this case, by the result of H. Takagi (cf. [2]), M is of the the form $\boldsymbol{S}^2\times\boldsymbol{E}^1/\Gamma$, where Γ is any one of the following group:

(1) $\{1\}$, (2) $\boldsymbol{Z}_2\times\{0\}$, (3) $\{1\}\times\{\beta\}$

(4) a group which is semi-direct product of the infinite cyclic group $<(-1, \beta)>$ generated by $(-1, \beta)$ and $\boldsymbol{Z}_2\times\{0\}$,

here, $\beta\neq 0$.

Secondary, we consider the case IV-(ii). Then, $\widetilde{M}$ is isometric with $\boldsymbol{H}^2\times\boldsymbol{E}^1$. And $I_0(\widetilde{M})$ is isomorphic with $SO(2, 1)\times\boldsymbol{R}^1$. In this case, by the result of H. Takagi, [2], M is of the form $\boldsymbol{H}^2\times\boldsymbol{E}^1/\Gamma$, where Γ is any one of the following groups:

(1) $\{1\}$, (2) $\{1\}\times\{\beta\}$

here $\beta\neq 0$.

3. Cases V-(i)

In this case, $\widetilde{M}$ is isometric with a certain group space.

First, we consider the case (i)-(2) (or (1), or (i)-(3)). Then, we can easily see that $I(\widetilde{M})$ is isomorphic with

$$\Theta=\left\{\begin{pmatrix} e^u & 0 & v \\ 0 & e^{-u} & w \\ 0 & 0 & 1\end{pmatrix}\in GL(3, \boldsymbol{R});\ u, v, w\in\boldsymbol{R}\right\}$$

or

$$\Theta'=\left\{\begin{pmatrix} e^{\beta w}e^{\sqrt{D_0}w\sqrt{-1}} & 0 & u+v\sqrt{-1} \\ 0 & e^{\beta w}e^{-\sqrt{D_0}w\sqrt{-1}} & u-v\sqrt{-1} \\ 0 & 0 & 1\end{pmatrix}\in GL(3, \boldsymbol{C});\ u, v, w\in\boldsymbol{R}\right\},$$

where $\beta=-(c+d)/2$, (cf. [1]).

Thus, $I(\widetilde{M})$ and hence, $\widetilde{M}$ is diffeomorphic with $\boldsymbol{R}^3$. In this case, M is isometric with the group space $I(\widetilde{M})$ with the left-invariant Riemannian metric as in [1].

Secondary, we consider the cases, (i)-$(4)_1$, (i)-$(4)_2$. Then, $\widetilde{M}$ is isometric with the group space $SU(2)$ (or $Spin$ (3), or $Sp(1)$) with a certain left-invariant Riemannian metric given by [1].

We put $e_1^0=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$, $e_2^0=\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0\end{pmatrix}$, $e_3^0=\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1}\end{pmatrix}$.

Then, we have

$$(3.1)\quad Ad(e_1^0)\,e_2^0=e_1^0e_2^0(e_1^0)^{-1}=-e_2^0\,,\qquad Ad(e_1^0)\,e_3^0=-e_3^0\,,$$

$$Ad(e_1^0)\,e_1^0=-e_1^0\,,$$

$$Ad(e_2^0)\,e_1^0=-e_1^0\,,\qquad Ad(e_2^0)\,e_2^0=-e_2^0\,,\qquad Ad(e_2^0)\,e_3^0=-e_3^0\,,$$

$$Ad(e_3^0)\,e_1^0=-e_1^0\,,\qquad Ad(e_3^0)\,e_2^0=-e_2^0\,,\qquad Ad(e_3^0)\,e_3^0=e_3^0$$

We see that the subgroup of $SU(2)$ which is generated by the elements, $\{e_1^0, e_2^0, e_3^0\}$, is isomorphic with $\boldsymbol{D}_2^*$. From (3.1), we see that

$$I(\widetilde{M})=\frac{SU(2)\times \boldsymbol{D}_2^*}{\boldsymbol{Z}_2}\,.$$

Here $SU(2)\times \boldsymbol{D}_2^*$ acts on $\boldsymbol{S}^3=SU(2)$ by the following way.

$$(3.2)\qquad \phi(g, k)(g_0)=g\,g_0k^{-1}\,,$$

Therefore, by making use of Theorem A, we can see that M is of the form $\boldsymbol{S}^3/\Gamma$, where Γ is any one of the followings:

$$(1)\ \{1\}\,,\qquad (2)\ \boldsymbol{Z}_2\,,\qquad (3)\ \boldsymbol{D}_2^*\,.$$

Remark. More precisely, in this case, M is one of the followings:

$$(1)\ \boldsymbol{S}^3/\{1\}=\boldsymbol{S}^3\,,\qquad (2)\ \boldsymbol{S}^3/\boldsymbol{Z}_2=SO(3)\,,$$

$$(3)\ \boldsymbol{S}^3/\boldsymbol{D}_2^*=SO(3)/\boldsymbol{Z}_2\times \boldsymbol{Z}_2\,.$$

Thirdly, we consider the cases, (i)-$(4)_5$, (i)-$(4)_6$, (i)-$(4)_7$. Then, M is isometric with the group space Σ with certain left-invariant Riemanian metric (cf. [1]), where Σ denotes the universal covering group of $SL(2, \boldsymbol{R})$. Then, for example, Σ can be constructed as follows. For any $g\in SL(2, \boldsymbol{R})$, let u be any continous curve in $SL(2, \boldsymbol{R})$ such that $u(0)=1$, $u(1)=g$, and $[u]$ be the equivalence class of all continuous curves v such that v is homotopic with u and $v(0)=u(0)=1$, $v(1)=u(1)=g$. For two continous curves, u, w: $[0, 1]\longrightarrow SL(2, \boldsymbol{R})$, we put $(u\cdot w)(t)=u(t)w(t)$, $t\in[0, 1]$. Furthermore, we shall define a multiplication on Σ by $[u]\cdot[w]=[u\cdot w]$, for any $[u]$, $[w]\in\Sigma$. Then, by this multiplication, Σ gives rise to a 3-dimensional Lie group wich is the universal covering group of $SL(2, \boldsymbol{R})$ with the covering projection $p: [u]\in\Sigma\rightarrow u(1)\in SL(2, \boldsymbol{R})$. The Riemannian structure on Σ corresponds to certain positive definite inner product on $\mathfrak{sl}(2, \boldsymbol{R})$(cf. [1]). Now, we put

$$e_1^0=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\qquad e_2^0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad e_3^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Then, we have

$$(3.3)\qquad Ad(e_1^0)\, e_1^0 = e_1^0\,, \qquad Ad(e_1^0)\, e_2^0 = -e_2^0\,, \qquad Ad(e_1^0)\, e_3^0 = -e_3^0\,.$$

The subgroup of $SL(2, \boldsymbol{R})$ which is generated by $\{e_1^0\}$ is isomorphic with $\boldsymbol{Z}_4$. The center of $SL(2, \boldsymbol{R})$ is $\boldsymbol{Z}_2=\{-1, 1\}$ and hence, the center of Σ which is generated by $\{[u_0]\}$ where

$$u_0(t)=\begin{pmatrix}\cos \pi t & -\sin \pi t\\ \sin \pi t & \cos \pi t\end{pmatrix}, \qquad t\in[0, 1].$$

Thus, the center of Σ is isomorphic with $\boldsymbol{Z}$. In this case, we can see that $I(\widetilde{M})$ is isomorphic with $\Sigma\times L$, where $L=p^{-1}(\boldsymbol{Z}_4)=p^{-1}(\{1, e_1^0, -1, -e_0{}^1\})(\cong\boldsymbol{Z})$. Here, $\Sigma\times L$ acts on Σ by the following way.

$$(3.4)\qquad \phi(g, k)g_0=gg_0k^{-1}, \qquad \text{for any } (g, k)\in\Sigma\times L,\ g_0\in\Sigma.$$

Therefore, by considering Theorem A, we can see that M is of the form Σ/Γ, where Γ is any one of the following groups:

(1) $\{1\}$, (2) $\boldsymbol{Z}$.

Lastly, consider the case (i)-$(4)_8$. Then, M is isometric with the group space with certain left-invariant Riemannian metric (cf. [1]). In this case, we can see that M is the above group space alone by the sake of Theorem A.

4. Cases V-(ii)-$(1)_1$~V-(ii)-$(1)_5$

First, consider the cases, (ii)-$(1)_1$, (ii)-$(1)_2$. Then, $\widetilde{M}$ is isometric with the group space $SU(2)$ with certain left-invariant Riemannian metric (cf. [1]). In this case, we may apply the similar arguments as the cases, (i)-$(4)_1$~(i)-$(4)_4$.

$$I(\widetilde{M})=\frac{SU(2)\times \boldsymbol{D}_2^*}{\boldsymbol{Z}_2}, \qquad \text{and furthermore,}$$

M is of the form S^3/Γ, where Γ is any one of the following groups:

(1) $\{1\}$, (2) $\boldsymbol{Z}_2$, (3) $\boldsymbol{D}_2^*$.

Secondary, consider the cases, (ii)-$(1)_3$, (ii)-$(1)_4$. Then, $\widetilde{M}$ is isometric with the group space with certain left-invariant Riemannin metric (cf. [1]). In this case, we may apply the similar arguments as the cases, (i)-$(4)_5$~(i)-$(4)_7$.
Thus, we see that M is of the form $\boldsymbol{R}^3/\Gamma$, where Γ is any one of the following groups:

(1) $\{1\}$, (2) $\boldsymbol{Z}$.

Lastly, consider the case (ii)-$(1)_5$. Then, $\widetilde{M}$ is isometric with group space $\boldsymbol{\Theta}$ with certain left-invariant Riemannian metric (cf. [1]). In this case, we may apply the similar arguments as the case (i)-$(4)_8$. Thus, we see that M is the above group space alone.

5. Cases V-(ii)-$(2)_1$~V-(ii)-$(2)_3$

First, consider the case (ii)-(2). Let G^* be the connected, simply connected Lie group with the Lie algebra $\mathfrak{i}(\widetilde{M})$ and K^* be the subgroup of G^* with the Lie algebra $\mathfrak{k}$. Then, we see that

$$G^*=SU(2)\times \boldsymbol{R}_+=\{(g,\ e^{\beta t})\in SU(2)\times \boldsymbol{R}_+;\ t\in \boldsymbol{R},\qquad \text{for some } \beta\neq 0\}$$

and

$$K^*=\left\{\left(\begin{pmatrix} e^{-t\sqrt{-1}/2} & 0 \\ 0 & e^{t\sqrt{-1}/2}\end{pmatrix},\ e^{\beta t}\right);\ t\in \boldsymbol{R}\right\},$$

and furthermore, $\widetilde{M}=G^*/K^*$, which is diffeomorphic with S^3 (cf. [1]).
Then, we can easily see that $\widetilde{M}=G^*/K^*=(G^*/\boldsymbol{Z})/(K^*/\boldsymbol{Z})=G/K$,
where $G=SU(2)\times U(1)$

$$=\left\{\left(g,\ \begin{pmatrix} e^{-u\sqrt{-1}} & 0 \\ 0 & e^{u\sqrt{-1}}\end{pmatrix}\right)\in SU(2)\times U(1);\ u=t/2,\ t\in \boldsymbol{R}\right\},$$

and $K=\left(\begin{pmatrix} e^{-u\sqrt{-1}} & 0 \\ 0 & e^{u\sqrt{-1}}\end{pmatrix},\ \begin{pmatrix} e^{-u\sqrt{-1}} & 0 \\ 0 & e^{u\sqrt{-1}}\end{pmatrix}\right)\in SU(2)\times U(1);\ u\in \boldsymbol{R}$.

Then, by making use of (3. 1), we can easily see that

$$I(\widetilde{M})=\frac{SU(2)\times \boldsymbol{D}_2^* U(1)}{\boldsymbol{Z}_2},$$ and the group $SU(2)\times \boldsymbol{D}_2^* U(1)$ acts on $SU(2)$

by the following way;

(5. 1) $\qquad \phi(g,\ k)g_0=gg_0k^{-1},\qquad$ for any $(g,\ k)\in SU(2)\times \boldsymbol{D}_2^* U(1),\ g_0\in SU(2)$.

Thus, considering Theorem A, we can show that M is of the form $SU(2)/\Gamma=S^3/\Gamma$, where Γ is any one of the following groups:

(1) $\{1\}$, $\quad$ (2) $\boldsymbol{Z}_{2m}$, $\quad$ (3) $\boldsymbol{D}_{2m}^*$,

for any positive interger m.

Secondary, consider the case (ii)-$(2)_2$. Let G^* be the connected, simply connected Lie group with the Lie algebra $\mathfrak{i}(\widetilde{M})$ and K^* be the subgroup of G^* with the Lie algebra $\mathfrak{k}$. Then, we see that $G^*=\Sigma\times L$,

where $L=p^{-1}(SO(2))=p^{-1}\left\{\begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u\end{pmatrix};\ u\in \boldsymbol{R}\right\}$

$$=\left\{[\hat{u}]\in\Sigma;\ \hat{u}(t)=\begin{pmatrix} \cos tu & -\sin tu \\ \sin tu & \cos tu\end{pmatrix},\ t\in[0,\ 1]\right\},$$

and $K^* = \{([\hat{u}], [\hat{u}]) \in \Sigma \times L;\ u \notin \boldsymbol{R}\}$.
And $\widetilde{M} = G^*/K^*$. In this case, $\Sigma \times L$ acts on Σ by the following way.

$$\text{(5.2)} \qquad \phi(g, k)g_0 = gg_0k^{-1}, \qquad \text{for any } (g, k) \in \Sigma \times L,\ g_0 \in \Sigma.$$

Therefore, by considering Theorem A, we can see that M is of the form $\boldsymbol{R}^3/\Gamma$, where Γ is any one of the following groups:

$$(1)\ \{1\}, \qquad (2)\ Z.$$

Lastly, consider the case (ii)-$(2)_3$. Then, $\widetilde{M}$ is isometric with the group of upper triangular matrices of degree 3, Ψ
$=\left\{\begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \boldsymbol{R});\ u, v, w \in \boldsymbol{R}\right\}$, with certain left-invariant Riemannian metric (cf. [1]). In this case, from the arguments in [1], we see that

$$G = I(\widetilde{M})$$
$$= \left\{\begin{pmatrix} 1 & b\cos t - a\sin t & b\sin t + a\cos t & c \\ 0 & \cos t & \sin t & -a \\ 0 & -\sin t & \cos t & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4, \boldsymbol{R});\ t, a, b, c \in \boldsymbol{R}\right\},$$

and

$$K(\text{the subgroup of } G = I(\widetilde{M}) \text{ with the Lie algebra } \mathfrak{k})$$
$$= \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 \\ 0 & -\sin t & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4, \boldsymbol{R});\ t \in \boldsymbol{R}\right\}.$$

Then, $G = I(\widetilde{M})$ acts on $\widetilde{M}$ by the following way.

$$\text{(5.3)} \qquad \phi(g)(x_1, x_2, x_3)$$
$$= (x_1\cos t - x_2\sin t + a,\ x_1\sin t + x_2\cos t + a,$$
$$-x_1(b\cos t - a\sin t) + x_2(b\sin t + a\cos t) + x_3 + c),$$

where

$$g = \begin{pmatrix} 1 & b\cos t - a\sin t & b\sin t + a\cos t & c \\ 0 & \cos t & \sin t & -a \\ 0 & -\sin t & \cos t & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let Γ be a discrete subgroup of G which acts freely and properly discontinuously on $\widetilde{M}=G/K$. Then, by (5. 3), we see that

$$\Gamma=\left\{\begin{pmatrix}1&0&0&nc\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\in G;\ n\in \boldsymbol{Z}\right\},\ \text{for some fixed } c\neq 0.$$

Therefore, by Theorem A, we can see that M is of the form $\boldsymbol{R}^3/\Gamma$, where Γ is any one of the following groups:

(1) $\{1\}$, (2) $\boldsymbol{Z}$.

Remark. Let Π be the product set of Ψ and $SO(2)$, say, $\Pi=\Psi\times SO(2)$. Now, we define a multiplication on Π by the following way.

$$\left(\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix},\begin{pmatrix}\cos t&\sin t\\-\sin t&\text{con } t\end{pmatrix}\right)\cdot\left(\begin{pmatrix}1&a'&b'\\0&1&c'\\0&0&1\end{pmatrix},\begin{pmatrix}\cos t'&\sin t'\\-\sin t'&\cos t'\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix}1&a'\cos t-b'\sin t+a&a'\sin t+b'\cos t+b\\0&1&c+c'-a'(b\cos t-a\sin t)+b'(b\sin t+a\cos t)\\0&0&1\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}\cos(t+t')&\sin(t+t')\\-\sin(t+t')&\cos(t+t')\end{pmatrix}\right).$$

Then, Π is a connected 4-dimensinoal Lie group, and furthermore, isomorphic with G by

$$\left(\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix},\begin{pmatrix}\cos t&\sin t\\-\sin t&\cos t\end{pmatrix}\right)\longleftrightarrow\begin{pmatrix}1&b\cos t-a\sin t&b\sin t+a\cos t&c\\0&\cos t&\sin t&-a\\0&-\sin t&\cos t&b\\0&0&0&1\end{pmatrix}$$

As a group of isometries of $M=\Psi$, Π acts on $M=\Psi$ by the following way.

$$\phi\left(\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix},\begin{pmatrix}\cos t&\sin t\\-\sin t&\cos t\end{pmatrix}\right)\begin{pmatrix}1&u&v\\0&1&w\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&u\cos t-v\sin t+a&u\sin t+v\cos t+b\\0&1&w+c-u(b\cos t-a\sin t)+v(b\sin t+a\cos t)\\0&0&1\end{pmatrix}$$

References

[1] K. SEKIGAWA. *3-dimensional homogeneous Riemannian manifolds* I. Sci. Rep. Niigata Univ., 14 (1977), 5–14.
[2] H. TAKAGI. *Conformally flat Riemannian manifolds admitting a transitive group of isometries*, II Tôhoku Math. J., 27 (1975), 445–451.
[3] J. A. WOLF. Spaces of constant curvature. McGraw-Hill, New York, 1967.