On 4-dimensional quasi-homogeneous affine algebraic varieties of reductive algebraic groups

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(Received November 10, 1977)

Introduction

A variety X is called, by definition, a quasi-homogeneous space of an algebraic group G if G acts on X morphically with one dense orbit whose complement is of dimension zero. In this note we shall classify 4-dimensional quasi-homogeneous affine algebraic varieties of reductive algebraic groups.

In this note all varieties and algegraic groups are considered over the field C of complex numbers. This note is organized as follows; section 1 contains preliminaries and in sections 2 and 3 we study possibilities of semi-simple part of the reductive group which acts on a variety quasi-homogeneously. In section 4, we show that 4-dimensional quasi-homogeneous spaces of reductive group are homogeneous or S-varieties (see section 1 for definition of S-variety), in section 5 we study homogeneous space and in section 6 we study S-varieties.

We always reserve the term "algebraic group" and "variety" for those group and for those variety, respectively, whose underlying varieties are affine, unless the contrary is expressly stated.

We shall use the following notations.

Let H be a linear algebraic group.

H^0=connected component of identity of H

$Rad\ H$=the radical of H

$Rad_u H$=the unipotent radical of H

$rk\ H$=rank of H=the dimension of a maximal torus of H

$H \cdot U$=the semi-direct product of H and U.

Let H act on X morphically.

$H_X=\{h \in H \mid h(x)=x \text{ for any } x \in X\}$ =ineffective kernel.

1. Preliminaries

In this section we assume a reductive group G acts on a variety X morphically and

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quasi-homogeneously. Let O_X be the dense orbit.

The following results are known.

PROPOSITION 1.1 ([5], 2. 3 Th. 4). *If $O_X \neq X$, then X is an S-variety, i.e. for any $x \in O_X$, the isotropy subgroup G_x contains a maximal unipotent subgroup of G.*

PROPOSITION 1.2 ([5], 3. 1 Prop. 2. 1). $rk\ G \leqq \dim X$.

PROPOSITION 1.3 ([8], 3 Th. 2). *If $\dim X = rk\ G$, then G is a direct product of a projective like group and torus and X is also a product of projective spaces and torus.*

PROPOSITION 1.4. *Let $R \neq \{e\}$ be a semi-simple group which acts on a variety Y almost effectively, i.e. $\dim R_Y = 0$. Then there is an observable subgroup Q of R (this meana that R/Q is quasi- affine, i.e. open subset of an affine variety) such that*

a) $1 \leqq \dim P/Q \leqq \dim Y$

b) *Q contains no normal subgroup of R of dimension* > 1.

PROOF. From almost effectivity of the action, there is an element y of Y such that $\dim R_y \lneqq \dim R$. Put $R_y = Q''$. Since $\overline{R/Q''}$ is a closed subset of Y, $\overline{R/Q''}$ is affine and hence Q'' is observable, because R/Q'' is open in $\overline{R/Q''}$. Assume Q'' contains a normal subgroup N of R of positive dimension. Let $\widetilde{R} = R_1 \times R_2 \times \cdots\cdots \times R_s$ (R_i; simple) be the covering group of R and $\pi : \widetilde{R} \rightarrow R$ the natural projection. Then $\widetilde{R}$ acts on Y morphically and almost effectively. It is clear that $\pi^{-1}(Q'')$ is observable. Let $\pi^{-1}(Q'')$ contains a simple factor of $\widetilde{R}$, say R_1. $\widetilde{R}/R_1$ contains a subgroup Q' which is isomorphic to $\pi^{-1}(Q'')/R_1$. Since $\widetilde{R}/R_2 \times \cdots\cdots \times R_s = R_1$ and $R_2 \times \cdots\cdots \times R_s/Q' = \widetilde{R}/\pi^{-1}(Q'')$ are quasi-affine, the following lemma shows that Q' is observable.

LEMMA ([3], p. 143) *Let K and L be subgroups of G. Assume G/L and L/K are quasi-affine. Then G/K is quasi-affine.*

Our proposition follows from the induction on s. This completes the proof.

We have the following

COROLLARY. *Let $G = R \cdot Rad\ G$. Then R contains a subgroup H with the following properties*

(i) *H is observable.*

(ii) $1 \leqq \dim R/H \leqq \dim X$

(iii) *H contains no normal subgroup of R of dimension* $\geqq 1$

(iv) $codim_R\ H \geqq rk\ R+1$.

PROOF. The statements (i), (ii) and (iii) follows from proposition immeadiately. To prove (iv) let R act on $\overline{R/H}$. Since $\overline{R/H}$ is affine and H contains no positive dimensional normal subgroup of R, the action of R is almost effective. Then we have $rk\,R \leqq \dim \overline{R/H} = \mathrm{codim}_R\ H$. The equality holds only if R/H is a projective variety and hence $R=H$ which is a contradiction. Thus we have $\mathrm{codim}_R\ H \geqq \mathrm{rk}\ R+1$. This completes the proof.

We recall some fundamental facts on Borel subgroups and maximal unipotent subgroups of a simple group.

The following results are fundamental.

THEOREM ([2]) *Let G be a semi-simle group and $\underline{g}$ its Lie algebra. Then* (1) *$\underline{g}$ has a generator $\{h_i, e_i, f_i; i=1, 2, \ldots\ldots, r\}$ with the following properties.*

(i) *$\{h_1, h_2, \ldots\ldots, h_r\}$ is a basis of a maximal diagonalizable subalgebra of $\underline{g}$, i.e. simple roots.*

(ii) *e_i(or f_i) is a root vecter corresponding to a positive (or negative, respectively) simple roots.*

(iii) $[h_i, e_i]=2e_i$, $[h_i, f_i]=-2f_i$.

(iv) $[e_i, f_i]=h_i$.

(2) *Let T be the maximal torus generated by $h_1, h_2, \ldots\ldots, h_r$. Then the Borel subgroup B which contains T is generated by $h_1, h_2, \ldots\ldots, h_r$ and $e_1, e_2, \ldots\ldots, e_r$.*

(3) *Every parabolic subgroup which contains B is generated by $\underline{b}$ and some f_i.*

Example. SL_4

roots: $\{x_p-x_p\}$ $p, q=1, 2, 3, 4.$ $x_1+x_2+x_3+x_4=0.$

simple roots: $a_1=x_1-x_2$, $a_2=x_2-x_3$, $a_3=x_3-x_4$.

positive roots: $a_1, a_2, a_3, a_1+a_2, a_2+a_3, a_1+a_2+a_3$.

Borel subgroup B; generated by a_1, a_2, a_3 and $e_{a_1}, e_{a_2}, e_{a_3}$.

Since $[e_{a_1}, e_{a_2}]=e_{a_1+a_2}$, $[e_{a_1}, e_{a_3}]=0$, $[e_{a_2}, e_{a_3}]=e_{a_2+a_3}$ and $[e_{a_1+a_2}, e_{a_3}]=e_{a_1+a_2+a_3}$, we have dim $B=9$. It follows from $B=T\cdot U(T$: a maximal torus, U : a maximal unipotemt subgroup) that dim $U=6$. By the same arguments we have the following table.

G	dim. of Borel subgroup	dim. of maximal unipotent subgroup
SL_4	9	6
B_3	9	6
Sp_3	11	8
SL_3	5	3
Sp_2	6	4
G_2	8	6
SL_2	2	1

2. Possibility of semi-simple part of G

In this section, X denotes a 4-dimensional variety on which a reductive group G acts quasi-homogeneously. Let $G=P\cdot$ Rad G be the Levi decomposition of G.

REMARK. The case when rk $P=0$ has been considered in [6]. We restrict ourself to the case in which rk$P\neq0$.

PROPOSITION. 2.1 $P\neq G_2, Sp_3, B_3$.

PROOF. We recall the following result ([2], Th. 30. 4)

THEOREM. *Let G be a reductive group. Then*

a) *if H is a maximal proper closed subgroup of G, then H^0 is reductive or parabolic.*

b) *a maximal unipotent subgroup of G is the unipotent radical of a Borel subgroup.*

The proof of $P \neq G_2$.

Consider the subgroup H in Corollary to Proposition 1. 4. Let $\widetilde{H}$ be the maximal proper closed subgroup which contains H°. Assume $\widetilde{H}$ is parabolic. Then it is clear that dim $P/H>4$. Assume $\widetilde{H}$ is reductive. Then $\widetilde{H}=L \cdot U$(L: semi-simple, U=Rad $\widetilde{H}$). By the table in section, 1, we have dim $U \leqq 8$. If rk $L=0$, then dim $\widetilde{H} \leqq 8$ and hence dim $P/H \geqq 6$. If rk $L=1$, and $U=T \cdot V$ (T: torus V: unipotent), then dim $T \leqq 1$ and dim $V \leqq 6$. Since L is locally isomorphic to SL_2, L contains 1-dimensional unipotent group and hence dim $V \leqq 5$. Then we have dim $\widetilde{H} \leqq 3+1+5=9$, and hence dim $P/H>5$. If rk $L=2$ then $L \sim A_2$ or $A_1 \times A_1$. Since A_2 or $A_1 \times A_1$ is a maximal subgroup of G_2 we have $\widetilde{H}=L$ and hence dim $P/H \geqslant 6$. Thus we have shown that there is no subgroup H of P such that dim $P/H \leqq 4$. This completes the proof.

The Proof of $P \neq Sp_3$.

Let H and $\widetilde{H}$ be the subgroups of Sp_3 as in the proof of $P \neq G_2$. Assume $\widetilde{H}$ is parabolic. We show that dim $P/H \geqq 5$. In fact let a_1, a_2 and a_3 be simple roots of Sp_3. Then $\widetilde{H}$ is generated as Lie algebra by $a_1, a_2, a_3, e_{a_1}, e_{a_2}, e_{a_3}$ and two of e_{-a_1}, e_{-a_2} and e_{-a_3}. It is easy to see that dim $\widetilde{H}=15$ and hence dim $P/H \geqq 5$. Next assume $\widetilde{H}$ is reductive. Put $\widetilde{H}=L \cdot$ Rad $\widetilde{H}$ and Rad $\widetilde{H}=T \cdot U$, where L is the semi-simple part of $\widetilde{H}$ and T a torus and U unipotent subgroup. From the table in section 1, it follows that dim $U \leqq 8$.

Case 1. rk $L=0$.

In this case we have dim $\widetilde{H} \leqq$ dim T+dim $U \leqq 11$. Therefore we have dim $P/H \geqq 9$.

Case 2. rk $L=1$.

In this case we have dim $L=3$, dim $T \leqq 3$ and dim $U \leqq 7$ and hence dim $\widetilde{H} \leqq 13$. Thus we have dim $P/H \geqq 5$.

Case 3. rk $L=2$.

In this case L is locally isomorphic to either $A_1 \times A_1$, A_2 or C_2.

Subcase 1. $L \sim A_1 \times A_1$.

Then we have dim $T \leqq 1$, dim $U \leqq 6$ and hence dim $\widetilde{H} \leqq 13$, which implies dim $P/H>4$.

Subcase 2. $L \sim A_2$.

We have dim $T \leqq 1$, dim $U \leqq 5$ and hence dim $\widetilde{H} \leqq 14$, which implies that dim $P/H>4$.

Subcase 3. $L \sim G_2$.

We have dim $T\leqq 1$, dim $U\leqq 4$ and hence dim $\widetilde{H}\leqq 15$, which implies that dim $P/H>4$.

Case 4. rk L=3.

If L is maximal, then $L\sim C_1\times C_2$ and dim $\widetilde{H}\leqq 13$. If L is not maximal then $L\sim C_1\times C_1\times C_1$ and dim $U\leqq 5$, thus we have dim $P/H>4$. Since the proof of $P\neq B_3$ is completely similar, we omit its proof. This completes the proof of Proposition 2. 1.

3. Subgroups of P with codimension $\leqq 4$

In this section we assume a reductive group G acts on a 4-dimensional affine irreducible variety X quasi-homogeneously and almost effectively. We shall study proper observable subgroup of P with codim $_PH\leqq 4$.

We recall the following theorem of Birkes.

THEOREM. ([1])

(1) *Let G be a reductive algebraic group and $\rho: G\longrightarrow GL(V)$ a rational representation. If G_x contains a maximal torus of G, then $G(x)$ is closed (we call this property for G the property B).*

(2) *Let an algebraic group G act on an affine variety X. If G has the property B, then $G(x)$ is closed for x such that G_x contains a maximal torus of G.*

We have the following

PROPOSITION 3. 1 *Let $D=Sp_2$ and C a proper observable subgroup of D with codim $_DC\leqq 4$. Then codim $_DC=4$ and there occur two possibilities*

i) $C\sim A_1\times A_1$

and

ii) *$C\sim C_2\times Rad_uC$ and D/C is an open orbit of an S-variety.*

PROOF. Let $C^0=L\cdot$ Rad C^0 be the Levi-decomposition.

Case 1. Rad $C^0=1$.

In this case we have $C^0=L$. Since rk $C^0\leqq 2$, L is locally isomorphic to A_1 or $A_1\times A_1$ and this implies that $L\sim A_1\times A_1$, because dim $D/C\leqq 4$.

Case 2. Rad $C^0\neq 1$.

Subcase 1. L=1.

In this case $C^0=T\cdot C_u{}^0$. It follows from the table in section 1 that dim $C_u{}^0\leqq 4$. Since dim $C\geqq 6$, rk $C^0=2$ and hence $C_u{}^0$ is a maximal unipotent subgroup, which implies that C^0 is a Borel subgroup. This contradicts to the fact C^0 is observable.

Subcase 2. L$\sim$SL$_2$.

In this case we have $2\leqq$dim Rad $C^0\leqq 6$ and rk *R*ad $C^0\leqq 1$. Assume rk Rad $C^0=1$. Then C^0 is of maximal rank. It follows from the theorem of Birkes that D/C^0 is affiine and hence C^0 is reductive which implies that dim $C^0=4$. This contradicts to the assumption that dim $D/C\leqq 4$. Next assume rk Rad $C^0=0$. Since dim $C=$dim $L+$dim Rad $C^0\geqq 6$, we have dim Rad $C^0\geqq 3$. Since Rad C^0 is unipotent and dimension of a maximal unipotent subgroup of Sp_2 is 4, dim Rad C^0 must be 3. Thus C^0 contains a maximal unipotent subgroup of D and hence D/C is an open orbit of an S-variety. This completes the proof.

In the following T^k denotes a k-dimensional torus, U a unipotent group, G_m the multiplicative group k^*, and G_a the additive group k.

PROPOSITION 3.2. *Let $D=SL_4$ and C a proper observable subgroup of D with codim $C\leqq 4$. Then codim $C=4$ and there occur the following two possibilities.*

i) *$C^0=L\cdot(T\cdot U)$, where $L\sim A_1\times A_1$ and dim $U=4$. D/C is an open orbit of an S-variety.*

ii) *$C^0=L\cdot U$, wher $L\sim A_1$ and dim $U=3$. D/C is an open orbit of an S-variety.*

PROOF. Let $C^0=L\cdot R$ ad C^0 be the Levi-decomposition.

Case 1. Rad $\mathbf{C^0=1}$.

In this case C^0 is semi-simple dim $C\leqq 10$ and hence codim $C\geqq 5$.

Case 2. Rad $\mathbf{C^0\neq 1}$.

Subcase 1. L=1.

In this case C^0 is solvable. Since a Borel subgroup of L is of dimension 9, codim C^0 $\geqq 6$.

Subcase 2. rk L=1.

In this case $L\sim A_1$ and hence rk Rad $C^0\leqq 2$. Put Rad $C^0=T\cdot U$. Since a maximal unipotent subgroup of SL_4 or SL_2 is of dimension 6 or 1 respectively, dim $U\leqq 5$. If rk C^0 $=2$, then dim $C\leqq 3+2+5=10$. If rkRad $C^0=1$, then dim $C\leqq 3+1+5=9$. Thus we have codim $C\geqq 5$.

Subcase 3. rk L=2.

In this case $L\sim A_1\times A_1$ or A_2. Assume $L\sim A_1\times A_1$. Then rk Rad $C^0\leqq 1$. Put Rad C^0 $=T\cdot U$. Then dim $U\leqq 4$ and hence dim $C\leqq 6+1+4=11$. The equality holds if Rad C^0 $=T\cdot U$ where dim $U=4$, which implies C^0 contains a maximal unipotent subgroup. Thus D/C is an open orbit of an S-variety.

Next assume $L\sim A_2$. Then rk Rad $C^0\leqq 1$ and dim $U\leqq 3$, andhence dim $C^0\leqq 8+1+3$ $=12$. It is clear that codim $C\leqq 4$ if and only if $C^0=L\cdot U$ or $L\cdot(T\cdot U)$, where dim $U=3$. Since $L\cdot(T\cdot U)$ is not observable, we have codim $C=4$ and $C^0=L\cdot U$ where U is of dim 3. Thus D/C is an open orbit of an S-variety. This completes the proof.

PROPOSITION 3.3. *Let $D=SL_3$ and C proper observable subgroup of D with codim $C\leqq 4$. Then we have the following two possibilities;*

i) *codim* $C=4$.

a) $C\sim A_1\times T^1$

b) $C\sim A_1\times U$, *dim* $U=1$

c) $C\sim T^1\times U$, dim $U=3$

ii) *codim* $C=3$ *and* $C\sim A_1\times U$, *dim* $U=2$.

PROOF. Let $C^0=L\cdot \text{Rad } C^0$ be the Levi-decomposition.

Case 1. L=1.

In this case C^0 is solvable and hence dim $C\leqq$ dim. of a Borel subgroup$=5$. Since C^0 is not a Borel subgroup, we have dim $C=4$. Put $C^0=T\cdot U$. Then dim $T=1$ and dim $U=3$ or dim $T=2$ and dim $U=2$. Since C^0 is observable, we have dim $T=1$ and dim $U=3$.

Case 2. $\mathbf{L\sim A_1}$.

In this case C^0 has the radical of rank 1 or 0.

Subcase 1. rk Rad $\mathbf{C^0=1}$.

In this case C^0 is reductive, since C^0 contains a maximal torus of D and hence Rad C^0 is a torus, which implies that dim $C=4$.

Subcase 2. rk Rad $\mathbf{C^0=0}$.

In this case Rad C^0 is unipotent. It is easy to see that dim Rad $C^0\leqq 2$, and hence dim $C=3$, 4 or 5. Thus we have $C\sim SL_2\cdot U$, where dim $U=1, 2$. This completes the proof.

PROPOSITION 3. 4. *Let* $D=SL_2\times SL_2$ *and* C *a proper observable subgroup of* D *with codim* $C\leqq 4$. *Then we have the following three possibilities;*

i) *codim* $C=4$.

a) $C\sim SL_2\times G_m\times G_m$ *and* D/C *is affine*

b) $C^0=T^2\cdot U$, *dim* $U=3$ *and* D/C *is an open orbit of an S-variety.*

c) $C\sim SL_2\times G_m\times G_a$

d) $C\sim SL_2\cdot C_u$, *dim* $C_u=2$.

ii) *codim* $C=3$.

a) $C\sim SL_2\times SL_2$

b) $C\sim SL_2\times G_m\times C_u$, *dim* $C_u=2$

iii) *codim* $C=2$.

a) $C\sim SL_2\times SL_2\times G_m$

b) $C\sim SL_2\times SL_2\times G_a$.

PROOF. Let $C^0=L\cdot \text{Rad } C^0$ be the Levi-decomposition.

Case 1. Rad $\mathbf{C^0=1}$.

Since rk $C^0\leqq 3$ and codim $C\leqq 4$, we have $C^0\sim SL_2\times SL_2$.

Case 2. Rad $C^0 \neq 1$.

Subcase 1. L=1.

In this case $C^0 = T \cdot C_u{}^0$. Since a maximal unipotent subgroup of D is of dimension 3, we have dm $C_u \leqq 3$. Moreover since dim $C \geqq 5$, we have $2 \leqq$ rk $C^0 \leqq 3$. Assume rk C^0 $=3$. It follows from Birkes' theorem that D/C is affine and hence C is reductive and C_u^0 $=1$, which contradicts to the fact codim $C \leqq 4$. Assume rk $C^0 = 2$. Then we have dim $C_u{}^0 \geqq 3$ and hence $C^0 = T \cdot C_u{}^0$ where dim $C_u = 3$.

Subcase 2. L$\sim SL_2$.

Since dim $C = 3 +$ dim Rad C^0, we have $2 \leqq$ dim Rad $C^0 \leqq 6$. Clearly rk Rad $C^\circ \leqq 2$ and $0 \leqq$ dim Rad ${}_uC^0 \leqq 2$. Assume rk Rad $C^0 = 2$. Since rk $C =$ rk D, Birkes' theorem implies that Rad $C^0$is 2-dimensional torus and hence dim $C \geqq 5$. Assume rk Rad $C^0 = 1$. Clearly dim Rad ${}_uC^0 \neq 0$. If dim Rad ${}_uC^0 = 1$ or 2, then Rad $C^0 = T.G_a$ or C^0 contains a maximal unipotent subgroup of D, respectively. Assume rk Rad $C^0 = 0$. Since Rad $C^0 =$ Rad ${}_uC^0$, dim $C^0 \geqq 5$ and dim Rad ${}_uC^0 \leqq 2$, we have dim Rad ${}_uC^0 = 2$. Thus C^0 contains a maximal unipotent subgroup of D and hence D/C is an open orbit of an S-variety.

Subcase 3. L$\sim SL_2$.

Since dim $C^0 = 6 +$ dim Rad C^0, we have $1 \leqq$ dim Rad $C^0 \leqq 2$ and $0 \leqq$ rk Rad $C^0 \leqq 1$. Clearly dim Rad ${}_uC^0 \leqq 1$. Assume rk Rad $C^0 = 1$. Then rk $C^0 = 3$ and hence C^0 is reductive. This implies Rad $D^0 = G_m$ and dim $C = 7$. Assume rk Rad $C^0 = 0$. Then Rad C^0 $= G_a$ and C^0 contains a maximal unipotent subgroup of D. This completes the proof.

PROPOSITION 3.5. *Let $D = SL_2 \times SL_3$ and C a proper observable subgroup of D with codim $C \leqq 4$. Then we have the following three possibilities;*

i) *codim $C = 4$.*
  a) $C \sim SL_2 \times G_m \times Rad_u\, C^0$
  b) $C \sim SL_2 \times SL_2 \times G_m$
  c) $C \sim SL_2 \times SL_2 \times Rad_u C^0$

ii) *codim $C = 3$.*
  a) $C \sim SL_3$
  b) $C \sim SL_2 \times SL_2 \times Rad_u C^0$

iii) *codim $C = 2$*
  a) $C \sim SL_2 \times G_m$
  b) $C \sim SL_3 \times G_a$.

PROOF. Let $C^0 = L \cdot$ Rad C^0 be the Levi-decomposition.

Case 1. Rad $C^0 = 1$.

In this case it is clear that $C \sim SL_3$.

Case 2. Rad $C^0 \neq 1$.

Subcase 1. L=1.

Put $C^0 = T \cdot C_u{}^0$. Cleary dim $C_u{}^0 \leqq 4$. Since dim $C \geqq 7$, we have rk $C^0 \geqq 3$ and hence rk $C^0 = 3$. Since rk $C^0 =$ rk D and D/C is affine, we have that C^0 is reductive, which is impossible.

Subcase 2. $L \sim SL_2$.

Since dim $C^0 = 3 +$ dim Rad C^0 and dim $C^0 = 7, 8, 9, 10$, we have $4 \leqq$ dim Rad $C^0 \leqq 7$. Moreover since rk $C^0 \leqq 3$, we have rk Rad $C^0 \leqq 2$. Assume rk Rad $C^0 = 2$. Then we have Rad $C^0 = T$ and dim $C^0 = 5$, which contradicts to our assumption. Assume rk Rad $C^0 = 1$. Put Rad $C^0 = T \cdot$Rad ${}_uC^0$. Cleary $3 \leqq$ dim Rad${}_uC^0 \leqq 6$ and dim Rad $C^0 = 3$. Assume rk Rad $C^0 = 0$. Then we have Rad $C^0 =$ Rad ${}_uC^0$ and dim Rad ${}_uC^0 \geqq 4$, which is impossible.

Subcase 3. $L \sim SL_3$.

It is easy to see that $1 \leqq$ dim Rad $C^0 \leqq 2$ and rk Rad $C^0 \leqq 1$. Assume rk Rad $C^0 = 1$. Then we have rk $C^0 = 3$ and hence C^0 is reductive, which implies $C^\circ \sim SL_3 \cdot G_m$. Assume rk Rad $C^0 = 0$. Then we have Rad $C^0 =$ Rad ${}_uC^0$ and $1 \leqq$ dim Rad ${}_uC^0 \leqq 2$. Since a maximal unipotent subgroup of D is of dimension 3, we have dim Rad${}_uC^0 = 1$ and hence dim $C^0 = 1$ and hence dim $C^0 = 9$. Thus D/C is an open orbit of an S-variety.

Sulcase 4. $L \sim SL_2 \times SL_2$.

Clearly we have $1 \leqq$ dim Rad$C^0 \leqq 4$ and rk Rad $C^0 \leqq 1$. Assume rk Rad $C^0 = 1$. Then we have rk $C^0 = 3$, and hence C^0 is reductive and Rad$C^0 = G_m$. Assume rk Rad $C^0 = 0$. Then we have $1 \leqq$ dim Rad${}_uC^0 \leqq 2$. If dim Rad${}_uC^0 = 1$ or 2, then $C^0 \sim SL_2 \times SL_2 \times$ Rad${}_uC^0$ or $C^\circ \sim SL_2 \times SL_2 \times$ Rad${}_uC^0$ respectively. This completes the proof.

4. 4-dimensional quasi-homogeneous space X of a reductive group G

At first we state some results about S-varieties of a connected linear algebraic group G which are used in the sequel.

We say that an irreducible affine variety X is an S-variety of G provided there is an open G-orbit O_X such that for any x of O_X the isotropy subgroup G_x contains a maximal unipotent subgroup of G. Clearly G may be assumed to be reductive. Let X be an S-variety of G.

(1) There are a rational representation $\rho : G \longrightarrow GL(V)$ and an equivariant embedding $\sigma : X \longrightarrow V$ such that $\sigma(X)$ is closed in V. Identify $\sigma(X)$ to X. Choose an element v of X such that $G(v)$ is open and G_v contains a maximal unipotent subgroup N of G. Let B be a Borel subgroup of G containning N. By considering V as a B-space, we have $v = v_1 + \dots + v_k$ where each v_i is the highest weight vector of an irreducible invariant subspace

V_i with the highest weight Λ_i and $V=V_1 \oplus \cdots\cdots \oplus V_k$. Then $X=\overline{G(v)}$ and we denote $X=X(\Lambda_1, \ldots\ldots, \Lambda_k)$. Moreover it is known ([7], Th. 6) that $k[X(\Lambda_1, \ldots\ldots, \Lambda_k)]=\sum S_M$ (summing up over $M \in \{\Lambda_1, \ldots\ldots, \Lambda_k\}$ = the semi-group with identity generated by $\Lambda_1, \ldots \ldots, \Lambda_k$) where S_M is the eigenspace with the eigenvalue M under the representation in $k[V]$ contragradient to ρ.

(2) We decompose $G=P\times Z$ in the direct product of a simply-connected semi-simple group P and the connected center Z. Under the above notations, let $H=G_v$, and let $\pi_i: V_i-\{0\} \longrightarrow PV_i$ (the projective space) be the canonical mapping. Consider $\pi=\pi_1\times\pi_2\times\cdots\cdots\times\pi_k: \prod(V_i-0)\longrightarrow\prod PV_i$. Then G acts naturally on $\prod PV_i$, and if we denote $G_{\pi(v)}$ by Q, Q clearly contains B. Let $\underline{p}$ be the Lie algebra of P, and choose a set of generators $\{h_i, e_i, f_i\}$ $i=1, 2, \ldots\ldots, l$ (l=rank P) such that (1) $h_1, \ldots\ldots, h_l$ form a basis of the Lie algebra $\underline{t}$ of the maximal torus of B, and (2) each e_i (or f_i) is a root vector corresponding to a positive (or negative) simple root. Then the Lie algebra $\underline{b}$ of B is generated by $\{h_i, e_i\}$ $i=1, 2, \ldots\ldots, l$, and the Lie algebra $\underline{q}$ of Q is generated by b and some of the f_i's. Let $E=\{i \mid f_i \in q\}$, we write $Q=Q_E$.

From now on, let G be a reductive group and P the semi-simple Levi factor of G.

PROPOSITION 4.1. *If rk $G=4$, G and X are 4-dimensional tori, and G acts on X by left translaton.* ([8]).

PROPOSITION 4.2. *If $P=SL_2\times SL_2\times SL_2$, then G is isomorphic to P and X is a non-homogeneous S-variety of G.*

PROOF. It follows from proposition (1.5) that G is isomorphic to P. Let $O_X=P/P_x$, and $P_x^0=L\cdot \mathrm{Rad}\, P_x^0$. Because of dim $P_x=5$, it is shown by proposition (3.4) that L must be isomorphic to SL_2 and Rad P_x^0 to one of the followings, i) $G_m\times G_m$, ii) $G_m\cdot G_a$, iii) $\mathrm{Rad}_u P_x^0$ and iv) $G_m\cdot (P_x^0)_u$.

Case i). Since rk P_x=rk P, L must be isomorphic to one of the factors of G. But this fact contradicts to almost effectivity of our action.

Case ii). Let $\varphi_i: L\longrightarrow P\longrightarrow SL_2^{(i)}$, $\psi_i: G_m\longrightarrow P\longrightarrow SL_2^{(i)}$, and $\eta_i: G_a\longrightarrow P\longrightarrow SL_2^{(i)}$ be the compositions of the inclusions and the i-th projections, $i=1, 2, 3$. We may assume that φ_1 is non-trivial. Moreover φ_2 may be also assumed to be non trivial. In fact, if both φ_2 and φ_3 are trivial, the subgroup L of P_x must contain one of the factors of G, contradicting to almost effectivity.

Then ψ_1 must be trivial. Assume that ψ_1 is non-trivial and consider the homomorphism $\Phi: L\cdot G_m\longrightarrow SL_2\times SL_2$ defined by $\Phi(l\cdot g)=(\varphi_1(g)\varphi_1(g), \varphi_2(l)\varphi_2(l))$. It is shown that Ker Φ is finite. In fact, since $\mathfrak{L}$ (Ker Φ) is an ideal of $\mathfrak{L}(L\cdot G_m)=\mathfrak{L}(L)\oplus\mathfrak{L}(G_m)$ of the form $\underline{l_1}\oplus \underline{l_2}$, it follows that Ker $\Phi\simeq K\times K_2$ where $K_1\lhd L$ and $K_2\lhd G_m$. Clearly K_1 is finite. On the other hand if K_2 is not finite, we have $K_2=G_m$ contradicting to that ψ_1 is non-trivial. Hence $L\cdot G_m$ is locally isomorphic to a subgroup of $SL_2\times SL_2$. But, if follows from rk $L\cdot G_m=2$ that L is isomorphic to the factor $SL_2^{(1)}$ and so φ_1 is trivial. This is a contradiction.

The similar arguments show that ψ_2 must be also trivial, and hence ψ_3 is non-trivial.

Thus it is shown as above that φ_3 is trivial.

Next we shall show that η_1 must be trivial. Assume that η_1 is non-trivial. Then the similar arguments to the above show that the homomorphism $L\cdot G_a \longrightarrow SL_2^{(1)}\times SL_2^{(2)}$ defined as above has the finite kernel. Hence we may consider $L\cdot G_a$ as a subgroup of $SL_2^{(1)}\times SL_2^{(2)}$ which contain a maximal unipotent subgroup of $SL_2^{(1)}\times SL_2^{(2)}$. Under the notations stated above, let $G=SL_2\times SL_2$ and $G_v=G=L\cdot G_a$. Since Q_E is a subgroup with maximal rank containing H, we have $Q_E=Q_1\times Q_2$. But it follows from $L\subset Q_E$ that at least one of the Q_i's must be isomorphic to SL_2, that is, L is one of the factors $SL_2^{(i)}$ of G containning such Q_i. This contradicts to almost effectivity of our action. Thus η_1 is trivial.

Similarly η_2 is shown to be also trivial. Hence η_3 must be non-trivial.

Now consider the restricted action on $SL_2^{(3)}$. Then it is easy to see that the isotropy subgroup is Rad $P_x^0=T\cdot G_a$ and the orbit space $SL_2^{(3)}/T\cdot G_a$ is a projective variety. This contradicts to affinness.

Case iii). In this case P_x contains a maximal unipotent subgroup of P and we have $Q_E=Q_1\times Q_2\times Q_3$ where $Q_i\subset SL_2^{(i)}$, since Q_E is a subgroup with maximal rank. Hence it follows from $P_x\lhd Q_E$ that P_x^0 must be decomposed into a direct sum. On the other hand $L\subset P_x^0\subset Q_E$. Hence we see that P_x^0 must be a factor of P. This is impossible.

Case iv). In this case P/P_x is an open orbit of S-variety X. Because of $Q_E\rhd P_x$, it is impossible that $P=Q_E$. Hence Q_E is a Borel subgroup of P. So, it follows from ([7], (36)) that $X-O_X\neq\phi$, and hence X is not homogeneous. This completes the proof.

PROPOSITION 4. 3. *The case $P=SL_2\times SL_3$ cannot occur.*

PROOF. It follows from proposition (1. 2) that G is isomorphic to P. Let $O_X=P/P_x$ and $P_x^0=L\cdot \text{Rad}\, P_x^0$. Because of dim $P_X=7$, it is shown in proposition (3. 5) that there are only three cases as follows; i) $L\sim SL_2$ and Rad $P_x^0=G_m\cdot$ Rad P_x^0 where dim $\text{Rad}_u P_x^0=3$, ii) $L\sim SL_2\times SL_2$ and Rad $P_x^0=G_m$, and iii) $L\sim SL_2\times SL_2$ and Rad $P_x^0=\text{Rad}_u P_x^0$ where dim $\text{Rad}_u P_x^0=1$.

Clearly the case ii) is impossible.

Case i). Since P_x contains a maximal unipotent subgroup of P, X is an S-variety. Clearly Q_E is of the form $P_1\times P_2$ where $P_1\subset SL_2$ and $P_2\subset SL_3$. Since P_x is a nomal subgroup of Q_E, it follows that $P_x^0\sim Q_1\times Q_2$ where $Q_i\subset P_i$. $Q_1\neq SL_2$, otherwise P_x contains a factor SL_2 of P, contradicting to almost effectivity. On the other hand $Q_2\not\sim SL_2\times G_m$, otherwise it follows from $P_x^0\sim SL_2\times G_m\times U$ ($U=\text{Rad}\ {}_uP_x^0$) that Q_1 is locally isomorphic to 3-dimensional unipotent subgroup U (this is impossible).

Let $N=\text{Ker}\,(SL_2\times G_m \longrightarrow P_x \longrightarrow P \longrightarrow SL_3)$. Since N is normal in $SL_2\times G_m$, the image $\pi(N)$ of N by the projection $\pi : SL_2\times G_m \longrightarrow SL_2$ is normal in SL_2 and hence $\pi(N^0)$ is either $\{1\}$ or SL_2.

Assume $\pi(N^0)=SL_2$. Then we have $3\leqq \dim N\leqq 4$, because of $\dim N=\dim \pi(N)+\dim G_m\cap N=\dim N\cap SL_2+\dim \pi'(N)$, where $\pi' : SL_2\times G_m \longrightarrow G_m$ the projection. If dim $N=3$, we have $N^0\cong\pi(N^0)\cong SL_2$. And if dim $N=4$, as it follows from dim $N\cap SL_2=3$ that

$N \cap SL_2$ and hence $N \subset SL_2$, we have $N^0 = SL_2 \times \{1\}$. In both cases $SL_2 \longrightarrow SL_3$ is trivial, this is impossible.

Thus we saw that $\pi(N^0) = \{1\}$ and hence $N^0 \cap G_m = N^0$. From this and $N^0 \cap SL_2 = \{1\}$ it follows that $N^0 \cong G_m$ and hence $G_m \longrightarrow SL_3$ must be trivial. On the other hand, because of $U \lhd P_x^0$ we have $U = U_1 \times U_2$ where U_1 is an 1-dimensional subgroup of Q_1 and U_2 is a 2-dimensional one of U_2. From these we have $Q_1 = G_m \times U_1$ and $Q_2 = SL_2 \times U_2$. Therefore we have an isomorphism $P/P_x \cong SL_2/G_m \cdot U_1 \times SL_3/(SL_2 \times U_2)$, but the first term $SL_2/G_m \cdot U_1$ is projective. This case is impossible.

Case iii). Let $P_x \sim SL_2^{(1)} \times SL_2^{(2)} \times N$, $\varphi_1 : SL_2^{(1)} \longrightarrow P \longrightarrow SL_2$, $\varphi_2 : SL_2^{(1)} \longrightarrow P \longrightarrow SL_3$, $\psi_1 : SL_2^{(2)} \longrightarrow P \longrightarrow SL_2$, $\psi_2 : SL_2^{(2)} \longrightarrow P \longrightarrow SL_3$, $\eta_1 : N \longrightarrow P \longrightarrow SL_2$, and $\eta_2 : N \longrightarrow P \longrightarrow SL_3$. If φ_2 is non-trivial, it is clear that a factor $SL_2^{(1)}$ of P_x must be isomorphic to some factor of P and hence the ineffective kernel of P must contain SL_2. This is a contradiction.

Thus φ_2 is not trivial and similarly ψ_2 is also shown to be non-trivial. On the other hand, since the kernel of the homomorphism $SL_2^{(1)} \times SL_{(2)}{}^2 \longrightarrow SL_3$ is normal in $SL_2^{(1)} \times SL_2^{(2)}$, it is either a finite group or some factor. But none of these cases is possible. This completes the proof.

PROPOSITION 4. 4. *The case $P = Sp_2 \times SL_2$ can be reduced to the case $P = Sp_2$.*

PROOF. Consider the restricted action on Sp_2, it follows from proposition (3. 1) that any proper observable subgroup of Sp_2 satisfying $\mathrm{codim}_{Sp_2} C \leqq 4$ is of codimension 4. Hence Sp_2 acts on X quasi-transitively.

PROPOSITION 4. 5. *If $P = SL_4$, X is an S-variety of SL_4 which is not homogeneous.*

PROOF. It follows from proposition (3. 2) that P_x is isomorphic to either $SL_2 \times SL_2 \times G_m \times U$ or $SL_3 \times U$. If $P_x \sim SL_2 \times SL_2 \times G_m \times U$, then we have $\mathrm{rk}\, P_x = \mathrm{rk}\, P$. Hence from theorem in [1] it folloows that P_x is reductive. This is a contradiction. Thereforo P_x is isomorphic to $SL \times U_3$ where $\dim U = 3$. Since a parabolic group containning P_x is of dimension 12, it follows from ([7], 36)) that $X - O_X$ consists of one point.

PROPOSITION 4. 6. *In the case $P = SL_2 \times SL_2$, there occurs the following cases*; 1) $G = SL_2 \times SL_2$ *and X is homogeneous, and* 2) $G = SL_2 \times SL_2 \times G_m$ *and X is either a homogeneous variety or a non-homogeneous S-variety.*

PROOF. Let $G = P \cdot \mathrm{Rad}\, G$, then we have $\dim \mathrm{Rad}\, G \leqq 1$ because $\mathrm{rk}\, G \leqq 4$ and $\mathrm{Rad}\, G$ is a torus by our assumption that G is a reductive group.

In the case $\mathrm{Rad}\, G = \{1\}$, it is clear that the subgroup of codimension 4 of $SL_2 \times SL_2$ is a maximal torus. Hence X is homogeneous.

In the case $\mathrm{Rad}\, G \neq \{1\}$, clearly $\mathrm{Rad}\, G = G_m$ and $\dim G_x = 3$. Consider the projection $\pi : G \longrightarrow G_m$. It induces the morphism $G/G_x \longrightarrow G_m/\pi(G_x)$ with fibre P/P_x of dimension $\geqq 2$. Hence by considering the restricted P-action, we have $m_P(X) \geqq 3$. If $m_P(X) = 4$, it is clear that P acts transitively on X. If $m_P(X) = 3$, it follows from $\dim P_x = 3$ that P_x is locally isomorphic to either SL_2 or $G_m \times N$ where N is a 2-dimensional unipotent gpoup. The similar arguments to above show that if $P_x \sim G_m \times N$, X is an S-variety and $X - O_X$ is not empty, and if $P_x \sim SL_2$, X is not an S-variety and $X = O_X$. This completes the proof.

PROPOSITION 4.7. *In the case $P=SL_3$, there occur the following;* 1) *$G=SL_3$ and X is a hemogeneous variety, and* 2) *$G=SL_3\times G_m$ and X is a non-homogeneous S-variety.*

POPOSITION 4.8. *In the case $P=SL_2$, there occur the followings;* 1) *$G=SL_2\times G_m$ and X is a homogeneous variety, and* 2) *$G=SL_2\times G_m\times G_m$ and X is either a homogeneous S-variety.*

Propositions (4.7) and (4.8) are proved in the same way as proposition (4.6).

PROPOSITION 4.9. *If $P=Sp_2$, X is a non homogeneous S-variety.*

PROOF. It is shown as in (4.4) that $G=Sp_2$. Since we have $G_x\sim SL_2\times U$ where U is a 3-dimensional unipotent group, it follows that the parabolic subgroup of Sp_2 containning G_x is of 7-dimensional and hence $X-O_X\neq\phi$. Q.E.D.

Summing up the results in this section. Let $G=P$ Rad G, there are nine cases at follows:

1)	$G=P=SL_4$	X an S-variety	$X-O_X\neq\phi$
2)	$G=P=SL_2\times SL_2\times SL_2$	X an S-variety	$X-O_X\neq\phi$
3)	$G=P=SL_3$	X homogeneous	
4)	$G=SL_3\times G_m$	X an S-variety	$X-O_X\neq\phi$
5)	$G=P=Sp_2$	X an S-variety	$X-O_X\neq\phi$
6)	$G=SL_2\times SL_2$	X homogeneous	
7)	$G=SL_2\times SL_2\times G_m$	X homogeneous	
8)	$G=SL_2\times G_m\times G_m$	X homogeneous or an S-variety	
9)	$G=SL_2\times G_m$	X homogeneous	

5. 4-dimensional homogeneous spaces

In this section we consider homogeneous affine spaces. In the preceding section it is shown that there are only five cases as follows; $G=SL_3$, $SL_2\times SL_2$, $SL_2\times SL_2\times G_m$, $SL_2\times G_m\times G_m$, $SL_2\times G_m$.

Case 1. G=SL$_3$.

Since SL_3 has only one 4-dimensional reductive subgroup $N(SL_2, SL_3)$, we have $G_x^0=N(SL_2, SL^3)$. The following proposition shows that $G_x=N(SL_2, SL_3)$.

PROPOSITION 5.1. $N(SL_2, SL_3)$ is a maximal subgroup of SL_3.

Indeed, it is shown directly that, if g is any element of SL_3 satisfying $gN(SL_2, SL_3)g^{-1}\subseteqq N(SL_2, SL_3)$, then g belongs to $N(SL_2, SL_3)$.

Case 2. G=SL$_2\times$SL$_2$.

In this case $G_x^0=G_m\times G_m$. Because of $N(G_m, SL_2)/G_m=Z_2$ and $G_x\subseteqq N(G_m\times G_m, G)$, G_x/G_x^0 is a subgroup of $N(G_m\times G_m, G)/(G_m\times G_m)=Z_2\times Z_2$ and hence it is one of the followings; 1×1, $Z_2\times 1$, $1\times Z_2$, Z_2 (diagonal), $Z_2\times Z_2$. Thus we see that X is one of the followings; $SL_2\times SL_2/G_m\times G_m$, $SL_2/N\times SL_2/G_m$, $SL_2/G_m\times SL_2/N$, $(SL_2/G_m\times SL_2/G_m)/Z_2$, $SL_2/N\times SL_2/N$, where $N=N(G_m, SL_2)$.

Case 3. $\mathbf{G=SL_2\times SL_2\times G_m}$.

In this case G_x is a 3-dimensional reductive group. Let $G_x{}^0=L\cdot \mathrm{Rad}\ G_x{}^0$, then L can not be $\{1\}$. Otherwise $G_x{}^0=G_m\times G_m\times G_m$, contradicting almost effectivity. Hence it is shown that L is isomorphic to SL_2, so that $G_x{}^0$ is isomorphic to SL_2. This implies that one of two morphisms $G_x{}^0\longrightarrow G\longrightarrow SL_2{}^{(i)}$ ($i=1, 2$) must be an isomorphism, say such $i=1$. Then we have the commutative diagram

$$\begin{array}{ccccc} SL_2{}^{(2)}\times G_m & \longrightarrow & G & \longrightarrow & SL_2{}^{(1)} \\ \cup & & \cup & & \cup \\ G_x\cap(SL_2{}^{(2)}\times G_m) & \longrightarrow & G_x & \longrightarrow & SL_2{}^{(1)} \end{array}$$

and hence there is an isomorphism between $(SL_2{}^{(2)}\times G_m)/(G_x\cap(SL_2{}^{(2)}\times G_m))$ and G/G_x. Therefore our case can be reduced to the case $G=SL_2\times G_m$.

Case 4. $\mathbf{G=SL_2\times G_m\times G_m}$.

Since $\dim G_x=1$, $G_x{}^0$ is a torus. If $\dim (SL_2\times G_m{}^{(i)})\cap G_x=0$ for $i=1, 2$, then $SL_2\times G_m{}^{(i)}$ acts transitively on X. Thus this case can be reduced to the case $G=SL_2\times G_m$. Hence we may assume that $\dim (SL_2\times G_m{}^{(i)})\cap G_x=1$ for $i=1$ and 2.

PROPOSITIO 5. 2. *Let K and H be algebraic subgroups of G and let $H\subset K$. Then the natural mophism $G/H\longrightarrow G/K$ is a fiber space associated to $G\longrightarrow G/K$.*

Indeed, since $G/H=G\times_K K/H$, this follows from the following results of J. P. Serre.

i) Let H be an algebraic subgroup of an algebraic group G and let $L=G/H$ be the homogeneous space. Then (H, G, L) is a principal fibre space ([9]. Prop. 3).

ii) Let P be a principal fiber space of H. If $G\longrightarrow G/H$ is locally trivial, then $P\times_G G/H\longrightarrow P/H$ is a locally trivial fiber space ([9] Prop. 8).

Applying this proposition to $G=G'\times G_m{}^{(2)}$ (let $G''=SL_2\times G_m{}^{(1)}$), $K=pr_1(H)\times G_m{}^{(2)}$ ($pr_1: G\longrightarrow G'$ the projection) and $H=G_x$, we have the fibering

$$(\mathrm{pr}_1(H)\times G_m{}^{(2)})/H\longrightarrow(G'\times G_m{}^{(2)})/H\longrightarrow(G'\times G_m{}^{(2)})/(\mathrm{pr}_1(H)\times G_m{}^{(2)})$$

where

$$(G'\times G_m{}^{(2)})/H=(G'\times G_m{}^{(2)})\underset{\mathrm{pr}_1(H)\times G_m{}^{(2)}}{\times}(\mathrm{pr}_1(H)\times G_m{}^{(2)})/H.$$

i) There is an isomorphism φ: $(\mathrm{pr}_1(H)\times G_m{}^{(2)})/H\simeq G_m{}^{(2)}/(H\cap G_m{}^{(2)})$.

PROOF. Define $\varphi([(x, g)])=[g_2 g_x]$ where g_x is an element of $G_m{}^{(2)}$ such that (x, g_x) belongs to H. If (x, g_x') is another element belonging to H, then $g_x{}^{-1}g_x'\in H\cap G_m{}^{(2)}$, because of $(x, g_x)^{-1}(x, g_x')=(x^{-1}, g_x{}^{-1})(x, g_x')=(1, g_x{}^{-1}g_x')\in H$. Therefore $[g_2 g_x]=[g_2 g_x']$, that is, our definition of φ is independent of the choice of g_x, since $(g_2 g_x)^{-1}(g_2 g_x')=g_x{}^{-1}g_x'$. On the other hand, if $[(x, g_2)]=[(x', g_2')]$, that is, $(x, g_2')^{-1}(x', g_2')=(x^{-1}x', g_2{}^{-1}g_2')\in H$, then $(g_2 g_x)^{-1}(g_2' g_x')=g_x{}^{-1}g_x' g_2{}^{-1}g_2'\in H\cap G_m{}^{(2)}$ and hence $[g_2 g_x]=[g_2' g_x']$. Therefore it was shown for our to be well defined.

From the definition φ is clearly surjective. On the other hand if $[g_2 g_h]=1$, i.e. $g_2 g_x$

$\in H$, then $g_2 \in H \cap G_m^{(2)}$ since $(x, g_2)=(x, 1)(1, g_2) \in H$, and hence $[(x, g_2)]=1$. Thus φ is an isomorphism.

ii) There is a $\mathrm{pr}_1(H)$-action on $G' \times (G_m^{(2)}/H \cap G_m^{(2)})$.

PROOF. Define $x(g', \overline{t})=g'x^{-1}, \overline{g_x t})$ for $x \in \mathrm{pr}_1(H)$ and $(g', \overline{t}) \in G' \times (G_m^{(2)}/H \cap G_m^{(2)})$. If (x, g_x) and (x, g_x') are two elements of H, then we have $(g_x t)^{-1}(g_x' t)=g_x^{-1} g_x' \in H \cap G_m^{(2)}$ and hence the above definition is independent of the choice of g_x.

iii) There is an isomorphism between $(G' \times G_m^{(2)})/H$ and $G' \times_{\mathrm{pr}(H)} G_m^{(2)}/(H \cap G_m^{(2)})$.

PROOF. Clearly

$$(G' \times G_m^{(2)})/H \cong (G' \times G_m^{(2)}) \times_{\mathrm{pr}_1(H) \times G_m^{(2)}} (\mathrm{pr}_1(H) \times G_m^{(2)})/H.$$

We can define a morphism

$$\psi : (G' \times G_m^{(2)}) \times (\mathrm{pr}_1(H) \times G_m^{(2)})/H \longrightarrow G' \times (G_m^{(2)}/H \cap G_m^{(2)})$$

by $\psi\{((g, t), (x, s))\}=(g, \overline{ts g_x})$. In fact, it is shown in the same way as in i) that ψ is independent of the choice of g_x. On the other hand, if $(\overline{x, s})=(\overline{x', s'})$, i.e. $(x^{-1}x', s^{-1}s') \in H$, then it follows that $g_{x^{-1}x'}=s^{-1}s'$ and hence $(tsg_x)^{-1}(ts' g_x')=g_x^{-1} g_{xx'} g_x' \in H \cap G_m^{(2_2)}$. Thus ψ is well defined.

Next we shall see that this ψ is equivariant. Indeed, for $(x, u) \in \mathrm{pr}_1(H) \times G_m$, we have $\psi\{(x, u)((g. t), (\overline{y, s}))\}=\psi((gx^{-1}, tu^{-1}), (\overline{xy, us}))=(gx^{-1}, \overline{tu^{-1} usg_{xy}})=(gx^{-1}, \overline{tsg_x g_y})=x(g, \overline{tsg_y})$.

Therefore ψ induces the morphism

$$\overline{\psi} : (G \times G_m^{(2)}) \times_{\mathrm{pr}_1(H) \times G_m^{(2)}} (\mathrm{pr}_1(H) \times G_m^{(2)})/H \longrightarrow G' \times G_m^{(2)}/H \cap G_m^{(2)},$$

which is clearly an isomorphism.

iv) Consequently $(G' \times G_m^{(2)})/H=G/H$ is a line bundle with zero section deleted over $G'/\mathrm{pr}_1(H)$. Thus we can also reduce our case to the case $G=SL \times G_m$.

Case 5. $\mathbf{G=SL_2 \times G_m}$

Clearly G_x is a finite group. It is shown that

$$(\mathrm{pr}(G_x) \times G_m)/G_x \longrightarrow (SL_2 \times G_m)/G_x \longrightarrow (SL_2 \times G_m)/(\mathrm{pr}(G_x) \times G_m)$$

is the fiber space associated to $G \longrightarrow G/(\mathrm{pr}(G_x) \times G_m)$. Thus, since we have $(\mathrm{pr}(G_x) \times G_m)/G_x \cong G_m/G_x$ and $(SL_2 \times G_m)/(\mathrm{pr}(G_x) \times G_m) \cong SL_2/\mathrm{pr}(G_x)$, it follows that $X=G/G_x$ is a line bundle with the zero section deleted over a 3-dimensional affine variety $SL_2/\mathrm{pr}(G_x)$.

But it is well known (for example, see [4]) that every finite subgroup of SL_2 is conjugate to one of the followings; i) cyclic group T_m of order m, $m=1, 2, \ldots\ldots$, ii) the binary dihedral group $\widetilde{D}_m$, $m=1, 2, \ldots\ldots$, iii) the binary tetrahedral group $\widetilde{T}$, iv) the binary octahedral group $\widetilde{O}$, and v) the binary icosahedral group $\widetilde{I}$. Here we employ the same notations as in [5]. The affine varieties S_3, S_4 and S_5 are, by definition, the homogeneous spaces $SL_2/\widetilde{T}$, $SL_2/\widetilde{O}$ and $SL_2/\widetilde{I}$, respectively. Let X_n be the line bundle over the affine variety $P^1\times P^1-\Delta=SL_2/G_m$ (see [6]) corresponding to $n\in\mathrm{Pic}\,(P^1\times P^1-\Delta)=Z$ and X_n^* $=X_n-$the zero section. Then it was shown in [5], section 6, that X_n^* is isomorphic to SL_2/T_n and $SL_2/\widetilde{D}_m$ is isomorphic to W_m which is, by definition, a quotient space of X_{2m}^* by a suitable involution.

Consequently the homogeneous space $X=(SL_2\times G_m)/G_x$ is a line bundle with the zero section deleted over a 3-dimensional affine variety which is isomorphic to one of the varieties, $X_n^*(n\neq 0)$, W_n, S_3, S_4 and S_5.

6. 4-dimensional quasi-homogeneous S-varieties

In this section we shall determine the 4-dimensional quasi-homogeneous S-varieties. In section 4, it was shown that there may occur only five cases as follows; $G=SL_4$, $SL_2\times SL_2\times SL_2$, SL_3, $SL_3\times G_m$, Sp_2, $SL_2\times G_m\times G_m$.

PROPOSITION 6.1. *The cases $G=SL_3\times G_m$ and $G=SL_2\times G_m\times G_m$ can not occur.*

PROOF. **Case $\mathbf{G=SL_3\times G_m}$.**

Clearly $H=G_x$ is 5-dimensional and dim $(H\cap SL_3)\geqq 4$. It is clear that the subgroups SL_3 of dimension larger than 3 are ones of the following types;

$$P=\left\{\begin{bmatrix} a & b & c\\ d & e & f\\ 0 & 0 & g\end{bmatrix}\right\},\quad Q=\left\{\begin{bmatrix} a & b & c\\ d & e & f\\ 0 & 0 & h\end{bmatrix};\ h_d=1\right\},\quad N=\left\{\begin{bmatrix} a & b & 0\\ c & d & 0\\ 0 & 0 & e\end{bmatrix}\right\}.$$

6-dimensional 5-dimensional 4-dimensional

Since X is an S-variety, H contains a maximal unipotent subgroup $U\times 1$ of G (here U is a maximal unipotent subgroup of SL_3). Hence $H\cap SL_3$ must contain Q. But $H\cap SL_3=Q$, otherwise $H\cap SL_3=P$ and hence dim $H\geqq 6$, contradicting to dim $H=5$. Now since P is a parabolic subgroup of SL_3 and $P'=P\times G_m\supsetneqq H$, it follows follows from [7] that, under the notations in [7], $\partial(G)=\dim G-\dim P'=2$ and hence $\mathrm{rk}_Q\,C=\dim X-\partial(G)=2$. From this it is impossible that X is quasi-homogeneous.

Case $\mathbf{G=SL_2\times G_m\times G_m}$.

Let $H=G_x$. As above H must contain a maximal unipotent subgroup $U\times 1\times 1$ of G(here U is such one of SL_2). Hence $H^0=U\times 1\times 1$ because of dim $H=1$. Similarly, since $P'=U\times G_m\times G_m$, $\partial(C)=2$ and $\mathrm{rk}\,_QC=2$. Thus X is not quasi-homogeneous.

Now we introduce the affine variety $V_{n_1,\dots\dots,n_s}(A)$ ([5]).

Let $n_1, n_2, \dots\dots\dots, n_s, m_1, m_2, \dots\dots, m_s$ be positive integers, and $X_1, \dots\dots X_{n_1}, Y_1, \dots$ $\dots Y_{n_2}, \dots\dots, Z_1, \dots\dots, Z_{n_s}$ be the coordinates of A^{n_1}, A^{n_2}, ..., A^{n_s} respectively. Consider the morphism

$$v^{m_1,\dots\dots,m_s}_{n_1,\dots\dots,n_s}:A^{n_1}\times\dots\dots\times A^{n_s}\longrightarrow A^N,\ N=\prod_{i=1}^{s}\binom{n_i+m_i-1}{m_i}$$

defined by

$$v^{m_1, \ldots\ldots, m_s}_{n_1, \ldots\ldots, n_s}(X_1, \ldots\ldots, X_{n_1}, Y_1, \ldots\ldots, Y_{n_2}, \ldots\ldots, Z_1, \ldots\ldots, Z_{n_s})$$

$$=(\ldots\ldots, X_1^{i_1}\ldots\ldots X_{n_1}^{i_{n_1}} Y_1^{j_1}\ldots\ldots Y_{n_2}^{j_{n_2}}\ldots\ldots Z_1^{k_1}\ldots\ldots Z_{n_s}^{k_{n_s}}, \ldots\ldots)$$

where $i_p \geqq 0$, $j_q \geqq 0$, $k_r \geqq 0$, $i_1+\ldots\ldots+i_{n_1}=m_1$, $j_1+\ldots\ldots+j_{n_2}=m_2, \ldots\ldots, k_1+\ldots\ldots+k_{n_s}=m_s$. For $n_1, \ldots\ldots, n_s$ positive integers and an $r\times s$-matrix $A=(a_{ij})$ of positive integers, we define the variety $V_{n_1, \ldots\ldots, n_s}(A)$ to be the closure of the image of the morphism

$$v^{a_{11}, \ldots\ldots, a_{1s}}_{n_1, \ldots\ldots, n_s}\times\ldots\ldots\times v^{a_{r1}, \ldots\ldots, a_{rs}}_{n_1, \ldots\ldots, n_s} : A^{n_1}\times\ldots\ldots\times A^{n_s}\longrightarrow A^{N_1}\times\ldots\ldots\times A^{N_r},$$

here $N_j=\prod_{i=1}^{s}\binom{n_i+a_{ji}-1}{a_{ji}}$.

Note that $k[V_{n_1, \ldots\ldots, n_s}(A)]$ s isomorphic to the subalgebra of $k[X_1, \ldots\ldots, X_{n_1}, \ldots\ldots Y_1, \ldots\ldots, Y_{n_2}, \ldots\ldots, Z_1, \ldots\ldots, Z_{n_s}]$ generated by the monomials $X_1^{i_1}\ldots X_{n_1}^{i_{n_2}}\, Y_1^{j_1}\ldots Y_{n_1}^{j_{n_2}}\ldots\ldots Z_1^{k_1}$ $Z_{n_s}^{k_n} s$, where $i_1+\ldots\ldots+i_{n_1}=a_{p1}$, $j_1+\ldots\ldots+x_{n_2}=a_{p2}, \ldots\ldots, k_1+\ldots\ldots+k_{n_s}=a_{ps}$ $(p=1, 2, \ldots\ldots, r)$.

PROPOSITION 6.2. *If G is either SL_4 or Sp_2, the 4-dimensional quasi-homogeneous S-variety of G is isomorphic to $V_4(B)$ where $B={}^t(n_1, \ldots\ldots, n_s)$ n_i; positive integers.*

PROOF. **Case G=SL$_4$.**

Consider the standard representation $\varphi: G\longrightarrow GL(4, C)$. Let σ_1, σ_2, and σ_3 be the basic weights, and let e_1, $e_1{}_\Lambda e_2$ and $e_1{}_\Lambda e_2{}_\Lambda e_3$ be the coresponding leading weight vectors. By easy calculation we have

$$(SL_4)_{e_1}=\left\{\begin{pmatrix}1&0&0&0\\ *&*&*&*\\ *&*&*&*\\ *&*&*&*\end{pmatrix}\right\}(SL_4)_{e_1\Lambda e_2}=\left\{\begin{pmatrix}a_{11}&a_{12}&0&0\\ a_{21}&a_{22}&0&0\\ *&*&*&*\\ *&*&*&*\end{pmatrix}\begin{matrix}\text{Either } a_{11}a_{22}=1 \text{ and } a_{12}=a_{21}=0,\\ \text{or}\\ a_{12}a_{21}=1 \text{ and } a_{11}=a_{22}=0.\end{matrix}\right\}$$

Clearly dim $(SL_4)_{e_1}=11$. Let $v=e_1\oplus e_{1\Lambda}e_2$, it follows that dim $(SL_4)_v=7$ and hence $X(\sigma_1+\sigma_2)=\overline{SL_4(v)}$ is 8-dimensional. Similar arguments shows that $X(l\sigma_1+m\sigma_2+n\sigma_3)$ is not 4-dimensional for any triple (l, m, n) of integers of which at least two are positive. Therefore, since $\text{rk}_Q G=1$, it follows that if X is of type $X(\Lambda_1, \ldots\ldots, \Lambda_s)$ where $\Lambda_i=l_i\sigma_1+m_i\sigma_2+n_i\sigma_3$, l_i, m_i, n_i; positive integers $(i=1, 2, \ldots\ldots, s)$, then Λ_i must be of type $l_i\sigma_1$ (all i), $m_i\sigma_2$ (all i) or $n_i\sigma_3$ (all i). On the other hand, since for any pair (σ_i, σ_j) there exists an automophism of Dynkin diagram such that $\sigma_i=\sigma_t{}^j$, it follows from lemma 8 in [5] that $X(n_1\sigma_i, \ldots\ldots, n_s\sigma_i)$ is isomorphic to $X(n_1\sigma_j, \ldots\ldots, n_s\sigma_j)$. So we consider only of the type $X(n_1\sigma_1, \ldots\ldots, n_s\sigma_1)$.

Consider the standard respresentation φ with the basic weight σ_1. Then the G-action on A^4 is contragradient to the action,

$$X_j\longrightarrow a_{1j}X_1\times a_{2j}X_2\times a_{3j}X_3\times a_{4j}X_4, \qquad j=1, 2, 3, 4.$$

where X_1, X_2, X_3 and X_4 are the coordinates of A^4.

LEMMA. *The A^4 with the above SL_4-action is the S-variety $X(\sigma_1)$ of SL_4.* ([5], lemma 8).

In fact, since G acts transitively on A^4-0, $X(\sigma_1)$ is $\overline{A^4-0}$.

By the lemma, it follows that $k[A^4]=k[X(\sigma_1)]=k[X_1, X_2, X_3, X_4]=\sum_{n=0}^{\infty} S_{n\sigma_1}$ where $S_{n\sigma_1}$ is the algebra generated by the monomials $X_1^{i_1}X_2^{i_2}X_3^{i_3}X_4^{i_4}$, $i_1+i_2+i_3+i_4=n$ ([7]). Therefore it follows from theorem 6 in [7] that $k[X(n_1\sigma_1, \ldots\ldots, n_s\sigma_1)]$ is generated by monomials $X_1^{i_1}X_2^{i_2}X_3^{i_3}X_4^{i_4}$, $i_1+i_2+i_3+i_4\in\{n_1, n_2, \ldots\ldots, n_s\}=$the semigroup generated by $n_1, \ldots\ldots, n_s$. Hence it follows from the definition of $V_4(B)$ ([7]) that $k[V_4(B)]$ is isomorphic to $k[X(n_1\sigma_1, \ldots\ldots, n_s\sigma_1)]$, for $B={}^t(n_1, n_2, \ldots\ldots, n_s)$. Thus we see $X(n_1\sigma_1, \ldots\ldots, n_s\sigma_1)$ $=V_4(B)$.

Case G=Sp₂.

Consider the standard representation $\psi: Sp_2\longrightarrow GL(4, C)$. Let τ_1 and τ_2 be the basic weights and let e_1 and $e_1\wedge e_2$ be the corresponding leading weight vectors. By the similar calculation to Case $G=SL_4$, $(Sp_2)_{e_1}$ and $(Sp_2)_{e_1\wedge e_2}$ are shown to be of the same forms in the above case. Hence $(Sp_2)_{e_1}$ is of codimension 4. By similar arguments to case $G=SL_4$ we may assume that X is $X(n_1\tau_1, n_2\tau_1, \ldots\ldots, n_s\tau_1)$.

We may consider the representation space of ψ as the vector space V spanned by the indeterminates x_1, x_2, x_3 and x_4. Since $n\tau_1$ is the basic weight of the n-th tensor product of ψ, the representation corresponding to $n\tau_1$ is the vector space V_n spanned by the monomials of degree n in x_1, x_2, x_3, x_4. Then $V=V_{n_1}\oplus\ldots\ldots\oplus V_{n_s}$. Regarding x_1 as the leading weight vector of τ_1 and let $v=x_1^{n_1}+\ldots\ldots+x_1^{n_s}$, we have

$$(Sp_2)_v=\left\{\begin{pmatrix} f & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} ; \begin{array}{l} f^d=1 \text{ where} \\ d \text{ is G. C. D. } (n_1, n_2, \ldots\ldots, n_s) \end{array}\right\}$$

On the other hand, if we cansider V as the representation space of the standard representation φ with the basic weight σ_1 and the leading weight vector x_1, it is easy to see that $(SL_4)_v$ for $v=x_1^{n_1}+\ldots\ldots+x_1^{n_s}$.

Thus, we have an inclusion: $Sp_2/(Sp_2)_v\longrightarrow SL_4/(SL_4)_v$ and $\dim Sp_2/(Sp_2)_v=\dim SL_4/(SL_4)_v$. If we denote by Y the S-variety $X(n_1\sigma_1, \ldots\ldots, n_s\sigma_1)$ of SL_4, this implies that X is isomorphic to Y. Consequently X is also isomorphic to $V_4(B)$. Q.E.D.

PROPOSITION 6.3. *If $G=SL_2\times SL_2\times SL_2$, the 4-dimensional quasi-homogeneous S-varieties of G are isomorphic to $V_{2,2,2}(A)$, where* $A={}^t\begin{bmatrix} l_1 & l_2 & \ldots\ldots & l_s \\ m_1 & m_2 & \ldots\ldots & m_s \\ n_1 & n_2 & \ldots\ldots & n_s \end{bmatrix}$, *$l_i$, m_i, n_i positive integers and rank $A=1$.*

PROOF. Let $\pi_i : G=SL_2\times SL_2\times SL_2 \longrightarrow SL_2$ be the i-th projection, $i=1, 2, 3$, and $\rho : SL_2 \longrightarrow GL(2, C)$ the standard representation. We consider the representation $\varphi : G: \longrightarrow GL(6, C)$ defined by the direct sum of the representations $\rho\circ\pi_i$ with the basic weight σ_i.

Consider any 4-dimensional S-variety $X(\Lambda_1, \ldots\ldots, \Lambda_s)$, $\Lambda_i=l_i\sigma_1+m_i\sigma_2+n_i\sigma_3 (i=1, 2, \ldots \ldots, s)$ of G. It is quasi-homogeneous, because we have $\delta(G)=3$ and hence $\mathrm{rk}_Q(C)=1$. Moreover $\mathrm{rk}_Q(C)=1$ implies that the polyhedral cone $K=Q+G$ spanned by the Λ_i,s is one dimensional and hence rank $A=1$.

By considering φ, we have a G-action on $A^6=A^2\oplus A^2\oplus A^2$, contragredient to the action

$$(X_1, X_2, Y_1, Y_2, Z_1, Z_2) \longrightarrow (X_1, X_2, Y_1, Y_2, Z_1, Z_2)\begin{bmatrix} (a_{ij}) & 0 & 0 \\ 0 & (b_{ij}) & 0 \\ 0 & 0 & (c_{ij}) \end{bmatrix}$$

for $(a_{ij})\times(b_{ij})\times(c_{ij})\in SL_2\times SL_2\times SL_2$, where X_1, X_2, Y_1, Y_2, Z_1 and Z_2 are the coordinates of A^6.

LEMMA. *The A^6 with the above G-action is the S-variety $X(\sigma_1, \sigma_2, \sigma_3)$ of G*, ([5], lemma 11).

It is easy to prove Lemma.

By the lemma we have that $k[A^6]=k[X\sigma_1, \sigma_2, \sigma_3]=k[X_1, X_2, Y_1, Y_2, Z_1, Z_2]=\sum_{l,m,n\geq 0} S_{l\sigma_1+m\sigma_2+n\sigma_3}$ where $S_{l\sigma_1+m\sigma_2+n\sigma_3}$ is the algebra generated by the monomials $X_1^{i_1}X_2^{i_2}Y_1^{j_1}Y_2^{j_2}Z_1^{k_1}Z_2^{k_2}$ where $i_p, j_q, k_r\geqq 0$, $i_1+i_2=l$, $j_1+j_2=m$, and $k_1+k_2=n$. Therefore by theorem 6 in [7[, we see that $k[X(\Lambda_1, \ldots\ldots, \Lambda_s)]$ is the algebra generated by the monomials $X_1^{i_1}X_2^{i_2}Y_1^{j_1}Y_2^{j_2}Z_1^{k_1}Z_2^{k_2}$ where $(i_1+i_2, j_1+j_2, k_1+k_2)\in\{(l_1, m_1, n_1), \ldots\ldots, (l_s, m_s, n_s)\}$ (the semigroup generated by these triples). Hence it follows from the definition of $V_{2,2,2}(A)$ that $k[X]$ is isomorphic to $k[V_{2,2,2}(A)]$. This implies ehat X is isomorphic to $V_{2,2,2}(A)$.

Q.E.D.

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