On the Torus degree of symmetry of SU(3) and G_2

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Introduction

In this note we shall consider the torus degree of symmetry of simple Lie groups $SU(3)$ and G_2, where the torus degree of symmetry of a manifold M, denoted by $T(M)$, is by definition the maximal dimension of torus which can act on the manifold M effectively (eee [3]).

We shall prove the following.

THEOREM A. $T(SU(3))=4$.

THEOREM B. $T(G_2)=4$.

This work is motivated by the following conjecture of W. Y. Hsiang ([3]);

The torus degree of symmetry of compact semi-simple Lie group G is equal to 2 rk G.

In the following we shall consider only differentiable actions and use the notations:

(1) $X \underset{A}{\sim} Y$ means $H^*(X:A) \cong H^*(Y:A)$

as algebras, where A is a commutative ring.

(2) Q denotes the field of rational numbers and Z_n a cyclic group of order n.

1. Statement of results

In this section we shall prove Theorems A and B modulo some propositions, which are proved in the subsequent sections.

In the first place we shall consider the case of $SU(3)$ and put $X=SU(3)$.

Suppose $T(X) \geqq 5$. Let a 5-dimensional torus T'' act on X by $\Phi: T'' \times X \to X$. From a result in [1], it follows that $\mathrm{rk}\,\Phi \leqq 2$, where $\mathrm{rk}\,\Phi = \min \{\dim T''/T_x'' : x \in X\}$. If $\mathrm{rk}\,\Phi = 0$ (respectively 1.), some 5-dimensional (respectively 4-dimensional) subtorus of T'' has a fixed point. Since $X \underset{Q}{\sim} S^3 \times S^5$, the fixed point set of any torus action has Q-cohomology ring of product of two odd dimensional spheres ([2]), and hence it is connected and at least 2-dimensional. It follows from the consideration of local representation at fixed point that this is impossible. Thus $\mathrm{rk}\,\Phi=2$, and hence some 3-dimensional subtorus T'

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has a fixed point. It can be shown that there is a one-dimensional subtorus T of T' which has 6-dimensional fixed point set. Consider the subgroup Z_2 of T. Since the restricted action of Z_2 on X preserves orientation and T acts effectively on X, dim $F(Z_2, X)$ must be 6, which implies that $F(T, X)$ is a component of $F(Z_2, X)$. In section 2, we shall prove the following

PROPOSITION 1. *There is no orientation preserving involution on X with fixed point set one of whose components has Q-cohomology ring of* $S^3\times S^5$ *or* $S^1\times S^5$.

It is clear that Proposition 1 implies Theorem A.

REMARK. In the proof of Proposition 1 we use only the fact that X has Q-cohomogy ring of $S^3\times S^5$ and Z_2-cohomology ring of $SU(3)$. Hence we have the following

THEOREM A′ *Let M be a manifold such that* $M\underset{Q}{\sim}S^3\times S^5$ *and* $M\underset{Z_2}{\sim}SU(3)$. *Then there is no one-dimensional torus action on M whose fixed point set is* 6*-dimensional.*

Next we shall consider the case of G_2. Put $X=G_2$. Suppose a 5-dimensional torus T'' act on X by $\Phi : T''\times X\to X$. As in the case of $SU(3)$, we have rk $\Phi\leqq 2$.

Case 1. rk $\Phi=0$.

Case 2. rk $\Phi\leqq 1$.

In these cases there is a subtorus T' of dimension 4 whose fixed point set $F(T', X)$ is not empty. It follows from the Borel formula that there is a corank one subtorus T_1 of T' such that dim $F(T_1, X)>$dim $F(T', X)$. Consider the action of T_1 obtained by the restriction. Since the action of T'' is effective, the same argument as above shows that there is a corank one subtorus T_2 of T_1 such that dim $F(T_2, X)>$dim $F(T_1, X)$. Thus we obtain a sequence of fixed point sets:

$$F(T', X)\subset F(T_1, X)\subset F(T_2, X)\subset \cdots\cdots \subset F(T_k, X)\subset X.$$

Cleary $k=4$. It is easy to see that there is a one dimensional subtorus T of T'' such that $F(T, X)$ is 6-dimensional or 10-dimensional.

Subcase 1. dim $F(T, X)=6$.

Take the subgroup Z_2 of T. Then $F(Z_2, X)$ is 6-dimensional, 8-dimensional or at least 10-dimensional. Let F_0 be a component of $F(Z_2, X)$ containing $F(Z_2, X)$. Assumedim $F_0=8$. Then in section 3, we shall prove the following

PROPOSITION 2. $F_0\underset{Z_2}{\sim}SU(3)$ *and* $F_0\underset{Q}{\sim}S^3\times S^5$.

Thus T acts on F_0 with 6-dimensional fixed point set, which is impossible by Theorem A′.

The case in which F_0 is 6-dimensional or at least 10-dimensional does not occure by the following

PROPOSITION 3. *In the above situation, there is no involution on X whose fixed point set is* 6*-dimensional and has Q-cohomology ring of product of two odd dimensional spheres.*

PROPOSITION 4. *In the above situation there is no involution on X whose fixed point set is at least* 10-*dimensional.*

Sucase 2. Dim $F(T, X)=10$.

This case is clearly impossible by Proposition 4.

Case 2. rk $\Phi=2$.

In this case there is a 3-dimensional torus T' of T'' such that $F(T', X)\neq\Phi$.

Consider a sequence of fixed point sets:

$$F(T', X)\subset F_1\subset F_2\subset X.$$

If dim $F_1=6$, there is a one dimensional subtorus T of T' such that dim $F(T, X)=6$. The same arguments as in subcase 1 of case 1 show that this is impossible. If dim $F_1>8$. then there is a one-dimensional subtorus T of T'' such that dim $F(T, X)\geqq 10$, which is impossible by Proposition 4. Thus it is sufficient to consider only the case in which every 2-dimensional subtorus of T' has at most 4-dimensional fixed point set and every one dimensional subtorus of T' has 8-dimensional fixed point set. Consider the action of a 2-dimensional subtorus T^2 obtained by restriction and apply the Borel formula at $x\in F(T^2, X)$. We have

$$\begin{aligned}&\dim X-\dim F(T^2, X)\\&=\sum_K\{\dim F(K, X)-\dim(F(T^2, X))\},\end{aligned}$$

where K denotes subtorus of T^2 of codimension 1, our assumptin shows that $10=a(8-4)$, where a is the number of K. This is clearly impossible. Thus we have proved Theorem B.

2. Proof of Proposition 1

In this section we shall consider an orientation preserving involution on $X=SU(3)$ with 6-dimensional fixed point set and prove Proposition 1 in section 1. Put $G=Z_2$ and recall $H^*(X:Z_2)=Z_2[a]/(a)\otimes\Lambda_{Z_2}(S_q^2 a)$, deg $a=3$.

In this section we ccnsider only Z_2-cohomology group unless otherwise stated.

LEMMA 1. *X is totally non-homologous to zero in the fibre bundle $X_G=X\times_G E_G\to B_G$.*

PROOF. Consider the spectral sequence of the fibration $X_G\longrightarrow B_G$. Since $E_2^{0,3}=E_4^{0,3}$, every element of $H^3(X)$ is transgressive and hence $Sq^2 a$ is also transgressive. Since the action of G on X has fixed point, the homomorphism $H^*(B)\longrightarrow H^*(X_G)$ is injective. Then the transgression is trivial. In fact consider the following commutative diagram;

$$\begin{array}{ccccc}H^3(X) & \xrightarrow{\ \delta\ } & H^4(X_G, X) & \xrightarrow{\ j^*\ } & H^4(X_G)\\ & & \uparrow & \nwarrow q^* & \uparrow \pi^*\\ & & H^4(B_G, X) & \cong & H^4(B_G).\end{array}$$

Let $\tau(x)=y$ (τ denotes the transgression). By definition of τ, we have $\delta(x)=q^*(y)$. Then $\pi^*(y)=j^*q^*(y)=j^*\delta(x)=0$. Since π^* is injective, $y=0$. Since $H^*(X)$ is generated by a and $Sq^2\, a$, the homorphism $i^*: H^*(X_G)\longrightarrow H^*(X)$ is injective. This completes the proof of lemma.

Find an element $\alpha\in H^3(X_G)$ such that $i^*(\alpha)=a$. From a result in [2] (Chap. VII. 1. 4) it follows that $H^*(X_G)$ is a free $H^*(B_G)$-module generated by α, $Sq^2\,\alpha$, and $\alpha S^2\alpha$. Let F_0 denote a 6-dimensional component of $F(G, X)$ and choose a point $x\in F_0$. Let $j_0: (F_0, x)_G \longrightarrow X_G$ be the inclusion. Then we have

(2.1) $\qquad j_0^*(\alpha)=1\otimes b_3+t\otimes b_2+t^2\otimes b_1,$

where $H^*(B_G)=Z_2[t]$ and $b_i\in H^i(F_0)$.

Lemma 2. $b_3^2=0$.

Proof. Since $a^2=0$, we have $i^*(\alpha^2)=0$ and hence $\alpha^2\in \mathrm{Ker}\, i^*=\langle H^+(B_G)\rangle$, i.e. $(j_0^*(\alpha))^2=1\otimes b_3^2+t^2\otimes b_1^2\in\langle H^*(B_G)\rangle$, which implies $b_3^2=0$. The completes the proof.

By the same arguments as in [2] (chap. VII), we can show that $H^*(F_0)$ is multiplicatively generated by b_3, b_2, b_1 and $S_q^2 b_3$. Note that $\dim_{Z_2} H^*(F(Z_2, X))=\dim_{Z_2} H^*(X)=4$. It follows from this that $H^*(F_0)$ is generated by b_2 and $F_0\sim CP_3$ or generated by b_3. Clearly both cases contradict to the structure of Q-cohomology ring of F_0.

This completes the proof of Proposition 1 in section 1.

In the above arguments we use only the fact $X\underset{Q}{\sim}S^3\times S^5$ and $X\underset{Z_2}{\sim}SU(3)$. Hence we have proved the Theorem A$'$.

3. Proof of Propositions 2, 3 and 4

In this section we shall prove Proposition 2, 3 and 4. Put $G=Z_2$, $X=G_2$ and recall $H^*(X; Z_2)=Z_2[a]/a^4\otimes\Lambda_{Z_2}(S_q^2 a)$, deg $a=3$. In this section all cohomology groups are on Z_2 unless otherwise stated. By the same argument as in section 2, we can prove the following

Lemma 1. *X is totally non-homologous to zero in the fibration $X_G\rightarrow B_G$.*

Find an element $\alpha\in H^3(X_G)$ such that $i^*(\alpha)=a$, where $i: X\rightarrow X_G$ inclusion. Denote $\beta=S_q^2\,\alpha$ and F_0 the component of $F(G, X)$ which contains $F(T, X)$. Choose a point $x\in F_0$ and denote $j_0: (F_0, x)_G\rightarrow (X_G, x_G)$ inclusion. We have

(1) $\qquad j_0^*(\alpha)=1\otimes b_3+t\otimes b_2+t^2\otimes b_1$

and

(2) $\qquad j_0^*(\beta)=j_0^*(S_q^2\alpha)=1\otimes S_q^2 b_3+t\otimes b_2^2+t^4\otimes b_1.$

Note $H^*(F_0)$ is generated as algebra by b_1, b_2, b_3 and $S_q^2 b_3$ and $\dim_{Z_2} H^*(F_0)\leqq 8$. By the same argument as in section 2 we can prove

LEMMA 2. $b_3^4=0$, $(S_q^2b_3)^2=0$, *and* $S_q^1b_3=0$.

Moreover we prove

LEMMA 3. *Assume* $b_3\neq 0$. *Then we have*

a) *if* $b_1=0$, *then* $b_2=0$.

b) *if* $b_1\neq 0$, *then* $b_2=0$ *or* $b_2=b_1^2\neq 0$.

c) *if* $b_1\neq 0$ *and* $b_2=0$, *then* $b_3=b_1^3$.

PROOF. Since $j_0{}^*$ is surjective in high degrees (see [2]. chap. VII), we have

(3) $$t^r\otimes b_3=j_0{}^*(A_1t^r\alpha+A_2t^{r-3}\alpha^2+A_3t^{r-6}\alpha^3+B_0t^{r-2}\beta+B_1t^{r-5}\alpha\beta+B_2t^{r-8}\alpha^2\beta+B_3t^{r-11}\alpha^3\beta),$$

where A_i and B_j are in Z_2, Clearly $A_1=1$.

We have

(4) $$t^r\otimes b_3-j_0{}^*(t^r\alpha)=t^{r+2}\otimes b_1+t^{r+1}\otimes b_2.$$

The left hand side of (4) is

$$t^r\otimes b_3-j_0{}^*(t^r\alpha)$$
$$=t^{r+2}\otimes B_0b_1+t^{r+1}\otimes(A_2b_1^2+B_1b_1^2)+t^r\otimes(A_2b_1^3+B_1b_1b_2+B_2b_1^3)+\cdots\cdots$$

Compairing coefficients of t^k, we have

$$b_1=B_0b_1$$
$$b_2=(A_2+B_1)b_1^2$$

and $$b_3=A_3b_1^3+B_1b_1b_2+B_2b_1^3.$$

It is now easy to show that lemma holds. This completes the proof.

Now we shall prove the Propositions 3 and 4 in section 1.

Case 1. $\dim F_0=6$.

Note that possible generator of $H^6(F_0)$ is b_1^6, $b_1^4b_2$, $b_1^3b_3$, $b_1^2b_2^2$, $b_1S_q^2b_3$, b_3^2 and b_3^2.

Subcase 1. b_1^6 is a generator of $H^6(F_0)$.

Clearly $\dim H^*(F_0)\geqq 7$. Suppose $\dim H^*(F_0)=7$.

Then there exsits a component F_1 of $F(G, X)$ such that $\dim H^*(F_1)=1$. Since F_1 is an orientable closed manifold, $F_1=\{pt\}$. Moreover since $F(G, X)$ is T-invariant, $F(T, X)=F(T, F(G, X))=F_0\cup F_1$, which cortradcts to the connectedness of $F(T, X)$. Thus we have $\dim H^*(F_0)=8$ and F_0 is connected and $F_0=F(T, X)$. It is known that $F_0\underset{Q}{\sim}S^1\times S^5$ or $F_0\underset{Q}{\sim}S^3\times S^3$. Clearly $\dim H^3(F_0)=2$. and b_1^3 and b_3 are generators of $H^3(F_0)$.

LEMMA 4. $b_1b_3=0$.

PROOF. It follows from lemma 3 that $b_2\neq 0$. We have

$$t^{r+1}\otimes b_2=\text{the right hand side of (3).}$$

Since $b_2=b_1^2$, $A_1+B_1+A_2=1$ and we have

$$(4)\qquad t^{r+1}\otimes b_2-j_0^*(A_1t^r\alpha+A_2t^{r-3}\alpha^2+B_1t^{r-5}\alpha\beta)$$
$$=j_0^*(A_3t^{r-6}\alpha^3+B_0t^{r-2}\beta+B_2t^{r-8}\alpha^2\beta+B_3t^{r-11}\alpha^3\beta).$$

Case of $\mathbf{A_1=1}$ and $\mathbf{B_1=A_2=0}$.

Clearly we have

$$\text{the left hand side of (4)}=t^{r+2}\otimes b_1+t^r\otimes b_3\,.$$

and hence $b_3=b_1^3$, which contradicts to our situation.

Case of $\mathbf{A_2=1}$ and $\mathbf{A_1=B_1=0}$.

we have

$$\text{the left hand side of (4)}=t^{r-1}\otimes b_1^4+t^{r-3}\otimes b_3^2\,.$$

and hence $B_0=0$ and $A_3+B_3=0$. Comparing the coefficients of t^{r-1}, we have a contradiction.

Case of $\mathbf{A_1=A_2=0}$ and $\mathbf{B_1=1}$

We have

the left hand side of (4)

$$=t^r\otimes b_1^3+t^{r-1}\otimes b_1b_3+t^{r-2}\otimes b_1^5+t^{r-3}\otimes(b_1^6+b_1S_q^2b^3)+\cdots\cdots$$

and hence $B_0=0$ and $A_3+B_2=1$. Moreover, by compairing of coefficients of t^i in (4) we have

(i) $b_1b_3=A_3b_1^4+B_3b_1^4$

(ii) $b_1^5=A_3(b_1^5+b_1^2b_3)+B_2b_1^5+B_3b_1^5$

(iii) $b_1^6+b_1S_q^2b_3=A_3b_1^6+B_2b_1^6+B_3(b_1^6+b_1^3b_3)$.

Suppose $A_3=1$ and $B_2=0$. If $b_1b_3=0$, then $B_3=1$. From (iii), it follows that $b_1^6=b_1S_q^1b_3$. Since $S_q^2(b_1b_3)=b_1S_q^2b_3$, we have $b_1S_q^2b_3=0$ and hence $b_1^6=0$, which is a contradiction. If $b_1b_3\neq 0$, then $B_3=0$ and $b_1b_3=b_1^4$, which implies $b_1^2b_3=b_1^5$. It follows from (ii) that $b_1^2b_3=0$. This is a contradiction. Suppose $A_3=0$ and $B_2=0$. It follows from (ii) that $B_3=0$ and hence $b_1b_3=0$

Case of $\mathbf{B_1=A_1=A_2=1}$.

We have

$$t^{r+2}\otimes b_1+t^r\otimes(b_1^3+b_3)+t^{r-1}\otimes(b_1b_3+b_1^4)$$
$$=t^{r+2}\otimes B_0b_1+t^r\otimes(A_3b_1^3+B_2b_1^3)+\cdots\cdots$$

Since $b_1^3\neq b_3$, we have $A_3b_1^3+B_2b_1^3=b_1^3+b_3$, which is clearly impossible. These arguments complete the proof.

The following proposition shows that subcase 1 does not hold.

PROPOSITION 5. $F_0 \underset{Q}{\not\sim} S^1\times S^5$ *and* $F_0 \underset{Q}{\sim} S^3\times S^3$.

PROOF. We suppose $F_0 \underset{Q}{\sim} S^1\times S^5$. We may assume that b_1 is mod 2 reduction of an element of $H^1(F_0: Z)$. Hence we have $b_1^2=S_q^1b_1=0$, which is a contradiction. Next we suppose $F_0 \underset{Q}{\sim} S^3\times S^3$. Then we may assume that b_1^3 and b_3 are mod 2 reductions of elements of $H^3(F_0, Z)$, $b_1^3=r(\gamma_1)$ and $b_3=r(\gamma_2)$, where $r: H^3(F_0: Z)\to H^3(F_0: Z_2)$ is mod 2 reduction. We can choose γ_1 and γ_2 such that $\gamma_1\gamma_2$ is a generator of $H^6(F_0: Z)$ and hence $r(\gamma_1\gamma_2)\neq 0$, which contradicts to the fact $r(\gamma_1)\, r(\gamma_2)=b_1^3b_3=0$. This completes the proof.

Subcase 2. $b_1^4b_2$ is a generator of $H^6(F_0)$.

Since dim $H^*(F_0)>9$ this case does not occur.

Subcase 3. $b_1^3b_3$ is a generator of $H^6(F_0)$.

By the same argument as in subcase 1, we can prove Proposition 5 for this case. Hence this case does not hold.

Subcase 4. $b_1^2b_2^2$ is a generator of $H^6(F_0)$.

It is easy to see that dim $H^*(F_0)>9$.

Subcase 5. $b_1S_q^2b_3$ is a generator of $H^6(F_0)$.

If $b_3=b_1^3$, then $S_q^2b_3=b_1^5$ and hence this case is reduced to subcase 1. Since $b_1S_q^2b_3=S_q^2(b_1\, b_3)$, we have $b_1b_3\neq 0$. If $b_1\, b_3=b_1^4$, then $S_q^2(b_1b_3)=0$. Hence we have $b_1b_3\neq b_1^4$. If $b_1^4\neq 0$, then dim $H^*(F_0)>8$. Thus we have $b_1^4=0$. By the same argument as in the proof of Lemma 4, we can prove that $b_1b_3=0$, which is clearly impossible.

Subcase 6. b_2^3 is a generator of $H^6(F_0)$.

It follows from lemma 3 that $b_3=0$. Assume $b_1\neq 0$. It is easy to see that $b_2=b_1^2$, which is a contradiction. Hence we have $b_1=0$, and $F_0 \underset{Z_2}{\sim} CP_3$, which contradicts to the structure of cohomology ring of F_0.

Subcase 7. b_3^2 is a generator of $H^6(F_0)$.

Since $b_3^2=S_q^3b_3=S_q^1S_q^2b_3\neq 0$, we have $S_q^2b_3=0$ and hence $b_1\neq 0$. It is easy to see that dim $H^*(F_0)=5$, 7 or 8. Assume dim $H^*(F_0)=5$ or 7. Then there is a component F_1 of

$F(Z_2, X)$ such that dim $H^*(F_1 : Z_2)=1$ or 2. Clearly in both cases the Euler characteristic of F_1 is not zero. Hence $F(T, F_1)\neq\phi$, which is a contradiction. Thus dim $H^*(F_0)$ must be 8. Assume $b_1S_q^2b_3\neq 0$. This case reduces to the subcase 5. If $b_1S_q^2b_3=0$, F_0 must have the same Q-cohomology ring of $S^3\times S^3$ and hence $b_1^3b_3\neq 0$, which reduces to the subcase 3.

Case 2. dim $F_0=10$.

Assume $b_3=0$. Note if $b_1\neq 0$, then $b_2=b_1^2\neq 0$ or $b_2=0$. Thus a generator of $H^{10}(F_0)$ is one of the following: b_1^{10} and b_2^5. In the case of b_1^{10}, dim $H^*(F_0)$ is clearly greater than 8, which is impossible. In the case of b_2^5, $F_0\underset{Z_2}{\sim}CP_5$ and hence $\chi(F_0)\neq 0$. Since $F(T, X)=F(T, F_0)$, we have $\chi(F(T, X))\neq 0$, which contradicts to the fact $F(T, X)$ has Q-cohomology ring of product of odd dimensional spheres. Assume $b_3\neq 0$. b_1 must be non-zero. We may assume $b_2\neq 0$, since $b_3=b_1^3$ if $b_2=0$. It is easy to see that dim $H^*(F_0)>9$.

Case 3. dim $F_0=12$.

By the same argument as case 2, it can be shown that this case does not occur, Summing up the above arguments, we have proved Propesitions 3 and 4 in section 1.

Case 4. dim $F_0=8$.

In case in which $b_3=0$, the same argument as in case 2 shows that this case does not occur. Now assume $b_3\neq 0$ and $b_1^3\neq b_3$. Note that $b_2=0$ or $b_2=b_1^2$. Then possible generators of $H^8(F_0)$ are $b_1^5b_3$, $b_1^2b_2^\varepsilon$, $b_1^3S_q^2b_3$ and $b_3S_q^2b_3$. In cases except the case of $b_3S_q^2b_3$, it is easy to see that dim $H^*(F_0)>8$. Consider the case of $b_3S_q^2b_3$. Then we way assume $b_1=0$; in other words $F_0\underset{Z_2}{\sim}SU(3)$. We shall prove $F_0\underset{Q}{\sim}SU(3)$. Suppose $H_3(F_0 : Z)$ is torsion group. Then, by Poincare duality, $H_5(F_0 : Z)\cong H^3(F_0 : Z)$ is also torsion group. Since $H^5(F_0 : Z)=\text{Hom}\,(H_5(F_0 : Z), Z)+\text{Ext}\,(H_4(F_0 : Z), Z)$, $H_5(F_0 : Z)$ is torsion group. Moreover, since $H^4(F_0)=H^6(F_0)=0$, the mod 2 reductions: $H^i(F_0 : Z)\to H^i(F_0)$ are surjective for $i=3, 5$. We put $b_3=r(\beta_1)$ and $S_q^2b_3=r(\beta_2)$. Since β_1 and β_2 are torsion elements, we have $\beta_1\beta_3=0$, which implies $b_3S_q^2b_3=r(\beta_1)r(\beta_2)=r(\beta_1\beta_2)=0$. Thus we have proved $F_0\underset{Q}{\sim}S^3\times S^5$. This proves Proposition 3 in section 1.

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