Note on the relations between smooth SU(6)-actions and rational Pontrjagin classes

By
Etsuo TSUKADA*

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0. Introduction and statement of a result

In [4], W.-C. Hsiang and W.-Y. Hsiang investigated smooth actions of classical groups on smooth manifolds with vanishing 1-st rational Pontrjagin classes. After them, E. A. Grove [2] studied smooth $SU(n)$-actions in more detail.

We note that their studies have the restrictions of dimensions of manifolds, and so it is reasonable that we want to remove those restrictions. But, of course, we need another assumption that a group, which acts on a manifold with vanishing 1-st and 2-nd rational Pontrjagin classes, is $SU(6)$. The reason which makes us take $SU(6)$ is only that $SU(6)$ has the classification of its semisimple subgroups in [3].

In this paper, we have all possibilities of identity components of principal isotropy subgroups of $SU(6)$ which acts on a manifold of arbitrary dimension with vanishing 1-st and 2-nd Pontrjagin classes.

THEOREM 0.1

Let $SU(6)$ act smoothly on a smooth manifold with vanishing 1-st and 2-nd rational Pontrjagin classes. Then the type of identity components of principal isotropy groups of this action is one of the followings.

$A_5\,(=SU(6))$,

A_4^1,

A_3^1, A_3^2, $A_2^1 \cdot A_1^1$, $A_1^1 \cdot A_1^1 \cdot A_1^1$,

A_2^1, A_2^2, A_2^5, $A_1^1 \cdot A_1^1$, $A_1^2 \cdot A_1^2$, $A_1^4 \cdot A_1^4$, B_2^1, B_2^2,

A_1^1, A_1^2, A_1^3, A_1^4, A_1^5, A_1^8, A_1^{10}, A_1^{11}, A_1^{20}, A_1^{35},

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* Niigata University

and $\{e\}$, *where* e *is the identity of* $SU(6)$.

1. Notations and preminary results

Let M be a smooth manifold, $P_j(M)$ and $P(M)$ the rational j-th and total Pontrjagin classes respectively.

Let G be $SU(n)$, H a closed connected subgroup of G with the inclusion λ: $H \longrightarrow G$. We denote a maximal torus of H by T_H. Let $\eta(T_H)$ be a T_H-universal bundle, $\tau_{\eta(T_H)}$ the transgression in $\eta(T_H)$. We write τ instead of $\tau_{\eta(T_H)}$.

Now we recall the following results in [2] which are useful for our purpose.

PROPOSITION 1. 1

Put $\Delta^+(H)$ *the positive root system of* H, *and* $\Omega(\lambda)$ *the weight system of* λ. *Then* $P_1(G/H)=0$ *if and only if there is a rational number* k *such that*

$$k\sum_{\omega\in\Omega(\lambda)}(\tau(\omega))^2=\sum_{\alpha\in\Delta^+(H)}(\tau(\alpha))^2.$$

We set $S^+(x_1, x_2, \cdots\cdots, x_k)$ the ring of symmetric polynomials with zero constant term.

PROPOSITION 1. 2

Suppose $P_1(G/H)=0$. *Then* $P_2(G/H)=0$ *if and only if* $\sigma_2(\tau(\alpha)^2)\in<S^+(\tau(\omega))>$: *the ideal in* $H^*(B_{T_H})$, *where* $\sigma(\tau(\alpha)^2)=\sigma_2(\tau(\alpha_2)^2, \cdots\cdots, \tau(\alpha_l)^2)$,

$$\Delta^+(H)=\{\alpha_1, \cdots\cdots, \alpha_l,\},$$

$$S^+(\tau(\omega))=S^+(\tau(\omega_1)), \cdots\cdots, \tau(\omega_m)),$$

$$\Omega(\lambda)=\{\omega_1, \cdots\cdots, \omega_m\}.$$

We have some corollaries of (1. 1) and (1. 2).

COROLLARY 1. 3

Let H *be a closed subgroup of rank* 1 *in* G. *Then* $P(G/H)$ *is trivial.*

COROLLARY 1. 4 ([2] pp. 342, lemma 2.7)

Let H *be a closed subgroup of* G *such that* $P_1(G/H)=0$. *Then either the identity component* H_0 *of* H *is a toral subgroup or* H *is semisimple.*

COROLLARY 1. 5 ([1])

Let T *be a toral subgroup of* G. *Then* $P(G/T)$ *is trivial.*

REMALK: In (1. 5), we may take G any compact Lie group.

2. Caluculations

In this section, we will clasify closed subgroups H of $SU(6)$ such that $SU(6)/H$ has the vanishing 1-st and 2-nd rational Pontrjagin classes (Proposition 2.2).

We have the following lemma to simplify our many caluculations.

LEMMA 2. 1

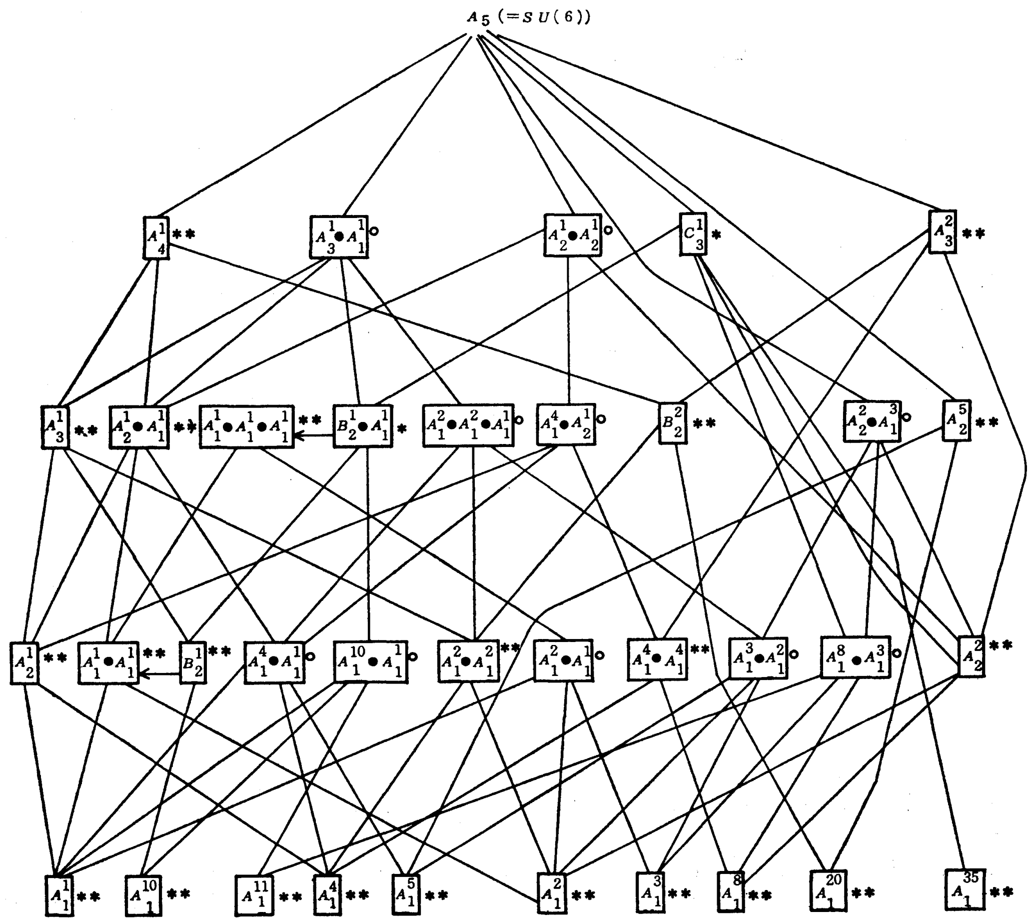
Let $K \subset H$ be subgroups of G, and K a regular subgroup of H. Then if $P_1(G/H)$ is vanishing, $P_1(G/K)$ is also vanishing. Moreover if $P_1(G/H)$ and $P_2(GH)$ are vanishing, $P_1(G/K)$ and $P_2(G/K)$ are also vanishing.

PROOF

We put $\mathfrak{t}_K$ the Lie algebra of T_K, and let $\lambda_1 : K \longrightarrow H$, $\lambda_2 : H \longrightarrow G$ and $\lambda = \lambda_2 \circ \lambda_1 : K \longrightarrow G$ be the inclusions. Then we have $\Delta^+(K) = \Delta^+(H)|\mathfrak{t}_K$ and $\Omega(\lambda) = \Omega(\lambda_2 \circ \lambda_1) = \Omega(\lambda_2)|\mathfrak{t}_K$. Therofore the lemma follows from (1.1) and (1.2) immediately. q.e.d.

PROPOSITION 2. 2

Let H be a semisimple closed subgroup of $SU(6)$. Then the 1-st and 2-nd rational Pontrjagin classes are following, where H° means that $P_1(SU(6)/H)$ is not zero, H^* means that



$P_1(SU(6)/H)=0$ and $P_2(SU(6)/H)\neq 0$ and H^{**} means that both $P_1(SU(6)/H)$ and $P_2(SU(6)/H)$ are zero.

PROOF

We prove it at only the case $H=B_2^2$. In this case, the defining matrix f of B_2^2 in A_5 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

(see [3]). Therefore we have $\Omega(\lambda)=\{y_1, y_2, -y_1, -y_2, 0, 0\}$, where y_1, y_2 are the canonical basis of $\mathfrak{t}_{B_2}$, and so we have

$$\Sigma\omega^2=2y_1^2+2y_2^2 .$$

Since $\Delta^+(B_2)=\{y_1+y_2, y_1-y_2, y_1, y_2\}$, we have

$$\Sigma\alpha^2=3y_1^2+3y_2^2 .$$

Henth $\frac{2}{3}\Sigma\omega^2=\Sigma\alpha^2$.

On the other hand, we have

$$\begin{aligned}\sigma_2(\alpha^2)&=\sigma_2((y_1+y_2)^2, (y_1-y_2)^2, y_1^2, y_2^2)\\ &=6y_1^4+6y_2^4+17y_1^2y_2^2\\ &=11(\sigma_2(\omega))^2-5\sigma_4(\omega) ,\end{aligned}$$

and so $\sigma_2(\alpha^2)$ is in $<S^+\omega>$.

Therefore we conclude $P_i(SU(6)/H)=0$ for $i=1, 2$ by (1.1) and (1.2).

Moreover we have $P_i(SU(6)/A_1^2\cdot A_1^2)=0$ for $i=1, 2$ by (2.1).

The remainded cases can be proved similarly. q.e.d.

3. Proof of Theorem 0.1

Let H be a principal isotropy group of G, and $i: G/H \longrightarrow M$ a inclusion. Then $i^*(P_j(M))=P_j(G/H)$ for any $j=0, 1, \cdots\cdots$. Therfore the theorem follows from (2.2).

q.e.d.

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