Transitive actions of compact connected Lie groups on symmetric spaces

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0. Introduction

Transitive actions of compact connected Lie groups on standard spheres have been studied by D. Montgomery–H. Sameleon [9] and A. Borel [2]. After them, W-Y. Hsiang–J. C. Su [8], A. L. Oniščik [12] and K. Abe–T. Watabe [1] have treated transitive actions on Grassmann and Stiefel manifolds.

In this paper we investigate transitive actions on every simply connected compact irreducible symmetric space M such that its Euler number $\chi(M)\neq 0$ and K is semisimple, where $M=I_0(M)/K$ as a symmetric space. Then we will show that such transitive action is unique and standard (Theorem 1. 2).

Finally in Apendix, we consider transitive actions on Grassmann manifolds $G_{2n,2k-1}$ $(2<k<n-1)$ which A. L. Oniščik has left. We will see easily that under some strong assumption, a simple transitive action, it is unique and standard (Theorem 6. 2).

I wish to thank Professor T. Watabe for his many helpful suggestions.

1. Notations and Main Theorem

For a topological space M, we denote the following notations. $H^*(M)$ is the cohomology with real coefficients and $P(M, t)$ is the Poincaré polynomial of M. The sum of the ranks of $\pi_{2k-1}(M)$ $k=1, 2, \cdots\cdots$ is called the *Oniščik rank* of M.

Let G be a compact connected Lie group, U its closed subgroup, $j: U\longrightarrow G$: inclusion. Let P_G, P_U be the spaces of the primitive elements of $H^*(G)$, $H^*(U)$ respectively. Then it is known that j induces the homomorphism $j^*: P_G\longrightarrow P_U$, and we denote by R and S the kernel and cokernel of j^* respectively. Note $P(R, t)$ and $P(S, t)$ are topological invariants for G/U (cf. [11], Theorem 1). Put $R^i=R\cap R^i_G$, and $S^i=S\cap P^i_U$. Then we have $R=\oplus_i R^i$ and $S=\oplus_i S^i$.

Now we consider a C^∞-manifold M which is a simply-connected compact irreducible symmetric space with the following properties.

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$$(*)\quad \begin{cases} \chi(M)\neq 0, \text{ and when } M \text{ can be obtained by } I_0(M)/K \\ \text{as a symmetric space, } K \text{ is semisimple.} \end{cases}$$

Then by E. Cartan's classification of irreducible symmetric spaces, M is one of the followings.

type	space
B_lI_{2m}	$SO(2l+1)/SO(2m)\times SO(2l-1-2m)$
B_lII	$SO(2l+1)/SO(2l)$
C_lII_m	$Sp(l)/Sp(m)\times Sp(l-m)$
D_lI_{2m}	$SO(2l)/SO(2m)\times SO(2l-2m)$
E_6II	$E_6/SU(2)\cdot SU(6)$
E_7V	$E_7/SU^*(8)$, $SU^*(8)=SU(8)/Z_2$
E_7VI	$E_7/SU(2)\cdot Spin(12)$
E_8VIII	$E_8/SO(16)$
E_8IX	$E_8/SU(2)\cdot E_7$
F_4I	$F_4/SU(2)\cdot Sp(3)$
F_4II	$F_4/Spin(9)$
G_2I	$G_2/SO(4)$

Table 1.

Definition. Let G and K be two compact connected Lie groups which act on M transitively and effectively, H and L isotropy subgroups of G and K respectively at some point of M, and $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$ and $\mathfrak{l}$ Lie algebras of G, K, H and L respectively. When there is an isomorphism $\varphi:\mathfrak{g}\longrightarrow\mathfrak{k}$ such that $\varphi(\mathfrak{h})=\mathfrak{l}$, we say that the action of G is *similar to* that of K.

Lemma 1. 1 ([14], p. 296)

Let M be any homogeneous space such that $\chi(M)\neq 0$, G a compact Lie group which acts on M transitively and effectively. Then the center $Z(G)$ of G is trivial.

Note: Let M be a simply-connected compact irreducible symmetric space with the property (*). Mereovere we represent M as a homogeneous space K/L in table 1. Then from (1. 1), the induced action of adK on M is always effective, where adK is adjoint group of K, that is, adK is $K/Z(K)$. This action is called the *standard* transitive action of M.

Theorem 1. 2

Let M be any simply-connected compact irreducible symmetric space (but B_3II) with the property (*), G a compact connected Lie group which acts on M transitively and effectively. Then the action of G is always similar to the standard transitive action of M.

Proof

For B_lI_{2m}, D_lI_{2m} and C_lI_m, we refer to [12], and for $B_lII=S^{2l}$ ($l\neq 3$), [9] and [2] have showed. Since Oniščik ranks of F_4II and G_2I are one, the theorem is true for them (cf. [11]).

Hence we have to prove the theorem for F_4I, E_6II, E_7V, E_7VI E_8VIII and E_8IX. Remaining sections will be spent to the proofs of the theorem for them.

NOTE: For $B_3II=S^6$, there is a non-standard transitive action $G_2/SU(3)$.

2. the Symmetric Space F_4I

We consider the symmetric space $F_4I=F_4/SU(2)\cdot Sp(3)=F_4/A_1\times C_3$ in this section.

Let T be a maximal torus of F_4, x_1, x_2, x_3, x_4 the canonical parameters of T. Then it is well-known that $H^*(B_T)$ is isomorphic to $R[x_1, x_2, x_3, x_4]$, where B_T is the classifying space of T.

Now we take the set $\{\pm x_i\ (i=1, 2, 3, 4),\ \pm x_i \pm x_j\ (1\leqq i<j\leqq 4),\ \frac{1}{2}(\pm x_1\pm x_2\pm x_3\pm x_4)\}$ as the root sysyem of F_4. Put $\Delta=\{\pm x_j\ (1\leqq i\leqq 4),\ \frac{1}{2}(\pm x_1\pm x_2\pm x_3\pm x_4)\}$, and for any positive integer k let $I_k=\frac{1}{2}\sum_{\alpha\in\Delta}\alpha^k$. Then we have $H^*(B_{F_4})\cong H^*(B_T)^{W(F_4)}\cong \boldsymbol{R}[I_2, I_6, I_8, I_{12}]$, where $W(F_4)$ is the Weyl group of F_4.

Set $\sigma_i(x^2)=\sigma_i(x_1^2, x_2^2, x_3^2, x_4^2)$ the i-th elementary symmetric polynomial. Then we have $I_2=3\sigma_1(x^2)$, $I_6=9\sigma_3(x^2)-\frac{3}{2}\sigma_2(x^2)\cdot\sigma_1(x^2)+9\sigma_1(x^2)^3$ (see [13], p. 316).

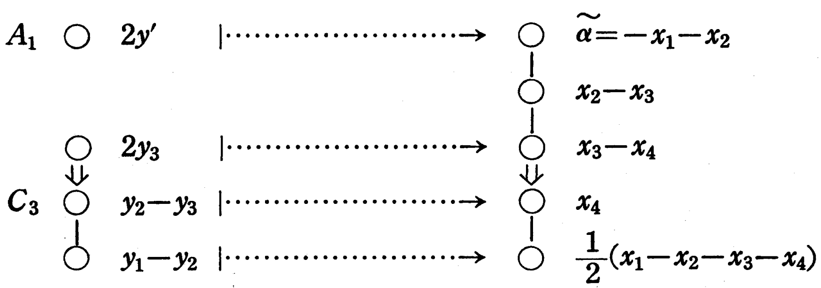
LEMMA 2. 1

For F_4I, we have $P(R, t)=t^{15}+t^{23}$ and $P(S, t)=t^3+t^7$.

PPOOF

In $F_4I=F_4/A_1\times C_3$, $A_1\oplus C_3$ (the Lie algebra of $A_1\times C_3$) is a regular subalgebra of F_4 [5], p. 142).

Let $T'=T_1\times T_2$ be a maximal torus of $A_1\times C_3$, and y', y_1, y_2, y_3 the canonical parameters of $A_1\times C_3$. Then the inclusion $A_1\times C_3\longrightarrow F_4$ can be represented by the following embedding of the Dynkin diagrams.



the Dynkin diagram of $A_1\times C_3$ the extended Dynkin diagram of F_4

It is easy to show that the defining matrix of $A_1\times C_3$ in F_4 is

$$f=\begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Hence we have

$$
\begin{aligned}
{}^t f : H^1(T) &\longrightarrow H^1(T') \\
x_1 &\longmapsto \frac{1}{2}(-y'+y_1) \\
x_2 &\longmapsto \frac{1}{2}(-y'-y_1) \\
x_3 &\longmapsto \frac{1}{2}(y_2+y_3) \\
x_4 &\longmapsto \frac{1}{2}(y_2-y_3)
\end{aligned}
$$

Here we need the following result in [3] (p. 178, Proposition 21. 3).

Let G be a compact Lie group, and U a closed subgroup of G. Put $Q_G=H^*(B_G)/D_G$, where D_G is the subspace of $H^*(B_G)$ spanned by $H^0(B_G)$ and decomposable elements of $H^*(B_G)$. Then the following diagram is commutative.

$$
\begin{array}{ccc}
P_G & \longrightarrow & P_U \\
\wr\big\downarrow\tau & & \wr\big\downarrow\tau \\
Q_G & \underset{\rho^*}{\longrightarrow} & Q_U
\end{array}
$$

where τ is transgression and ρ^* is the map induced by $\rho^*(U, G)$.

Therefore we have to investigate the map $\rho^* : Q_{F_4} \longrightarrow Q_{A_1\times C_3}$ where $Q_{F_4}=\boldsymbol{R}I_2+\boldsymbol{R}I_6+\boldsymbol{R}I_8+\boldsymbol{R}I_{12}$ and $Q_{A_1\times C_3}=\boldsymbol{R}y'^2+\boldsymbol{R}\sigma_1(y^2)+\boldsymbol{R}\sigma_2(y^2)+\boldsymbol{R}\sigma_3(y^2)$.

By simple verifications, we obtain $\rho^*(I_2)\equiv y'^2+\sigma_1(y^2)$, $\rho^*(I_6)\equiv\sigma_3(y^2)$ and $\rho^*(I_8)\equiv\rho^*(I_{12})\equiv 0 \mod D_{A_1\times C_3}$. Consequently *Ker.* $\rho^*=\boldsymbol{R}I_8+\boldsymbol{R}I_{12}$ and *Coker.* $\rho^*=\boldsymbol{R}(y'^2-\sigma_1(y^2))+\boldsymbol{R}\sigma_2(y^2)$. Hence we have $P(R, t)=t^{15}+t^{23}$ and $P(s, t)=t^3+t^7$. q.e.d.

NOTE: From (2. 1), Oniščik rank of F_4I is $P(R, 1)=2$. This can be also taken from that Oniščik rank of F_4/C_3 is two and that of $F_4/A_1\times C_3$ is not one (cf. [11]).

LEMMA 2. 2

Let $M=K/L$ be a homogeneous space such that $\chi(M)\neq 0$, K compact and simple, L semi-simple and its length $\leqq 2$, and dim. $M\leqq 12$. Then (K, L) has the following possibilities.

(K, L)		*dimension*
	(B_k, D_k) $k=2, 3, 4, 5, 6$	$2k$
	$(B_3, B_1\times D_2)$	11
	$(C_4, C_1\times C_3)$, $(C_3, C_1\times C_2)$, $(C_2, C_1\times C_1)$	8, 8, 4
	(G_2, A_2), $(G_2, A_1\times A_1)$	6, 8

PROOF.

Using results in [4], we can prove easily.

For two polynomials $f(t)=a_0+a_1t+\cdots+a_nt^n$ and $g(t)=b_0+b_1t+\cdots+b_mt^m$ of t with real coefficients, we write $f(t)\gg g(t)$ if $n\geqq m$ and $a_i>b_i$ for $i=0, 1, \cdots, n$, where we put $b_j=0$ for $j>m$.

Let G be a compact connected Lie group. We denote $k(G)$ the integer such that $2k(G)+1$ is a maximal stratification power for th space P_G.

PROPOSITIOO 2. 3

Let G be a compact connected Lie group such that G acts on the symmetric space F_4I transitively and effectively. Then the action of G is similar to the standard action of F_4I.

PROOF.

Let $G=G_1\times G_2\times \cdots \times G_n$, G_i simple, and $U=U_1\times U_2\times \cdots \times U_n$ an isotropy subgroup, $U_i\subset G_i$. Then there is just one G_i such that $k(F_4)=k(G_i)=11$, and U is semisiple. Moreover all G_j is not of type A_l ([12], p. 406, Lemma 5).

Now let $i=1$. Then $\mathfrak{g}_1$ is B_6, C_6, D_7, F_4, or E_6. Since the length of $\mathfrak{u}_1\leqq 2$, the possibilities of $(\mathfrak{g}_1, \mathfrak{u}_1)$ are following.

$\mathfrak{g}_1$	B_6	C_6	D_7	F_4	E_6
$\mathfrak{u}_1$	D_6			B_4, D_4	
	$B_i\times D_{6-i}$	$C_i\times C_{6-i}$	$D_i\times D_{7-i}$	$B_i\times D_{4-i}$	$A_1\times A_5$
	$i=1, 2, 3, 4$	$i=1, 2, 3$	$i=2, 3$	$i=1, 2$	
	$D_i\times D_{6-i}$			$D_2\times D_2$	
	$i=2, 3$			$A_1\times C_3$	
				$A_2\times A_2$	

Comparing $P(P\mathfrak{g}_1, t)$ and $P(P\mathfrak{u}_1, t)$ with (2. 1), we can cancell above possibilities mostly. After all, the following cases remain, $(C_6, C_1\times C_5)$, (F_4, B_4), $(F_4, A_1\times C_3)$.

We assume $(\mathfrak{g}_1, \mathfrak{u}_1)=(C_6, C_1\times C_5)$. Put $F_4I=[C_6/C_1\times C_5]\times M_2\times M'$. Then $dim.\ M_2 \leqq dim.\ F_4I-dim.\ C_6/C_1\times C_5=28-20=8$. Hence (2. 2) concludes that $(\mathfrak{g}_2, \mathfrak{u}_2)=(B_2, D_2)$, (B_3, D_3), (B_4, D_4), $(C_4, C_1\times C_3)$, $(C_3, C_1\times C_2)$, $(C_2, C_1\times C_1)$, (G_2, A_2) or $(G_2, A_1\times A_1)$.

If we assume $(\mathfrak{g}_2, \mathfrak{u}_2)=(B_2, D_2)$, (B_3, D_3), $(C_3, C_1\times C_2)$, $(C_2, C_1\times C_1)$, (G_2, A_2) or (G_2, A_2) or $(G_2, A_1\times A_1)$, we have $P(R_2, t)\gg t^7, t^{11}, t^{11}, t^7, t^{11}$ or t^{11} respectively. It contradicts (2. 1).

If $(\mathfrak{g}_2, \mathfrak{u}_2)=(B_4, D_4)$ or $(C_4, C_1\times C_3)$, we have $P(S_2, t)\gg t^3$. Sinc $P(S_1, t)\gg t^3$, $P(S_1+S_2, t) \gg 2t^3$. This contradicts (2. 1). Therefore we can except the case $(\mathfrak{g}_1, \mathfrak{u}_1)=(C_6, C_1\times C_5)$.

If we assume $(\mathfrak{g}_1, \mathfrak{u}_1)=(F_4, B_4)$, we can show contradictions in the same way as above.

Consequently we have $(\mathfrak{g}_1, \mathfrak{u}_1)=(F_4, A_1\times C_3)$. q.e.d.

3. the Symmetric Space E_6II

In this section we study the transitive action on the symmetric space $E_6II=E_6/SU(2)\cdot SU(6)=E_6/A_1\times A_5$.

Let $\alpha_i=x_i-x_{i+1}$, $i=1, 2, \cdots, 5$, $\alpha_6=x_4+x_5+x_6$ be the simple roots of E_6. Put

$$a_i=x_i+\frac{1}{3}(x_1+x_2+x_3+x_4+x_5+x_6),\ i=1, 2, \cdots, 6$$

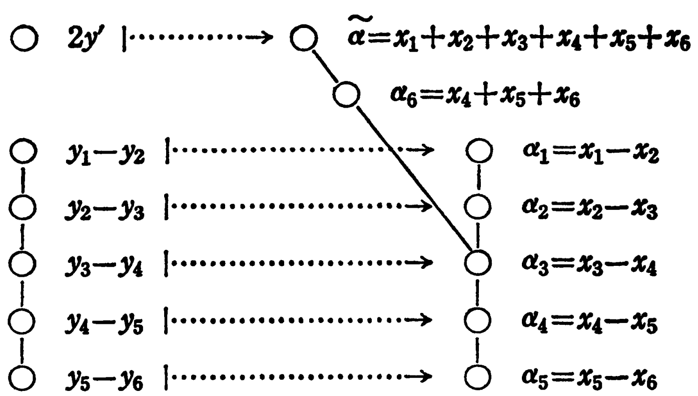
$$b_i=x_i-\frac{2}{3}(x_1+x_2+x_3+x_4+x_5+x_6),\ i=1, 2, \cdots, 6$$

$$c_{ij}=-x_i-x_j+\frac{1}{3}(x_1+x_2+x_3+x_4+x_5+x_6),\ i, j=1, 2, \cdots, 6$$

and $$I_k=\frac{1}{2}\left(\sum_i a_i^k+\sum_i b_i^k+\sum_i c_{ij}^k\right).$$

Then it is known $H^*(B_{E6})\cong H^*(B_T)^{W(E6)}\cong \boldsymbol{R}[I_2, I_5, I_6, I_8, I_9, I_{10}]$, where T is a maximal torus of E_6 (cf. [13]).

Now we consider the case E_6II as in the proof of (2. 1). Since $A_1\oplus A_5$ (the Lie algebra of $A_1\times A_5$) is a regular subalgebra of E_6, the embedding of the Dynkin diagrams is following.



the Dynkin diagram of $A_1\times A_5$ | the extended Dynkin diagram of E_6

Hence the defining matrix of $A_1\times A_5$ in E_6 is

$$f=\begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3\\ 5 & -1 & -1 & -1 & -1 & -1\\ -1 & 5 & -1 & -1 & -1 & -1\\ -1 & -1 & 5 & -1 & -1 & -1\\ -1 & -1 & -1 & 5 & -1 & -1\\ -1 & -1 & -1 & -1 & 5 & -1\\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix}$$

Consequently

$$\begin{aligned} {}^t f: H^1(T) &\longrightarrow H^1(T')\\ x_1 &\longmapsto y_1+y'\\ x_2 &\longmapsto y_2+y'\\ &\cdots\cdots\\ x_6 &\longmapsto y_6+y' \end{aligned}$$

where T' is a maximal torus of $A_1\times A_5$. Note that the reration $y_1+y_2+y_3+y_4+y_5+y_6=0$ holds.

Now we put

$$\eta=\frac{1}{2}(x_1+x_2+x_3+x_4+x_5+x_6)$$

$$\xi_i=x_i-\frac{1}{3}\eta=x_i-\frac{1}{6}(x_1+x_2+x_3+x_4+x_5+x_6),\ i=1,2,\cdots,6$$

and
$$\nu=\frac{1}{2}(x_4+x_5+x_6).$$

Then we we have

$$\begin{cases}\eta \longmapsto y' \\ \xi_i \longmapsto y_i,\ i=1,2,\cdots,6\end{cases}$$

Moreover by [5], p. 777 it holds

$$I_k=\frac{1}{2}\left(\sum a_i^k+\sum b_i^k+\sum c_{ij}^k\right)$$

$$=\sum_{j=0}^{[\frac{k}{2}]}\binom{k}{2j}s_{k-2j}\,\eta^{2j}+\frac{(-1)^k}{2}\left\{(6-2^{k-1})s_k+\frac{1}{2}\sum_{r=0}^{k-2}\binom{k}{r}s_r\,s_{k-r}\right\},$$

where $s_k=\xi_1^k+\xi_2^k+\cdots+\xi_6^k$.

Therefore

$$(**)\qquad I_k\longmapsto\sum_{j=0}^{[\frac{k}{2}]}\binom{k}{2j}s'_{k-2}\,y'^{2j}+\frac{(-1)^k}{2}\left\{(6-2^{k-1})s'_k\right.$$

$$\left.+\frac{1}{2}\sum_{r=2}^{k-2}\binom{k}{r}s'_r\,s'_{k-r}\right\}$$

where $s'_k=y_1^k+y_2^k+\cdots+y_6^k$.

LEMMA 3. 1

For E_6II, we have $P(R,t)=t^{11}+t^{17}+t^{23}$ and $P(S,t)=t^3+t^5+t^7$. Therefore Oniščk rank of E_6II is three.

PROOF.

Using (**), it can be shown that $\rho^*(I_2)\equiv 6(y'^3-\sigma_2(y))$, $\rho^*(I_5)\equiv -36\sigma_5(y)$, $\rho^*(I_6)\equiv 150\sigma_6(y)$ and $\rho^*(I_8)\equiv\rho^*(I_{12})\equiv 0$ *mod* D_{E_6}. Consequently $Ker.\rho^*=\boldsymbol{R}I_8+\boldsymbol{R}I_9+\boldsymbol{R}I_{12}$ and $Coker.\ \rho^*=\boldsymbol{R}(y'^2+\sigma_2(y))+\boldsymbol{R}\sigma_3(y)+\boldsymbol{R}\sigma_4(y)$. Taking them back by transgression, we conclude that $P(R,t)=t^{15}+t^{17}+t^{23}$ and $P(S,t)=t^3+t^5+t^7$. q.e.d.

PROPOSITION 3. 2

Let G be a compact connected Lie group which acts on the symmetric space E_6II effectively and transitively. Then the action of G is similar to the standard transitive action of E_6II.

PROOF.

As the proof of (2. 3) we can show that the possibilities of $(\mathfrak{g}_1, \mathfrak{u}_1)$ are $(C_6, C_1\times C_5)$, (F_4, B_4), $(F_4, B_1\times B_3)$ and $(E_6, A_1\times A_5)$.

Assum that $(\mathfrak{g}_1, \mathfrak{u}_1)=(C_6, C_1\times C_5)$, (F_4, B_4) or $(F_4, B_1\times B_3)$. Then $P(R_1, t)$ has no term of t^{17}. Therefore there is some integer i such that $k(G_i)=8$. Then G_i is of type A_8. But this contradicts the fact that every G_j is not of type A_l. Hence we have $(\mathfrak{g}_1, \mathfrak{u}_1)=(E_6, A_1\times A_5)$. q.e.d.

4. the Symmetric Spaces E_7V and E_7VI

(1) $E_7V=E_7/SU^*(8)=E_7/A_7$.

Let G be a compact connected Lie group which acts on the symmetric space E_7V. As in section 2 and 3, we take $(\mathfrak{g}_1, \mathfrak{u}_1)$. Then $\mathfrak{g}_1$ is B_9, C_9, D_{10} or E_7, since $k(E_7)=k(G_1)=17$. Moreover we have the length of $\mathfrak{u}_1\leqq 2$.

By [4], the possibilities of $(\mathfrak{g}_1, \mathfrak{u}_1)$ are following.

$\mathfrak{g}_1$	B_9	C_9	D_{10}	E_7
$\mathfrak{u}_1$	D_9			A_7
	$B_i\times D_{9-i}$	$C_i\times C_{9-i}$	$D_i\times D_{10-i}$	$A_1\times D_6$
	$i=1, 2, \cdots, 7$	$i=2, 3, 4$	$i=2, 3, 4, 5$	
	$D_i\times D_{9-i}$			$A_2\times A_5$
	$i=2, 3, 4$			

(i) the case $\mathfrak{g}_1=B_9$.

If $(\mathfrak{g}_1, \mathfrak{u}_1)=(B_9, D_9)$, we have $P(S_1, t)\gg t^{17}$. This contradicts the fact $P(P_{A_7}, t)\gg P(S, t)\gg P(S_1, t)$.

If $(\mathfrak{g}_1, \mathfrak{u}_1)=(B_9, B_i\times D_{9-i})$ for $i=1, 2, \cdots, 7$, we have $P(R_1, t)\gg t^{31}$. However $P(P_{E_7}, t)$ has no term of t^{31}. Therefore it is impossible.

If $(\mathfrak{g}_1, \mathfrak{u}_1)=(B_9, D_i\times D_{9-i})$ for $i=2, 3, 4$, we have $P(R_1, t)\gg t^{31}$. It is a contradiction.

(ii) the case $\mathfrak{g}_1=D_9$.

If $(\mathfrak{g}_1, \mathfrak{u}_1)=(C_9, C_i\times C_{9-i})$ $i=2, 3, 4$, we have $P(R_1, t)\gg t^{31}$. It is a contradiction.

(iii). the case $\mathfrak{g}_1=D_{10}$.

If $(\mathfrak{g}_1, \mathfrak{u}_1)=(D_{10}, D_i\times D_{10-i})$ for $i=2, 3, 4$, we have $P(R_1, t)\gg t^{31}$. It is is impossible.

(iv) the case $\mathfrak{g}_1=E_7$.

We assume $(\mathfrak{g}_1, \mathfrak{u}_1)=(E_7, A_2\times A_5)$. Then we have $P(S_1, t)\gg 2t^5$. But $P(P_{A_5}, t)$ has no term $2t^5$. Hence it is impossible.

Therfore we conclude that the possibilities of $(\mathfrak{g}_1, \mathfrak{u}_1)$ are $(C_9, C_1\times C_8)$, (E_7, A_7) and $(E_7, A_1\times D_6)$.

PROPOSITION 4. 1

Let G be a compact connected Lie group which acts on the symmetric space E_7V transitively and effectively. Then the action of G is similar to the standard action of E_7V.

PROOF.

It is sufficient to show that $(\mathfrak{g}_1, \mathfrak{u}_1)$ can be neither $(C_9, C_1\times C_8)$ nor $(E_7, A_1\times D_6)$.

We set $M=E_7/A_7$ and $M_1=G_1/U_1$. If M_1 is $C_9/C_1\times C_8$, we can take $\pi_9(M_1)=\boldsymbol{Z}_2$. On the other hand $\pi_9(M_1)=\boldsymbol{Z}$, and so it is impossible. If M_1 is $E_7/A_1\times D_6$, we have $\pi_9(M_1)=\boldsymbol{Z}_3+\boldsymbol{Z}_2$. It is impossible. q.e.d.

NOTE: About homotopy groups of Lie groups, we refer to the tables in "Mathematics Dictionary" (in Japanese) Iwanami, 1968.

(2) $E_7VI=E_7/SU(2)\cdot Spin(12)=E_7/A_1\times D_6$.

Since $P(P_{A_1\times D_6},t)=2t^5+\cdots$, we can see that the length of $\mathfrak{u}_1\leqq 3$. Now we assume that the length of $\mathfrak{u}_1$ is just three. Then the possibilities of $(\mathfrak{g}_1, \mathfrak{u}_1)$ are following.

$\mathfrak{g}_1$	B_9	C_9	D_{10}	E_{71}
$\mathfrak{u}_1$	$B_i\times D_j\times D_k$ $D_i\times D_j\times D_k$ $i+j+k=9$	$C_i\times C_j\times C_k$ $i+j+k=9$	$D_i\times D_j\times D_k$ $i+j+k=10$	$A_1\times D_i\times D_{6-i}$ $i=2, 3$

If $\mathfrak{g}_1=B_9$, C_9 or D_{10}, we have $P(R_1, t)\gg t^{31}$. But $P(P_{E_7}, t)$ has no term of t^{31}, it is a contradiction.

Now we asume that $(\mathfrak{g}_1, \mathfrak{u}_1)$ is $(E_7, A_1\times D_i\times D_{6-i})$ for $i=2, 3$. Then it can be shown that $P(S_1, t)\gg 3t^3$ for $i=2$ and $P(S_1, t)\gg 2t^5$ for $i=3$. So it is impossible.

Hence we conclude the length of $\mathfrak{u}_1\leqq 2$.

As in (1), we can see that the possibilities of $(\mathfrak{g}_1, \mathfrak{u}_1)$ are $(C_9, C_1\times C_8)$, (E_7, A_7) and $(E_7, A_1\times D_6)$.

PROPOSITION 4.2

Let G be a compact connected Lie group which acts on the symmetric space E_7VI transitively and effectively. Then action of G is similar to the standard transitive action of E_7VI.

PROOF

We can see easily that $\pi_9(E_7VI)=\boldsymbol{Z}_3+\boldsymbol{Z}_2$ and $\pi_9(E_7V)=\boldsymbol{Z}$. Therefore we omit the case (E_7, A_7).

Now we put $M_1=C_9/C_1\times C_8$. Then we can take that there is some i for $i\geqq 2$ such that $k(G_i)=13$ and the length of $\mathfrak{u}_i\leqq 2$. Set $i=2$. Then the possibilities of $(\mathfrak{g}_2, \mathfrak{u}_2)$ is following.

$\mathfrak{g}_2$	B_7	C_7	D_8
$\mathfrak{u}_2$	D_7 $B_i\times D_{7-i}$ $i=1, 2, \cdots, 5$ $D_i\times D_{7-i}$ $i=2, 3$	$C_i\times C_{7-i}$ $i=1, 2, 3$	$D_i\times D_{8-i}$ $i=2, 3, 4$

(i) the case $\mathfrak{g}_2=B_7$.

If $(\mathfrak{g}_2, \mathfrak{u}_2)=B_7, D_7)$, then $P(S_2, t)\gg t^{13}$. It is impossible.

If $(\mathfrak{g}_2, \mathfrak{u}_2)=(B_7, B_i\times D_{7-i})$ for $i=2, 3, 4$, then we have

$$P(S_2, t)\gg\begin{cases} t^9 & \text{if } i=2 \\ 2t^7 & \text{if } i=3 \\ t^5 & \text{if } i=4\ . \end{cases}$$

Therefore it is impossible.

If $(\mathfrak{g}_2, \mathfrak{u}_2)=(B_7, D_i\times D_{9-i})$ for $i=2, 3$, then we have

$$P(S_2, t)\gg\begin{cases} t^9 & \text{if } i=3 \\ t^5 & \text{if } i=4\ . \end{cases}$$

Hence it is a contrdiction.

(ii) the case $\mathfrak{g}_2=D_8$.

If $(\mathfrak{g}_2, \mathfrak{u}_2)=(D_8, D_i\times D_{8-i})$ for $i=3, 4$, then we have

$$P(S_2, t)\gg\begin{cases} t^9 & \text{if } i=3 \\ 3t^7 & \text{if } i=4\ . \end{cases}$$

Therefore it is impossible.

From (i) and (ii), we see that the possibilities of $(\mathfrak{g}_2, \mathfrak{u}_2)$ are $(B_7, B_1\times D_6)$, $(B_7, B_5\times D_2)$, $(C_7, C_1\times C_6)$, $(C_7, C_2\times C_5)$, $(C_7, C_3\times C_4)$ and $(D_8, D_2\times D_6)$.

Now we put $M_2=G_2/U_2$, then $E_7VI=M_1\times M_2\times M'$.

If $M_2=B_7/B_1\times D_6$, Then the facts, $\pi_{10}(E_7VI)=\boldsymbol{Z}_3+\boldsymbol{Z}_2$ and $\pi_{10}(M_1)=\pi_{10}(C_9/C_1\times C_8)=\boldsymbol{Z}_3$, follow that $\pi_{10}(M_2)$ is trivial or $\boldsymbol{Z}_2$. Considering the homotopy exact sequence of the fibre bundle $(B_7, M_2; B_1\times D_6)$:

$$\pi_{10}(B_7)\longrightarrow\pi_{10}(M_2)\longrightarrow\pi_9(B_1\times D_6)\longrightarrow\pi_9(E_7),$$

we take a contradiction.

As in above we can take contradictions for the cases $(C_7, C_1\times C_6)$ and $(D_8, D_2\times D_6)$. Hence the two cases $(C_7, C_2\times C_5)$ and $(C_7, C_3\times C_4)$ remain. But for them we have contradictions by comparing their dimensions. For example, if $M_2=C_7/C_2\times C_5$, then dim. $M_2=40$. Since dim. $E_7VI=64$ and dim. $M_1=$dim. $C_9/C_1\times C_8=38$, we have dim. $E_7VI <$dim. M_1+dim. M_2 Obviously it is a contradiction. q.e.d.

5. the Symmetric Spaces E_8VIII and E_8IX

(1) $E_8VIII=E_8/SO(16)=E_8/D_8$.

As in the proof of section 4, we cancel the most possibilities of $(\mathfrak{g}_1, \mathfrak{u}_1)$, and remain only three cases $(C_{15}, C_1\times C_{14})$, (E_8, D_8) and $(E_8, A_1\times E_7)$.

PROPOSION 5. 1

Let G be a compact connected Lie group which acts on the symmetric space E_8VIII transi-

tively and effectively. Then the action of G is similar to the standard transitive action of $E_8 VIII$.

PROOF

It is sufficient to say that $(\mathfrak{g}_1, \mathfrak{u}_1)$ cannot be neither $(C_{15}, C_1\times C_{14})$ nor $(E_8, A_1\times E_7)$. This is led from the fact that $\pi_{10}(E_8 VIII)=\boldsymbol{Z}_2$, $\pi_{10}(C_{15}/C_1\times C_{14})=\boldsymbol{Z}_3$ and $\pi_{10}(E_8/A_1\times E_7)=\boldsymbol{Z}_3$. q.e.d.

(2) $E_8IX=E_8/SU(2)\cdot E_7$.

We need the following lemma.

LEMMA 5. 2

The intgral homology of E_8IX *has* $\boldsymbol{Z}_2$*-torsion.*

PROOF

In the symmetric space $E_8IX=E_8/SU(2)\cdot E_7$, we have $SU(2)\cap E_7=\boldsymbol{Z}_2$. Therefore by the homotopy exact sequence of the fibre bundle $(SU(2)\times E_7,\ SU(2)\cdot E_7;\ \boldsymbol{Z}_2)$, we have $\pi_1(SU(2)\cdot E_7)=\boldsymbol{Z}_2$. Moreover we can take $\pi_2(E_8IX)=\boldsymbol{Z}_2$ and $\pi_1(E_8IX)=\pi_0(E_8IX)=0$. Using the Hurewicz isomorphism theorem, we have $H_2(E_8IX)=\boldsymbol{Z}_2$.

In the same way of section 4, we see the length of $\mathfrak{u}_1\leqq 2$, and moreover the possibilities of $(\mathfrak{g}_1, \mathfrak{u}_1)$ are $(C_{15}, C_1\times C_{14})$, (E_8, D_8) and $(E_8, A_1\times E_7)$.

PROPOSITION 5. 3

Let G be a compact connected Lie group which acts on the symmetric space E_8IX *transitively and effectively. Then the action of G is similar to the standard transitive action of* E_8IX.

PROOF

Since $\pi_{10}(E_8IX)=\boldsymbol{Z}_3$, and $\pi_{10}(E_8/D_8)=\boldsymbol{Z}_2$, we cancel the case (E_8, D_8).

Now we assume $(\mathfrak{g}_1, \mathfrak{u}_1)=(C_{15}, C_1\times C_{14})$. Then we have $P(R_1, t)=t^{59}$ and so there is just one i such that $k(G_i)=23$. We set $i=2$. Then we have the length of $\mathfrak{g}_2\leqq 2$, and by the same consideration of section 4, we see that the remaining possibilities of $(\mathfrak{g}_2, \mathfrak{u}_2)$ are (B_{12}, D_{12}) and $(C_{12}, C_1\times C_{11})$.

Now we put $M=E_8IX$, $M_1=C_{15}/C_1\times C_{14}$ and $M_2=G_2/U_2$, then we have $\pi_{10}(M_2)=\boldsymbol{Z}_3$. Therefore

$$\begin{aligned}\pi_{10}(M)&=\pi_{10}(M_1)+\pi_{10}(M_2)+\pi_{10}(M'')\\&=\boldsymbol{Z}_3+\boldsymbol{Z}_3+\pi_{10}(M'')\end{aligned}$$

This contradicts $\pi_{10}(E_8IX)=\boldsymbol{Z}_3$. Hence we can omit the case $(\mathfrak{g}_2, \mathfrak{u}_2)=(C_{12}, C_1\times C_{11})$.

Now we assume $(\mathfrak{g}_2, \mathfrak{u}_2)=(B_{12}, D_{12})$. Here we note $M''\neq\phi$, where $M=M_1\times M_2\times M''$. As the above consideration, we see $M=M_1\times M_2\times M_3\times M'''$, where $M_3=G_3/U_3$ and $(\mathfrak{g}_3, \mathfrak{u}_3)=(B_{10}, D_{10})$. Then it is easy to see that dim.$M''=12$. Hence we can use (2. 2), and so we have $M''=B_6/D_6$.

After all, we can take that

$$M=[C_{15}/C_1\times C_{14}]\times B_{12}/D_{12}]\times[B_{10}/D_{10}]\times[B_6/D_6],$$

that is,

$$M=[Sp(15)/Sp(1)\times Sp(14)]\times S^{24}\times S^{20}\times S^{12}.$$

Since it has torsion-free homology, we can see by (5. 2) that it is impossible. q.e.d.

6. Appendix

In this section we consider transitive actions of a compact Lie group G on Grassmann manifolds $G_{2n,\,2k-1}(2<k<n-1)$. Here we note that these manifolds have zero Eular number, and therefore the classification of transitive actions on them is more difficult than before. So we assume that G is simple. Then we can use the following lemma.

LEMMA 6. 1 ([11], p. 169, Theorem 7)

Let M be a homogeneous manifold G/H where G is a compact simple Lie group of type B_n, C_n or D_{n+1} and H is a closed subgroup of G. If G' is a compact simple Lie group which acts on M transitively and effectively, then G' is of type B_n, C_n or D_{n+1}.

THEOREM 6. 2

Let G be a compact simple Lie group which acts on a Grassmann manifold $G_{2n,\,2k-1}$ $(2<k<n-1)$ transitively and effectively. Then the action of G is similar to the standard transitive action.

PROOF.

From above lemma, G is B_{n-1}, C_{n-1} or D_n. On the other hand by simple verifications we have

$$P(R,\,t)=\begin{cases} t^{(4n-k)+3}+t^{(4n-k)+7}+\ \cdots\ +t^{4n-3}+t^{2n-1} & (n\geqq 2k-1)\\ t^{4k-1}+t^{4k+3}+\ \cdots\ +t^{4n-3}+t^{2n-1} & (n\leqq 2k-1)\end{cases}$$

Therefore we see that there is a non-zero element in $R\subset P_G$ such that its degree is $2n-1$. So G is of type D_n.

From [11] theorem 1, we have

$$\frac{P(G,\,t)}{P(H,\,t)}=\frac{P(SO(2n),\,t)}{P(SO(2(n-k)+1)\times SO(2k-1),\,t)}$$

Hence

$$\begin{aligned}P(H,\,t)&=P(SO(2(n-k)+1)\times SO(2k-1),\,t)\\ &=P(SO(2(n-k)+1),\,t)+P(SO(2k-1),\,t).\end{aligned}$$

Therefore we conclude that H is $B_{n-k}\times B_{n-k}$, $C_{n-k}\times B_{k-1}$, $B_{n-k}\times C_{k-1}$ or $C_{n-k}\times C_{k-1}$.

Now we consider an irreducible orthogonal representation

$$\phi:C_{n-k}\times B_k\longrightarrow D_m.$$

We set the complexification $\phi^C:C_{n-k}\times B_k\longrightarrow D_m^C$ of ϕ. Then we have $\phi^C=\phi_1+\phi_2$, where ϕ_1 and ϕ_2 are complex representations of C_{n-k} and B_k respectively. Since dimensions of non-trivial orthogonal representations are more than $4k-1$, we have

$m=\dim_R\phi=\dim_C\phi^C=\dim_C\phi_1+\dim_C\phi_2\geqq 2(n-k)+k-1=2n-k-1$. For $k<n-1$, we have $2n-k-1>n$, i.e. $m>n$. Therefore H is not $C_{n-k}\times B_{k-1}$. As in above we can see that H is neither $B_{n-k}\times C_{k-1}$ nor $C_{n-k}\times C_{k-1}$. Hence we have H is $B_{n-k}\times B_k$. Moreover non-trivial homomorphism $B_{n-k}\times B_k\longrightarrow D_n$ is only a standard inclusion, and so the action of G is similar to the standard transitive action. q.e.d.

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