On the quasi-spannability of graphs

By

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1. Definitions and general results

In [1], Pippenger-Golumbic defined the *inducibility* of k-vertex graphs. Similarly to [1], we will define the *'quasi'-spannability* of k-edge graphs.

In this paper, a *graph* is an undirected graph with no loops, no multiple edges and no isolated vertices. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively.

Let G, H be graphs such that $|E(G)|=k$ and $|E(H)|=n$ where $k<n$. $s(G, H)$ denote the number of subgraphs of H, which are isomorphic to G. Since this number lies between 0 and, $\binom{n}{k}$, we normalize it by setting $S(G, H)=\frac{s(G, H)}{\binom{n}{k}}$. Let $s(G, H)$ denote the maximum, taken over all n-edge graphs H, of $s(G, H)$. A subgraph of H which is isomorphic to G, if necessary by adding some isolated vertices, is a spanning graph of H. Thus, we define the *n-quasi-spannability* of G to be $S(G, n)=\frac{s(G, n)}{\binom{n}{k}}$.

THEOREM 1. *For any k-edge graph G and any $n>k$, $S(G, n+1)\leqq S(G, n)$.*

PROOF. Let H be a $(n+1)$-edge graph such that $s(G, H)=s(G, n+1)$. Each subgraph of H which is isomorphic to G has k edges, and $|E(H)|=n+1$. Hence, there is some edge e of H such that $m(e)\leqq k\cdot\frac{s(G, n+1)}{n+1}$ where $m(e)$ denotes the number of subgraphs of H which are isomorphic to G and contain e. If we delete this edge (and all isolated vertices), the resulting graph $H-e$ has n edges, and we have:

$$s(G, n)\geqq s(G, H-e)\geqq s(G, n+1)-k\cdot\frac{s(G, n+1)}{n+1}.$$

Hence,

$$\frac{s(G, n)}{\binom{n}{k}}\geqq\frac{s(G, n+1)}{\binom{n}{k}\frac{n+1}{n+1-k}}=\frac{s(G, n+1}{\binom{n+1}{k}}.$$

This completes the proof of Theorem 1.

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By Theorem 1, it follows that the sequence $S(G, n)$ is nonincreasing and bounded below by 0. Thus, we define the *quasi-spannability* of G to be $S(G)=\lim_{n\to\infty} S(G, n)$.

For graphs G, G', $G\cup G'$ denotes the disjoint sum of G and G', and $2G=G\cup G$ etc. K_n and $L_n(=K_{1,n})$ denote the complete n-vertex graph and the n-claw, respectively. P_n and C_n denote the path and the cycle with n edges, respectively.

The following theorems are obtained easily:

Theorem 2. (1) $S(kP_1)=1$.
(2) $S(L_k)=1$.

In fact, it is clear that $s(kP_1, nP_1)=\binom{n}{k}$ and $s(L_k, L_n)=\binom{n}{k}$.

Theorem 3. (1) $S(P_k, k+1)=1$.
(2) $S(hP_2\cup P_1, 2h+2)=1$.

In fact, it is clear that $s(P_k, C_{k+1})=k$ and $s(hP_2\cup P_1, (h+1)P_2)=2h+2$.

Theorem 4. (1) $s(K_k, K_m)=\binom{n}{k}$.

(2) $$s(L_h\cup L_{k-h}, L_r\cup L_{n-r})=\begin{cases}\binom{r}{h}\binom{n-r}{k-h}+\binom{r}{k-h}\binom{n-r}{h}, & h\neq k-h\\ \binom{r}{h}\binom{n-r}{h}, & h=k-h\end{cases}$$

For $n>k$, there is unique (m, r) such that $\binom{m}{2}+r=n$, $0\leqq r<m$. K_m*L_r denotes the graph which is shown in Fig. 1—— $V(K_m*L_r)=V(K_m)\cup\{v\}$, $E(K_m*L_r)=E(K_m)\cup EL$ where $EL=\{e_1, \dots, e_r\}$.

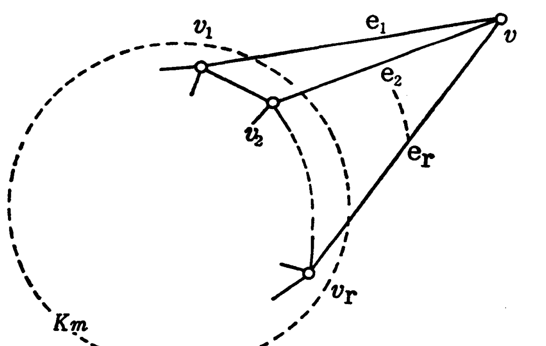

Fig. 1

Theorem 5. $s(P_k, K_m*L_r)=\frac{1}{2}\,{}_mP_{k+1}+r\cdot{}_{m-1}P_{k-1}+\binom{r}{2}(k-1)\cdot{}_{m-2}P_{k-2}$.

Proof. The first term is equal to $s(P_k, K_m)$——the half of the number of permutations of $(k+1)$ elements in $V(K_m)$. The second term is the number of P_k's such that $|E(P_k)\cap EL|=1$: Let e_i be the said edge. If $v_{i_1}, \dots, v_{i_{k-1}}$ are $(k-1)$ elements in $V(K_m)-\{v_i\}$, then there is one required P_k——the path $(v\ v_1\ v_{i_1}\cdots v_{i_{k-1}})$. The last term is the number of P_k's such that $|E(P_k)\cap EL|=2$: Let e_i, $e_{i'}$ be the said edges. If $v_{i_1}, \dots, v_{i_{k-2}}$ are $(k-2)$ elements in $V(K_m)-\{v_i, v_{i'}\}$, then there are $(k-1)$ required P_k's——the paths $(v_{i_1}\cdots v_{i_j}\ v_i\ v\ v_{i'}\ v_{i_{j+1}}\cdots v_{i_{k-2}})$ $(j=0, 1, \dots, k-2)$.

2. Results for small k's

In this section, we consider the quasi-spannability of k-edge graphs for $k=1$, 2, 3 and 4.

Case of $k=1$. By Theorem 2, we have: $S(P_1)=1$.

Case of $k=2$. By Theorem 2, we have: $S(2P_1)=1$ and $S(P_2)=1$.

Case of $k=3$. There are five 3-edge graphs——$3P_1$, L_3, $P_2 P_1$, P_3 and K_3.

(1) $S(3P_1)=1$ and $S(L_3)=1$ by Theorem 2.

(2) $S(P_2 \cup P_1) \geqq \frac{3}{4}$. In fact, by Theorem 4 (2),

$$s(P_2 \cup P_1, L_r \cup L_{n-r}) = \binom{r}{2}\binom{n-r}{1} + \binom{r}{1}\binom{n-r}{2} = \frac{r(n-r)(n-2)}{2}.$$

This number is maximum for $r=\left[\frac{n}{2}\right]$: that is,

$$s(P_2 \cup P_1, 2L_r) = \frac{1}{8} n^2 (n-2) \qquad \text{for } n=2r,$$

$$s(P_2 \cup P_1, L_r \cup L_{r+1}) = \frac{1}{8}(n+1)(n-1)(n-2) \qquad \text{for } n=2r+1$$

Hence, we have:

$$S(P_2 \cup P_1, n) \geqq \begin{cases} \dfrac{3n}{4(n-1)} & \text{for even } n \\ \dfrac{3(n+1)}{4n} & \text{for odd } n \end{cases}$$

(3) By Theorem 5,

$$s(P_3, K_m * L_r) = \frac{1}{2} m(m-1)(m-2)(m-3) + r(m-2)(m+r-2).$$

Hence, we have: $\lim_{m\to\infty} S(P_3, K_m * L_r)=0$

(4) $s(K_3, K_m * L_r) = \binom{m}{3} + \binom{r}{2}$. In fact, the first term is equal to $s(K_3, K_m)$. The second term is the number of K_3's such that $|E(K_3) \cap EL|=2$: Let e_i, $e_{i'}$ be the said edges. Then there is one required K_3——the cycle $(v\, v_i\, v_{i'}\, v)$. Hence, we have:
$\lim_{n\to\infty} S(K_3, K_m * L_r)=0$.

Case of $k=4$. There are eleven 4-edge graphs——$4P_1$, L_4, $2P_2$, $L_3 \cup P_1$, P_4, C_4, $K_3 * P_1$, T_4 (See Fig. 2), $P_3 \cup P_1$, $K_3 \cup P_1$ and $P_2 \cup 2P_1$.

(1) $S(4P_1)=1$ and $S(L_4)=1$ by Theorem 2.

(2) $S(2P_2) \geqq \frac{3}{8}$. In fact, by Theorem 4 (2),

$$s(2P_2, L_r \cup L_{n-r}) = \binom{r}{2}\binom{n-r}{2} = \frac{1}{4} r(r-1)(n-r)(n-r-1).$$

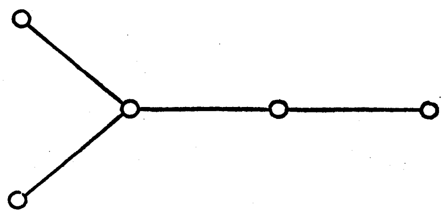

Fig. 2

This number is maximum for $r=\left[\frac{n}{2}\right]$: that is,

$$s(2P_2, 2L_r) = \frac{1}{64} n^2 (n-2)^2 \qquad \text{for } n=2r,$$

$$s(2P_2, L_r \cup L_{r+1}) = \frac{1}{64}(n+1)(n-1)^2(n-3) \qquad \text{for } n=2r+1.$$

Hence, we have:

$$s(2P_2, n) \geqq \begin{cases} \dfrac{3n(n-2)}{8(n-1)(n-3)} & \text{for even } n, \\ \dfrac{3(n+1)(n-1)}{8n(n-2)} & \text{for odd } n. \end{cases}$$

(3) $S(L_3 \cup P_1) \geqq \frac{1}{2}$. In fact, by Theorem 4 (2),

$$s(L_3 \cup P_1, L_r \cup L_{n-r}) = \binom{r}{3}\binom{n-r}{1} + \binom{r}{1}\binom{n-r}{3} = \frac{1}{6} r(n-r)((r-1)(r-2) + (n-r-1)(n-r-2)).$$

This number is maximum for $r = \left[\frac{n+1+\sqrt{3n-4}}{2}\right]$ and $n-r = \left[\frac{n+1-\sqrt{3n-4}}{2}\right]$. For these r and $n-r$, the coefficient of n^4 in $s(L_3 \cup P_1, L_r \cup L_{n-r})$ is equal to $\frac{1}{48}$. On the other hand, the coefficient of n^4 in $\binom{n}{4}$ is equal to $\frac{1}{24}$. Hence, we have: $S(L_3 \cup P_1) \geqq \frac{1}{2}$.

(4) By Theorem 5,

$$s(P_4, K_m * L_r) = \frac{1}{2} m(m-1)(m-2)(m-3)(m-4) + \frac{1}{2} r(m-2)(m-3)(2m+3r-5).$$

Hence, we have: $\lim_{m \to \infty} S(P_4, K_m * K_r) = 0$.

(5) $s(C_4, K_m * L_r) = 3\binom{m}{4} + \binom{r}{2}(n-2)$. In fact, the first term is equal to $s(C_4, K_m)$: Let $v_1, \cdots, v_4$ be four vertices of K_m. Then there are three required C_4's——the cycles $(v_1 v_2 v_3 v_4 v_1)$, $(v_1 v_2 v_4 v_3 v_1)$ and $(v_1 v_3 v_2 v_4 v_1)$. The second term is the number of C_4's such that $|E(C_4) \cap EL| = 2$: Let $e_i, e_{i'}$ be the said edges. If v_j is an element in $V(K_m) - \{v_i, v_{i'}\}$, then there is one required C_4——the cycle $(v v_i v_j v_{i'} v)$. Hence, we have: $\lim_{n \to \infty} S(C_4, K_m * L_r) = 0$.

(6) $s(K_3 * P_1, K_m * L_r) = 3\binom{m}{3}(m-3) + r\binom{m-1}{2} + 2\binom{r}{2}(m-2) + 3\binom{r}{3}$. In fact, the first term is equal to $s(K_3 * P_1, K_m)$: Let v_1, v_2, v_3 be three vertices of K_m, and v_j be an element in $V(K_m) - \{v_1, v_2, v_3\}$. Then there are three required $K_3 * P_1$'s——the graphs $(v_1 v_2 v_3 v_1) * (v_i v_j)$ $(i = 1, 2, 3)$. The second term is the number of $K_3 * P_1$'s such that $|E(K_3 * P_1) \cap EL| = 1$: Let e_i be the said edge, and $v_j, v_{j'}$ be two elements in $V(K_m) - \{v_i\}$. Then there is one required $K_3 * P_1$——the graph $(v_i v_j v_{j'} v_i) * (v v_i)$. The third term is the number of $K_3 * P_1$'s such that $|E(K_3 * P_1) \cap EL| = 2$: Let $e_i, e_{i'}$ be the said edges, and v_j be an element in $V(K_m) - \{v_i, v_{i'}\}$. Then there are two required $K_3 * P_1$'s——the graphs $(v v_i v_{i'} v) * (v_i v_j)$ and $(v v_i v_{i'} v) * (v_{i'} v_j)$. The last term is the number of $K_3 * P_1$'s such that $|E(K_3 * P_1) \cap EL| = 3$: Let $e_i, e_{i'}, e_{i''}$ be the said edges. Then there are three required $K_3 * P_1$'s——the graphs $(v v_i v_{i'} v) * (v v_{i''})$, $(v v_{i'} v_{i''} v) * (v v_i)$ and $(v v_{i''} v_i v) * (v v_{i'})$. Hence, we have: $\lim =_{n \to \infty} S(K_3 * P_1, K_m * L_r) = 0$.

(7) Let $L_r ** L_{n-r-1}$ denote the graph which is shown in Fig. 3.

Since $T_4=L_1{**}L_2$, we have:

$$s(T_4,\ L_r{**}L_{n-r-1})=\binom{r}{2}\binom{n-r-1}{1}+\binom{r}{1}\binom{n-r-1}{2}=\frac{1}{2}\,r(n-r-1)\,(n-3).$$

This number is maximum for $r=\left[\frac{n-1}{2}\right]$: that is,

$$s(T_4,\ L_r{**}L_{r-1})=\frac{1}{8}\,n(n-2)\,(n-3)\qquad \text{for } n=2r,$$

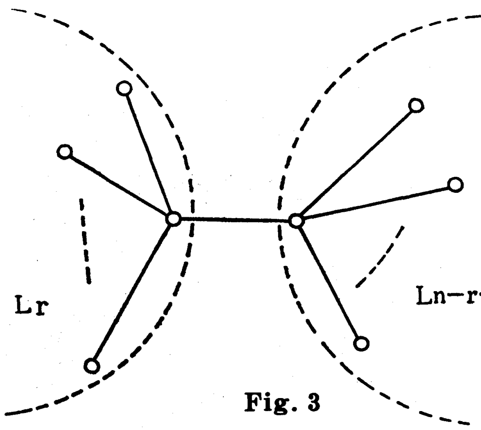

Fig. 3

$$s(T_4,\ L_r{**}L_r)=\frac{1}{8}\,(n-1)^2(n-3)\qquad \text{for } n=2r+1.$$

Hence, we have: $\lim_{n\to\infty}S(T_4,\ L_r{**}L)=0$ were $L=L_{r-1}$ or L_r.

(8) Since $P_3=L_1{**}L_1$, we have:

$$s(P_3\cup P_1,\ L_r{**}L_{r'}\cup L_t{**}L_{t'})=rr'(t+t'+1)+(r+r'+1)tt'.$$

Hence, for example, we have:

$$s(P_3\cup P_1,\ 2(L_r{**}L_{r-1}))=\frac{1}{16}\,n^2(n-4)\qquad \text{for } n=4r.$$

$$s(P_3\cup P_1,\ L_r{**}L_{r-1}\cup L_r{**}L_r)=\frac{1}{16}\,(n-1)\,(n^2-3n-3)\qquad \text{for } n=4r+1,$$

$$s(P_3\cup P_1,\ 2(L_r{**}L_r))=\frac{1}{16}\,n(n-2)^2\qquad \text{for } n=4r+2,$$

$$s(P_3\cup P_1,\ L_r{**}L_r\cup L_r{**}L_{r+1})=\frac{1}{16}(n+1)\,(n-2)\,(n-3)\qquad \text{for } n=4r+3.$$

In these cases, $\lim_{n\to\infty}S(P_3\cup P_1,\ \text{——})=0$.

(9) For n, there is unique $(n,\ r)$ such that (i) $n=(m-1)m+r\,(0\leqq r<m)$, or (ii) $n=m^2+r$ $(0\leqq r<m)$.

(i) $$s(K_3\cup P_1,\ K_m\cup K_m{*}L_r)=\binom{m}{3}\left(\binom{m}{2}+r\right)+\binom{m}{3}\binom{m-3}{2}+\left(\binom{m}{3}+\binom{r}{2}\right)\binom{m}{2}+\binom{m}{3}\binom{m-3}{2}+r\binom{m-1}{3}+\binom{r}{2}\binom{m-2}{2}.$$

In fact, the first three terms are equal to $s(K_3,\ K_m)\cdot s(P_1,\ K_m{*}L_r)$, $s(K_3\cup P_1,\ K_m)$ and $s(K_3,\ K_m{*}L_r)\cdot s(P_1,\ K_m)$, respectively. Also, the sum of the other terms is equal to $s(K_3\cup P_1,\ K_m{*}L_r)$——The fourth term is equal to $s(K_3\cup P_1,\ K_m)$. The fifth term is the number of $K_3\cup P_1$'s such that K_3 is a subgraph of K_m and P_1 is some e_i. The last term is the number of $K_3\cup P_1$'s such that $|E(K_3\cup P_1)\cap EL|=2$: Let e_i, $e_{i'}$ be the said edges, and v_j, $v_{j'}$ be two elements of $V(K_m)-\{v_i,\ v_{i'}\}$, then there is one required $K_3\cup P_1$——the graph $(v v_i v_i v_{i'})\cup(v_j v_{j'})$.

(ii) Similarly to (i), we have:

$$s(K_3 \cup P_1, K_{m+1} \cup K_m * L_r) = \binom{m+1}{3}\left(\binom{m}{2} + r\right) + \binom{m+1}{3}\binom{m-2}{2} + \left(\binom{m}{3}\right.$$

$$\left. + \binom{r}{2}\right)\binom{m+1}{2} + \binom{m}{3}\binom{m-3}{2} + r\binom{m-1}{3} + \binom{r}{2}\binom{m-2}{2}.$$

Hence, in these cases, $\lim_{n\to\infty} S(K_3 \cup P_1, \text{——}) = 0$.

(10) $s(P_2 \cup 2P_1, C_n) = n\left(\binom{n-4}{2} - (n-5)\right) = \frac{1}{2} n(n-5)(n-6)$. In fact, let $C_n = (v_1 v_2 \cdots v_n v_1)$ and $P_2 = (v_1 v_2 v_3)$. Then the number of subgraphe of $P_{n-4} = (v_4 v_5 \cdots v_n)$ which are isomorphic to $2P_1$ is equal to $\binom{n-4}{2} - (n-5)$. Hence, we have: $\lim_{n\to\infty} S(P_2 \cup 2P_1, C_n) = 0$.

Reference

[1] PIPPENGER, N. and GOLUMBIC, M. C.: *The inducibility of graphs.* J. Comb. Th. (B) 19 (1975) 189–203.