3-dimensional homogeneous Riemannian manifolds I

By

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0. Introduction

Let (M, g) be a Riemannian manifold. Let ∇^o, R^o, $(R^o)_1$ and S^o be the Riemannian, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. The k-th covariant differential of a tensor field K with respect to ∇^o is denoted by $(\nabla^o)^k K$ and $(\nabla^o)^0 K=K$, by definition. Manifolds and tensor fields are assumed to be of class C^∞ unless otherwise stated. For any integer $n\geqq 0$ and tangent vectors or vector fields on M, we adopt a notation:

$$((\nabla^o)^n_V R^o)=(V_n,\ V_{n-1},\cdots\cdots,V_1;\ (\nabla^o)R^o)$$

$$=(V_n^{j_n} V_{n-1}^{j_{n-1}}\cdots\cdots V_1^{j_1}\nabla^o_{j_n}\nabla^o_{j_{n-1}}\cdots\cdots\nabla^o_{j_1}(R^o)_{kji}{}^h),$$

where $V_n^{j_n}$ etc., are components of V_n, etc., and $\nabla^o{}_{j_n}\ \nabla^o{}_{j_{n-1}}\cdots\cdots\nabla^o{}_{j_1}(R^o)_{kji}{}^h$ are components of the k-th covariant differential of R^o in local coordinates. For each $x, y\in M$, a linear isomorphism Φ of the tangent space $T_x(M)$ onto $T_y(M)$ is naturally extended to a linear isomorphism of the tensor algebra $T(T_x(M))$ onto $T(T_y(M))$, which is also denoted by Φ. Now, we assume that (M, g) is a homogeneous Riemannian space, i. e., that (M, g) admits a transitive group of isometries.
Then, for every integer $n\geqq 0$, the following condition $P(n)$ is satisfied:

$P(n)$; For each $x, y\in M$, there exists a linear isometry of $T_x(M)$ onto $T_y(M)$ such that

$$\Phi((\nabla^o)^k R^o)_x=((\nabla^o)^k R^o)_y, \qquad \text{for} \quad k=0, 1, \cdots\cdots, n.$$

In fact, Φ is given by putting $\Phi=(d\varphi)_x$, where φ is an isometry which maps x to y. I. M. Singer (of. [4]) dealt with the converse problem and proved the following

THEOREM A. *Let (M, g) be a connected, simply connected and complete Riemannian space. If (M, g) satisfies the condition $P(n)$ for certain n, then it is Riemannian homogeneous.*

In this Theorem, the minimum of such integers n, depends on (M, g), though it is smaller than $m(m-1)/2+1$, where $m=\dim M$. The proof of the above Theorem in [4] is based

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on the following

THEOREM B. (W. Ambrose and I. M. Singer, [1]) *Let (M, g) be a connected, simply connected and complete Riemannian manifold. (M, g) is Riemannian homogeneous if and only if there exists a skew-symmetric tensor field T of type* (1, 2) *on M satisfying*

$$\nabla^o_X R^o = T(X)\cdot R^o, \tag{0. 1}$$

$$\nabla^o_X T = T(X)\cdot T, \tag{0. 2}$$

for any tangent vector $X \in T_x(M)$, $x \in M$.
Concerning with Theorem A, the present author has proved the following (cf. [3])

THEOREM C. *Let (M, g) be a 3-dimensional connected simply connected and complete Riemannian manifold satisfying the condition $P(1)$. Then (M, g) is Riemannian homogeneous, and furthermore, it is isometric with one of* (I) $\boldsymbol{S}^3$, (II) $\boldsymbol{E}^3$, (III) $\boldsymbol{H}^3$, (IV) $M^2\times \boldsymbol{E}^1$, (V) *group manifolds with certain left-invariant metrics which ane not symmetric ones, where $\boldsymbol{S}^m$, $\boldsymbol{E}^m$, $\boldsymbol{H}^m$ denote m-dimensional sphere, euclidean space, hyperbolic space with canonical metrics, respectively, and $M^2=\boldsymbol{S}^2$ or $\boldsymbol{H}^2$ in* (IV).

In this paper, we shall give a list of Lie algebras of Lie groups of full isometries which act effectively and transitively on 3-dimensional connected, simply connected Riemannian manifolds by means of the eigenvalues of the Ricci transformation $(R^o)^1$, etc. (cf. Table in §2).

1. Preliminaries

Let (M, g) be a connected m-dimensional Riemannian manifold with the Riemannian connection ∇^o and $I(M, g)$ ($I_o(M, g)$, resp.) be the full group of the isometries (the identity component of $I(M, g)$, resp.). Let $\boldsymbol{O}(M, g)$ be the orthonormal frame bundle over M with the projection π. Let $\boldsymbol{R}^m$ be an m-dimensional real number space and (v_i), $v_i=(0,\dots, 0, \overset{i}{1}, 0, \dots\dots, 0)\in \boldsymbol{R}^m$, $i=1, 2, \dots\dots, m$, be the canonical basis for $\boldsymbol{R}^m$. Then, each $\gamma=(x; X_1, \dots\dots, X_m)\in \boldsymbol{O}(M, g)$ can be regarded as a linear isomorphism from $\boldsymbol{R}^m$ cnto $T_x(M)$ such that $\gamma(v_i)=X_i$, $i=1, 2, \dots\dots, m$. Let $O(m)$ ($SO(m)$, resp.) be orthogonal group of degree m (a special orthogonal group of degree m, resp.) and $\mathfrak{o}(m)=\mathfrak{so}(m)$ be the Lie algebra of $O(m)$ (or $SO(m)$). More generally, let $SO(n, m-n)$ be the group of metrices in $SL(m, \boldsymbol{R})$ which leave invariant the quadratic form $-u_1^2-\dots\dots-u_n^2+u_{n+1}^2+\dots\dots+u_m^2$, and $\mathfrak{so}(n, m-n)$ be its Lie algebra. Now, we define linear mappings $f_{ij}(i<j)$ on $\boldsymbol{R}^m$ by

$$f_{ij}(v_i)=-v_j,\ f_{ij}(v_j)=v_i, \quad \text{and} \quad f_{ij}(v_k)=0, \quad \text{for} \quad k\neq i, j.$$

Furthermore, we put $f_{ji}=-f_{ij}(i<j)$ and $f_{ii}=0$. Then $(f_{ij})\ (i<j)$ is a basis for $\mathfrak{so}(m)$. Let F_{ij} be the fundamental vector fields on $O(M, g)$ corresponding to $f_{ij}\in\mathfrak{so}(m)$, $i, j=1, 2, \dots\dots, m$ (cf. [2]), and $\mathfrak{F}$ be the Lie algebra of all vector fields on $\boldsymbol{O}(M, g)$ of the form

$\sum_{j<k} b_{jk}F_{jk}$, where b_{jk} are any real numbers. Now, let ∇ be a metric linear connection on M with respect to g. Then we may associate with each $v \in \boldsymbol{R}^m$ a horizontal vector field $E(v)$ on $\boldsymbol{O}(M, g)$ with respect to ∇ as follows. For each $\gamma \in \boldsymbol{O}(M, g)$, $(E(v))_\gamma$ is the unique horizontal vector at γ such that $\pi((E(v)_\gamma) = \gamma(v)$. $E(v)$ is called the standard horizontal vector field corresponding to $v \in \boldsymbol{R}^m$. We put $E_i = E(v_i)$, $i = 1, 2, \ldots\ldots, m$. Then, (E_i) are lineary independent horizontal vectors at each $\gamma \in \boldsymbol{O}(M, g)$. Let $\mathfrak{E}$ be the m-dimensional vector space of all vector fields on $\boldsymbol{O}(M, g)$ of the form $\sum_i a_i E_i$ where a_i are any real numbers. Then, $\mathfrak{F}$ acts on $\mathfrak{E}$ by

$$[F_{ij}, E(v)] = E(f_{ij}v), \qquad i, j = 1, 2, \ldots\ldots, m. \tag{1. 1}$$

On the other hand, if we denote by $\widetilde{X}$ the horizontal lift of a vector field X on M with respect to ∇, then we have (cf. [2])

$$[\widetilde{X}, \widetilde{Y}]_\gamma = (\widetilde{[X, Y]})_\gamma - \gamma^{-1} \circ R(X, Y) \circ \gamma, \qquad \gamma \in \boldsymbol{O}(M, g), \tag{1. 2}$$

where R denotes the curvature tensor field with respect to ∇.

Now, we assume that (M, g) is simply connected, complete, and that there exists a skew-symmetric tensor field T of type (1. 2) on M satisfying (0. 1) and (0. 2). Using the tensor field T, we may construct a metric linear connection ∇ by $\nabla_X = \nabla^o_X - T(X)$. We now fix a point $\varepsilon = (x_o; e^o_1, \ldots\ldots, e^o_m) \in \boldsymbol{O}(M, g)$. Let G be the holonomy subbundle of $\boldsymbol{O}(M, g)$ through ε with respect to ∇. Then, W. Ambrose and I. M. Singer proved that G has a Lie group structure with the identity ε and acts transitively on M as a group of isometries of (M, g) by $\gamma'(x) = \pi(\gamma'\gamma)$, $\gamma, \gamma' \in G$, $\pi(\gamma) = x$. The Lie algebra of G can be expressed by means of $\mathfrak{F}$ and $\mathfrak{E}$. This is nothing but the assertion of Theorem B in §1.

2. Statement of main result

Let (M, g) be a connected, simply connected 3-dimensional homogeneous Riemannian manifold. dim $M=3$ implies

$$R^o(X, Y) = (R^o)^1 X \wedge Y + X \wedge (R^o)^1 Y - (S^o/2) X \wedge Y, \tag{2. 1}$$

for all $X, Y \in T_x(M)$, $x \in M$.

We now fix a point $x_o \in M$. At x_o, we may choose an orthonormal basis (e^o_i) in $T_{x_o}(M)$ such that

$$(R^o)^1 e^o_i = k_i e^o_i, \qquad 1 \leqq i \leqq 3. \tag{2. 2}$$

By the homogeneity of (M, g), k_i are constant on M. (2. 1) and (2. 2) imply

$$\begin{aligned} R^o(e^o_1, e^o_2) &= ((k_1+k_2+k_3)/2) e^o_1 \wedge e^o_2, \\ R^o(e^o_2, e^o_3) &= ((k_2+k_3-k_1)/2) e^o_2 \wedge e^o_3, \\ R^o(e^o_3, e^o_1) &= ((k_3+k_1-k_2)/2) e^o_3 \wedge e^o_1. \end{aligned} \tag{2. 3}$$

For our purpose, it is sufficient to deal with the case (V) in Theorem C. The details for (V) is mentioned as follows. In this case, (M, g) can be regarded as a connected, simply connected 3-dimensional group manifold $\widehat{G}$ with left-invariant metric g induced by a positive-definite inner product $<,>$ on $\mathfrak{g}$ (Lie algebra of $\widehat{G}$) such that $\mathfrak{g}=\mathrm{span}_{\mathbf{R}}(e_1, e_2, e_3)$, where $\langle e_i, e_j\rangle=\delta_{ij}$, $i, j=1, 2, 3$, and furthermore, e_i are related by

$$(2.4)\qquad \begin{aligned}[e_1, e_2]&=(b-a)e_3+ee_1-pe_2,\\ [e_2, e_3]&=(b-h)e_1+de_2-fe_3,\\ [e_3, e_1]&=(h-a)e_2+qe_3-ce_1,\end{aligned}$$

here $[\ ,\ ]$ denotes the Lie bracket operation on $\mathfrak{g}$, and the real numbers, $a, b, c, d, e, f, h, p, q$, in (2.4) are determined by the following manner:

(i) $k_1\neq k_2,\quad k_2\neq k_3,\quad k_3\neq k_1.$

(i)-(1) $c=d=e=f=0,\quad p, q\neq 0,$

$a=(k_1-k_3)r,\quad b=(k_3+k_1-2k_2)r,\quad h=(2k_2-k_3-k_1)r,\quad p=(k_3-k_2)t,$

$q=(k_2-k_1)t,$

where $r^2=(k_1{}^2-k_2k_3+k_2{}^3-k_1k_2)/(2(k_1-k_3)^2(k_3+k_1-2k_2)),$

$t^2=-(k_3+k_1)/(k_1-k_3)^2.$

(i)-(2) $e=f=p=q=0,\quad c, d\neq 0,$

$a=(k_1+k_2-2k_3)r,\quad b=(k_1+k_2-2k_3)r,\quad h=(k_2-k_1)r,\quad c=(k_1-k_3)t,$

$d=(k_3-k_2)t,$

where $r^2=(k_1{}^2-k_2k_3+k_2{}^2-k_3k_1)/(2(k_1-k_2)^2(k_1+k_2-2k_3)),$

$t^2=-(k_1+k_2)/(k_1-k_2)^2.$

(i)-(3) $c=d=p=q=0,\quad e, f\neq 0,$

$a=(k_2+k_3-2k_1)r,\quad b=(k_2-k_3)r,\quad h=(k_2+k_3-2k_1)r,\quad e=(k_1-k_3)t,$

$f=(k_2-k_1)t,$

where $r^2=(k_2{}^2-k_3k_1+k_3{}^2-k_1k_2)/(2(k_2-k_3)^2(k_2+k_3-2k_1)),$

$t^2=-(k_2+k_3)/(k_2-k_3)^2.$

(i)-(4) $c=d=e=f=p=q=0,$

$a^2=k_1k_3/2k_2,\quad b^2=k_2k_3/2k_1,\quad h^2=k_1k_2/2k_3,$

where $ab=-k_3,\quad bh=k_2,\quad ah=-k_1.$

(ii) $k_1=k_2,\quad k_3\neq k_1(k_2),\quad k_3\neq 0.$

(ii)-(1) $c=d=e=f=p=q=0.$

$h=0,\quad ab=(2k_1-k_3)/2=-k_3/2,$

where $a+b\neq 0.$

(ii)-(2) $c=d=e=f=p=q=0,$

$a+b=0,\quad a^2=k_3/2,\quad h=-k_1/2a.$

Since we may regard $(e_i)=(e_1, e_2, e_3)$ as a left-invariant orthonormal frame field on $M=\widehat{G}$, we may put

$$\nabla_{e_i}e_j=\sum_{k=1}^{3} b_{ijk}e_k, \tag{2.5}$$

where $b_{ijk}=-b_{ikj}$.
Then, by (2. 4), we have

$$b_{132}=a,\quad b_{231}=b,\quad b_{212}=h,\quad b_{121}=e,\quad b_{212}=p,\quad b_{131}=c, \tag{2.6}$$

$$b_{232}=d,\quad b_{313}=q,\quad b_{323}=f.$$

Let B be the Killing form of $\mathfrak{g}$. Then, corresponding to the respective cases, B takes the following forms:

(i)-(1)

$$[B(e_i, e_j)]=\begin{pmatrix} (k_1^2+k_3^2-2k_2^2)/(2k_2-k_1-k_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

where $(k_1^2-k_2k_3+k_3^2-k_1k_2)(k_1+k_3-2k_2)>0,\quad k_1+k_3<0.$

(i)-(2)

$$[B(e_i, e_j)]=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (k_1^2+k_2^2-2k_3^2)/(2k_3-k_1-k_2) \end{pmatrix} \tag{2.8}$$

where $(k_1^2-k_2k_3+k_2^2-k_1k_3)(k_1+k_2-2k_3)>0,\quad k_1+k_2<0.$

(i)-(3)

$$[B(e_i, e_j)]=\begin{pmatrix} 0 & 0 & 0 \\ 0 & (k_2^2+k_3^2-2k_1^2)/(2k_1-k_2-k_3) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.9}$$

where $(k_2^2-k_1k_3+k_3^2-k_1k_2)(k_2+k_3-2k_1)>0,\quad k_2+k_3<0.$

(i)-(4)

$$[B(e_i, e_j)]=\begin{pmatrix} -(k_1+k_2)(k_2+k_3)/k_2 & 0 & 0 \\ 0 & -(k_1+k_2)(k_3+k_1)/k_1 & 0 \\ 0 & 0 & -(k_2+k_3)(k_3+k_1)/k_3 \end{pmatrix} \tag{2.10}$$

where $k_1k_2k_3>0$.

$$(2.11)\qquad [B(e_i, e_j)]=\begin{pmatrix} 2a(b-a) & 0 & 0 \\ 0 & 2b(a-b) & 0 \\ 0 & 0 & 2ab \end{pmatrix}$$

where $(2k_1-k_3)/2=ab$, $k_3/2=-ab$, $a+b\neq 0$, $ab\neq 0$.

$$(2.12)\qquad [B(e_i, e_j)]=\begin{pmatrix} -2(k_1+k_3) & 0 & 0 \\ 0 & -2(k_1+k_3) & 0 \\ 0 & 0 & -(k_1+k_3)^2/k_3 \end{pmatrix}$$

where $k_3=a^2$.

On the other hand, by the well known Iwasawa's decomposition theorem, we have

LEMMA 2. 1. *Let G be a connected, simply connected 3-dimensional Lie group. Then, it is diffeomorphic with $\boldsymbol{R}^3$ or S^3.*

First, we consider the cases, (i)-(1), (i)-(2), (i)-(3). Without loss of essentiality, for example, it is sufficient to deal with the case (i)-(2). In this case, we see that $\mathfrak{g}$ is isomorphic with the Lie subalgebra $\mathfrak{a}(a+h, a-h, c, d)$ of $\mathfrak{gl}(3, \boldsymbol{R})$ given by

$$\mathfrak{a}(a+h, a-h, c, d)=\mathrm{span}_{\boldsymbol{R}}(e_1^o, e_2^o, e_3^o),$$

where

$$e_1^o=\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & a-h \\ 0 & 0 & 0 \end{pmatrix},\quad e_2^o=\begin{pmatrix} 0 & 0 & a+h \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3^o=\begin{pmatrix} c & -(a+h) & 0 \\ -(a-h) & -d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, $\hat{G}=I_0(M, g)$ acts simply transitively on $M=\hat{G}$. From (i)-(2), $\mathfrak{g}$ is solvable, and hence, by Lemma 2. 1, M is diffeomorphic with $\boldsymbol{R}^3$.

Secondary, we consider the case (i)-(4). From (2. 10), $\mathfrak{g}$ is semi-simple (and hence, simple) if and only if $(k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0$. Furthermore, if (i)-$(4)_1$ $k_1>0$, $k_2<0$, $k_3<0$, $k_1+k_2>0$, $k_3+k_1>0$, or (i)-$(4)_2$ $k_1<0$, $k_2<0$, $k_3>0$, $k_2+k_3>0$, $k_3+k_1>0$, or (i)-$(4)_3$ $k_1<0$, $k_2>0$, $k_3<0$, $k_1+k_2>0$, $k_2+k_3>0$, or (i)-$(4)_4$ $k_1>0$, $k_2>0$, $k_3>0$, then $\mathfrak{g}\cong\mathfrak{so}(3)$, and if (i)-$(4)_5$ $k_1>0$, $k_2<0$, $k_3<0$, $(k_1+k_2)(k_3+k_1)<0$, or (i)-$(4)_6$ $k_1<0$, $k_2<0$, $k_3>0$, $(k_2+k_3)(k_3+k_1)<0$, or (i)-$(4)_7$ $k_1<0$, $k_2>0$, $k_3<0$, $(k_2+k_3)(k_1+k_2)<0$, then $\mathfrak{g}\cong\mathfrak{so}(2, 1)$ $(\cong\mathfrak{sl}(2, \boldsymbol{R}))$.

And if (i)-$(4)_8$ $(k_1+k_2)(k_2+k_3)(k_1+k_3)=0$, then, we can see that $\mathfrak{g}\cong\mathfrak{a}(1, 1, 0, 0)$.

For the case (i)-(4), since $\hat{G}=I_0(M, g)$ acts simply transitively on $M=\hat{G}$, by Lemma 2. 1. M is diffeomorphic with S^3 for the cases, (i)-$(4)_1\sim$(i)-$(4)_4$, and with $\boldsymbol{R}^3$ for the cases, (i)-$(4)_5\sim$(i)-$(4)_8$. Thirdly, we consider the case (ii)-(1). Then, from (2. 11) $\mathfrak{g}$ is simple if and only if $a\neq b$. And furthermore, if (ii)-$(1)_1$ $a>0$, $b<0$, or (ii)-$(1)_2$ $a<0$, $b>0$, then $\mathfrak{g}\cong\mathfrak{so}(3)$, and if (ii)-$(1)_3$ $a>0$, $b>0$, or (ii)-$(1)_4$ $a<0$, $b<0$, then $\mathfrak{g}\cong\mathfrak{so}(2, 1)$ ($\cong\mathfrak{sl}(2, \boldsymbol{R})$). And if (ii)-$(1)_5$ $a=b$, then we can see that $\mathfrak{g}\cong\mathfrak{a}(1, 1, 0, 0)$. In the case (ii)-(1), since $\widehat{G}=I_o(M, g)$ acts simply transitively on $M=\widehat{G}$, by Lemma 2.1, M is diffeomorphic with $\boldsymbol{S}^3$ for the cases, (ii)-$(1)_1$, (ii)-$(1)_2$, and with $\boldsymbol{R}^3$ for the cases, (ii)-$(1)_3\sim$ (ii)-$(1)_5$, Lastly, we consider the case (ii)-(2). In this case, by the following arguments, we can see that G does not coincides with $I_o(M, g)$. From (ii)-(2) and (2.6), we have

$$(2.13)\qquad b_{1\,21}=b_{1\,31}=b_{2\,12}=b_{2\,32}=b_{3\,13}=b_{3\,23}=0,$$

$$b_{1\,32}=a,\quad b_{2\,31}=-a,\quad b_{3\,12}=-k_1/2a.$$

For any real number w, we define a skew-symmetric tensor field T_w of type (1, 2) on M by

$$(2.14)\qquad T_w(e_1)=a e_2\wedge e_3,\quad T_w(e_2)=a e_3\wedge e_1,\quad T_w(e_3)=w e_1\wedge e_2.$$

Then, we can easily show that T_w satisfies (0.1) and (0.2) in Theorem B. Conversely, any tensor field T of type (1.2) on M satisfying (0.1) and (0.2) is given by the above fashion. We put $T^*=T_w(w\neq k_1/2a)$ and $T=T_{k_1/2a}$. And we define the corresponding metric linear connections ∇^*, ∇ on M by $\nabla^*_X=\nabla^o_X-T^*(X)$, and $\nabla_X=\nabla^o_X-T(X)$, respectively. Then, by the definition of ∇ and (2.13), we have $\nabla_{e_i}e_j=0$, $i, j=1, 2, 3$. Let G^* and G be the holonomy subbundles of $O(M, g)$ through $\varepsilon=(\hat{\varepsilon}; (e_1)_{\hat{\varepsilon}}, (e_2)_{\hat{\varepsilon}}, (e_3)_{\hat{\varepsilon}})$ with respect to ∇^* and ∇, respectively, where $\hat{\varepsilon}$ is the identity of $\widehat{G}$. Then, we see that $\mathfrak{g}^*$(Lie algebra of G^*) is isomorphic with $\mathfrak{i}(M, g)$ (Lie algebra of $I_o(M, g)$), and the Lie algebra $\mathfrak{k}$ of the (linear) isotropy subgroup at ε is isomorphic with the holonomy group of ∇^* with reference point ε, and hence, with $\mathfrak{so}(2)$ ($\subset\mathfrak{so}(3)$) (cf. [2], [3], [4]). Furthermore, we can identify G with $\widehat{G}$ through π. Let $E^*(v)$ and $E(v)$ be the fundamental horizontal vector fields on $O(M, g)$ with respect to ∇^* and ∇, respectively. Then, we have

$$(2.15)\qquad E_1^*=E_1,\quad E_2^*=E_2,\quad E_3^*=E_3+((k_1-2aw)/2)A,$$

where $E_i^*=E^*(v_i)$, $E_i=E(v_i)$, $i=1, 2, 3$, and $A=F_{12}$. Thus, $\mathfrak{g}^*=\mathfrak{i}(M, g)=\mathrm{span}_{\boldsymbol{R}}(A, E_1^*, E_2^*, E_3^*)=\mathrm{span}_{\boldsymbol{R}}(A, E_1, E_2, E_3)$. Along G, we have

$$(2.16)\qquad E_1=\widehat{e_1}-aF_{23},\quad E_2=\widehat{e_2}-aF_{31},\quad E_3=\widehat{e_3}-(k_1/2a)F_{12},$$

where $\widehat{e_i}$ denotes the horizontal lift of e_i with respect to ∇^o, $i=1, 2, 3$.

From (2.16), by making use of (1.1), (1.2) and the property, $[F_{ij}, \widehat{e_k}]=0$, $i, j, k=1, 2, 3$, we can see that $\mathfrak{i}(M, g)$ is isomorphic with the following Lie algebra:

$$\mathfrak{g}^*=\mathrm{span}_{\boldsymbol{R}}(A, e_1, e_2, e_3),$$

where

$$(2.17)\qquad [A, e_1]=-e_2,\quad [A, e_2]=e_1,\quad [A, e_3]=0,\quad [e_1, e_2]=-2a e_3,$$

$$[e_2, e_3]=-((k_1+k_3)/2a)e_1, \quad [e_3, e_1]=-((k_1+k_3)/2a)e_2.$$

First, we assume that $k_1+k_3\neq 0$. We put $A^o=A-(2a/(k_1+k_3))e_3$. Then, $\mathfrak{g}^*=\mathrm{span}_{\boldsymbol{R}}(A^o, e_1, e_2, e_3)$ and furthermore,

$$(2.18) \qquad [A^o, e_i]=0, \quad i=1, 2, 3.$$

Thus, $\mathfrak{i}(M, g)=\mathfrak{g}^*$ is isomorphic with $\mathfrak{g}_o+\mathfrak{g}$ (direct sum), where $\mathfrak{g}=\mathrm{span}_{\boldsymbol{R}}(e_1, e_2, e_3)$ (=Lie algebra of G) and $\mathfrak{g}_o=\mathrm{span}_{\boldsymbol{R}}(A-(2a/(k_1+k_3))e_3)$. From (2.12) and (2.17), we see that $\mathfrak{g}\cong\mathfrak{so}(3)$ for the case (ii)-$(2)_1$ $k_1+k_3>0$, and $\mathfrak{g}\cong\mathfrak{so}(2,1)$ for the case (ii)-$(2)_2$ $k_1+k_3<0$. Next, we assume that $k_1+k_3=0$. Then, from (2.17), we see that $\mathfrak{i}(M, g)=\mathfrak{g}^*$ is isomorphic with $\mathfrak{g}_1+\mathfrak{g}$ (semi-direct sum), where $\mathfrak{g}_1=\mathrm{span}_{\boldsymbol{R}}(A)$. In this case, $\mathfrak{g}$ is isomorphic with the following Lie subalgebra $\mathfrak{b}$ of $\mathfrak{gl}(3, \boldsymbol{R})$ given by

$$\mathfrak{b}=\mathrm{span}_{\boldsymbol{R}}(e_1^o, e_2^o, e_3^o),$$

where

$$e_1^o=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \quad e_2^o=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \quad e_3^o=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

Thus, from the above arguments and Lemma 2.1, M is diffeomorphic with $\boldsymbol{S}^3$ for the case (ii)-$(2)_1$, and with $\boldsymbol{R}^3$ for the cases, (ii)-$(2)_2$, (ii)-$(2)_3$.

REMARK 1. The Lie algebra $\mathfrak{g}_1+\mathfrak{b}$ (semi-direct sum) is isomorphic with the following Lie subalgebra $\mathfrak{c}$ of $\mathfrak{gl}(4, \boldsymbol{R})$ given by

$$\mathfrak{c}=\mathrm{span}_{\boldsymbol{R}}(A, e^o{}_1, e^o{}_2, e^o{}_3),$$

where

$$A=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}, \quad e_1^o=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$e_2^o=\begin{pmatrix}0&-1&0&0\\0&0&0&0\\0&0&0&-1\\0&0&0&0\end{pmatrix}, \quad e_3^o=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

REMARK 2. In the case (i)-(2), if $(k_1^2-k_2k_3+k_2^2-k_1k_3)(k_1+k_2-2k_3)>0$, $k_1+k_2<0$, and (1) $(2k_3-k_1-k_2)((k_1-k_2)^2-2k_3(2k_3-k_1-k_2))\geqq 0$, ((2) $(2k_3-k_1-k_2)((k_1-k_2)^2-2k_3(2k_3-k_1-k_2)<0$, resp.), then $\mathfrak{a}(a+h, a-h, c, d)\cong\mathfrak{a}(1, 1, 0, 0)$ or $\mathfrak{a}(0, 0, -1, 1)$ ($\mathfrak{a}(a+h, a-h, c, d)$ is isomorphic with the following Lie subalgebra $\mathfrak{a}$ of $\mathfrak{gl}(3, \boldsymbol{C})$ given by

$$\mathfrak{a}=\mathrm{span}_{\boldsymbol{R}}(e_1^o, e_2^o, e_3^o),$$

where

$$e_1^o=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -\sqrt{-1}\\ 0 & 0 & 0\end{pmatrix},\qquad e_2^o=\begin{pmatrix}0 & 0 & \sqrt{-1}\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix},$$

$$e_3^o=\begin{pmatrix}(-(c+d)+\sqrt{D_o}\ \sqrt{-1})/2 & 0 & 0\\ 0 & (-(c+d)-\sqrt{D_o}\ \sqrt{-1})/2 & 0\\ 0 & 0 & 0\end{pmatrix},$$

were $D_o=(2k_3(2k_3-k_1-k_2)-(k_1-k_2)^2)/(2k_3-k_1-k_2)$, resp.).

REMARK 3. The homogeneous Riemannian space in the case IV-(ii) can be expressed by some group manifold with left-invariant metric as follows:

Let P be the group of triangular matrices of degree 2 furnished with the Riemannian metric g induced by the positive-definite inner product $<, >$ on $\mathfrak{p}$ (Lie algebra of P) such that

$$\mathfrak{p}=\mathrm{span}_{\boldsymbol{R}}(e_1^o, e_2^o, e_3^o),\quad \text{where}\quad <e_i^o, e_j^o>=\delta_{ij},\quad i, j=1, 2, 3,$$

and

$$e_1^o=(\sqrt{-k_1/2})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad e_2^o=(\sqrt{-k_1/2})\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

$$e_3^o=(\sqrt{-k_1/2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad k_1<0.$$

The, (P, g) is isometric with $\boldsymbol{H}^2(k_1)\times\boldsymbol{E}^1$.

Summing up the arguments in this section, we have the following table:

Cases		$\mathfrak{i}(M, g)$	$\mathfrak{k}$	M
I		$\mathfrak{so}(4)$	$\mathfrak{so}(3)$	$\boldsymbol{S}^3$
II		$\mathfrak{so}(3)+\boldsymbol{R}^3$ (semi-direct sum)	$\mathfrak{so}(3)$	$\boldsymbol{R}^3$
III		$\mathfrak{so}(3, 1)$	$\mathfrak{so}(3)$	$\boldsymbol{R}^3$
IV	(i)	$\mathfrak{so}(3)+\boldsymbol{R}$ (direct sum)	$\mathfrak{so}(2)$	$\boldsymbol{S}^2\times\boldsymbol{R}$
	(ii)	$\mathfrak{so}(2, 1)+\boldsymbol{R}$ (direct sum)	$\mathfrak{so}(2)$	$\boldsymbol{R}^3$
V	(i)-(1)	$\mathfrak{a}(-a-b, a-b, p, q)$		
	(i)-(2)	$\mathfrak{a}(a+h, a-h, c, d)$	0	$\boldsymbol{R}^3$
	(i)-(3)	$\mathfrak{a}(-h-b, b-h, e, f)$		
	(i)-$(4)_1$ (i)-$(4)_2$ (i)-$(4)_3$ (i)-$(4)_4$	$\mathfrak{so}(3)$	0	$\boldsymbol{S}^3$

Cases		$\mathfrak{i}(M, g)$	$\mathfrak{k}$	M
	(i)-$(4)_5$ (i)-$(4)_6$ (i)-$(4)_7$	$\mathfrak{so}(2, 1)$	0	$\boldsymbol{R}^3$
	(i)-$(4)_8$	$\mathfrak{a}(1, 1, 0, 0)$	0	$\boldsymbol{R}^3$
	(ii)-$(4)_1$ (ii)-$(1)_2$	$\mathfrak{so}(3)$	0	$\boldsymbol{S}^3$
	(ii)-$(1)_3$ (ii)-$(1)_4$	$\mathfrak{so}(2, 1)$	0	$\boldsymbol{R}^3$
	(ii)-$(1)_5$	$\mathfrak{a}(1, 1, 0, 0)$	0	$\boldsymbol{R}^3$
	(ii)-$(2)_1$	$\boldsymbol{R}+\mathfrak{so}(3)$ (direct sum)	$\mathfrak{so}(2)$	$\boldsymbol{S}^3$
	(ii)-$(2)_2$	$\boldsymbol{R}+\mathfrak{so}(2, 1)$ (direct sum)	$\mathfrak{so}(2)$	$\boldsymbol{R}^3$
	(ii)-$(2)_3$	$\mathfrak{so}(2)+\mathfrak{b}$ (semi-direct sum), or $\mathfrak{c}$	$\mathfrak{so}(2)$	$\boldsymbol{R}^3$

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