On maximality of $H^{\infty}(\alpha)$ in finite von Neumann algebras

By

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1. Introduction

Let M be a finite von Neumann algebra and $\{\alpha_t\}_{t\in R}$ be a flow on M which we mean a σ-weakly continuous one-parameter group of *-automorphisms of M. Suppose that there exists a α_t-invariant, faithful, normal, finite trace τ on M such that $\tau(1)=1$. We construct the Banach space $L^p(M, \tau)$ $(1\leq p\leq\infty)$ in the sense as Segal [7]. For each p, $1\leq p<\infty$, $\{\alpha_t\}_{t\in R}$ extends uniquely to a strongly continuous representation of R of isometries on $L^p(M, \tau)$. We denote this extension to each $L^p(M, \tau)$ by $\{\alpha_t\}_{t\in R}$ also. Therefore we can consider a spectrum of an element in $L^p(M, \tau)$ according to Arveson [1]. Let $H^p(\alpha)$ be the set of all elements of $L^p(M, \tau)$ with non-negative spectrum with respect to $\{\alpha_t\}_{t\in R}$ and $H_0{}^p(\alpha)$ be the L^p-norm closure (σ-weakly closure if $p=\infty$) of the set of all elements of $L^p(M, \tau)$ with positive spectrum with respect to $\{\alpha_t\}_{t\in R}$. We shall call $H^p(\alpha)$ the non-commutative Hardy space. Then the structure of $H^p(\alpha)$ was investigated by Saito [6] and, in particular, the structure of $H^{\infty}(\alpha)$ was studied by Kawamura and Tomiyama [2], Loebl and Muhly [3] and Saito [5]. They showed that $H^{\infty}(\alpha)$ is a maximal subdiagonal algebra if the family of α_t-invariant normal states of M separates the non-negative elements of M. However, we don't know whether $H^{\infty}(\alpha)$ is maximal as a σ-weakly closed subalgebra of M. On the other hand, the Hardy space H^{∞}, which is the space of bounded analytic functions in the open unit disk, is a maximal σ-weakly closed subalgebra of L^{∞} of the unit circle. Therefore, in this note, we shall investigate the maximality of $H^{\infty}(\alpha)$ as a σ-weakly closed subalgebra of M.

2. Maximality of $H^{\infty}(\alpha)$

Keep the notations as §1 and suppose $\{\alpha_t\}_{t\in R}$ is ergodic in the sense that, for $x\in M$, $\alpha_t(x)=x$ for all $t\in R$ implies $x=\lambda 1$ for some complex number λ. For $a\in M$, the least projection of all the projections q in M such that $qa=a$ is called the left support of a and is denoted by $l(a)$. For a subset S of $L^p(M, \tau)$, $[S]_p$ is the closed (σ-weakly closed if $p=\infty$) linear span of S in $L^p(M, \tau)$.

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Our goal in this note is the following theorem.

THEOREM *Keep the notations as above. If $l(x)=1$ for every non-zero element x in $H^\infty(\alpha)$, then $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M.*

PROOF. Suppose that $H^\infty(\alpha)$ is not a maximal σ-weakly closed subalgebra of M and let B be a proper σ-weakly closed subalgebra of M which contains $H^\infty(\alpha)$ properly. Then we shall prove that $L^2(M, \tau)\neq[B]_2$. Since $B\subsetneqq M$, there exists a non-zero element a of $L^1(M, \tau)$ such that $\tau(a^* y)=0$ for every $y\in B$. As B contains $H^\infty(\alpha)$, we have $a^*\in H_0^1(\alpha)$ by [6, Proposition 2.7] Let $a=u|a|$ be the polar decomposition of a and let $|a|=\int_0^\infty \lambda\, de_\lambda$ be the spectral decomposition of $|a|$. How we define a function $f(\lambda)=\min(1, 1/\lambda)$, $\lambda\geqq 0$. Putting $k=f(|a|)$, k is a self-adjoint operator in M and the invertible element k^{-1} of k belongs to $L^1(M, \tau)$. Furthermore we have $k\notin[kH_0^\infty(\alpha)]_\infty$. Then $[kH^\infty(\alpha)]_\infty$ is a right simply invariant subspace in the sense that

$$[[kH^\infty(\alpha)]_\infty H_0^\infty(\alpha)]_\infty\subsetneqq[kH^\infty(\alpha)]_\infty.$$

According to [6, Theorem 5.1], there exists a unitary element v of M such that $[kH^\infty(\alpha)]_\infty=vH^\infty(\alpha)$. Thus there exists an element y of $H^\infty(\alpha)$ such that $k=vy$. Then we have

$$ay^*=u|a|kv=u\int_0^\infty\lambda\, de_\lambda\int_0^\infty f(\lambda)\, de_\lambda\, v=u\int_0^\infty\lambda f(\lambda)\, de_\lambda\, v.$$

Since $\lambda f(\lambda)$ is bounded, we have $ay^*\in M$. Furthermore for every $z\in[B]_2$,

$$(z, ay^*)=\tau((ay^*)^*z)=\tau(ya^* z)=\tau(a^* zy)=0.$$

Thus $ay^*\in[B]_2^\perp\cap M$ where $[B]_2^\perp$ is the orthogonal complement of $[B]_2$ in $L^2(M, \tau)$. Therefore we have $L^2(M, \tau)\neq[B]_2$.

By [6, Proposition 2.7], we have $[B]_2^\perp\subset H_0^2(\alpha)^*$. Furthermore there exists a projection p of $B\cap B^*$ such that $0<p<1$. We choose an element x in B which is invertible in B but which does not belong to $H^\infty(\alpha)$. Such a choice is possible since a Banach algebra with identity is spanned by its invertible elements and since $B\neq M$. As $x\in M\cap M^{-1}$, $[xH^\infty(\alpha)]_\infty$ is a right simply invariant subspace of M. Thus there exists a unitary element u of M such that $[xH^\infty(\alpha)]_\infty=uH^\infty(\alpha)$. Then there exists $y\in H^\infty(\alpha)\cap H^\infty(\alpha)^{-1}$ such that $x=uy$. Since $x\notin H^\infty(\alpha)$, u is not a scalar multiple of 1 and $u^{-1}=u^*\in B$. Therefore B contains all the spectral projections of u. Since u is not a scalar multiple of 1, there exists a spectral projection p of u as a unitary operator such that $0<p<1$.

Since $p\in B$, we have $[B]_2^\perp p\subset[B]_2^\perp\subset H_0^2(\alpha)^*$. Putting $b=ay^*p(=ay^*(1-p)$ if $ay^*p=0)$, we have $l(b^*)\leqq p<1$ and $b^*\in H_0^\infty(\alpha)$. This is a contradiction. Therefore $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M.

Q. E. D.

References

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