Differentiable Circle Group Action on Homotopy Complex Projective 3–Spaces

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Introduction

Let M be an $H\mathbb{C}P^3$; in other wards M is a simply connected 6–manifold with the same homotopy type as the standard complex projective 3–spaces.

We will denote by T^1 the circle group, by (T^1, M) the action on M and by $F(T^1, M)$ the fixed point set of T^1 action on M. $X \underset{R}{\sim} P^h(n)$ means that the cohomology ring $H^*(X; R)$ is isomorphic to $R[a]/(a^{h+1})$, where $n = deg\ a$.

By a result in [1] (chap. VII, 5–1), it follows that there are following four cases.

(a) $F(T^1, M) \underset{z}{\sim} \mathbb{C}P^2 + \{point\}$

(b) $F(T^1, M) \underset{z}{\sim} S^2 + S^2$

(c) $F(T^1, M) \underset{z}{\sim} S^2 + \{point\} + \{point\}$

(d) $F(T^1, M) \underset{z}{\sim} \{point\} + \{point\} + \{point\} + \{point\}$

In this paper, we shall consider the cases (a) and (b).

We have the following.

THEOREM 1. *If* $F(T^1, M) \underset{z}{\sim} \mathbb{C}P^2 + \{point\}$ *or* $F(T^1, M) \underset{z}{\sim} S^2 + S^2$, *then M is diffeomorphic to* $\mathbb{C}P^3$.

THEOREM 2. T^2 *cannot act effectively on exotic complex projective* 3*–spaces.*

In the following all actions are asumed to be differentiable.

1. The main lemma

LEMMA 1–1. *If M contains a submaniforld A such that* $A \underset{Q}{\sim} \mathbb{C}P^2$, *then M is diffeomorphic to* $\mathbb{C}P^3$.

PROOF. Let $\nu = (E, p, A)$ be the normal bundle of A in M, and $((E, E^0), p, A, (\mathbb{D}^2, S^1))$ be pair of disk bundle and sphere bundle associated to ν.

It is known that

(1) $\chi(\nu)=i^*D^{-1}(i_*[A])$

where $\chi(\nu)$ is the Euler class of ν, $[A]$ is the fundamental homology class of A and D is the Poincare duality map. (see [6])

Let α(γ, respectively) be a generator of $H^2(A)/tor$ ($H^2(M)$, respectively), where tor denotes the torsion group of $H^2(A)$, let β be the dual base of $[A](\in H_4(A))$ and let k, l and m be integers such that satisfy $i^*(\gamma)=m\alpha+t$, $\alpha^2=k\beta$ and $p_1(M)=l\gamma^2$, where $t\in Tor$ and $p_1(M)$ is the first Pontrjagin class of M.

It follows immediately that

(2) $i^*(\gamma^2)=i^*(\gamma)^2=m^2\alpha^2=km^2\beta.$

It is easy to see that $i_*[A]=km^2c$, where $c(\in H_4(M))$ is the dual base of $\gamma^2(\in H^4(M))$. Hence, we have

$$i^*D^{-1}(i_*[A])=i^*(km^2\gamma)=km^3\alpha+km^2t.$$

And

(3) $p_1(\nu)=\chi^2(\nu)=k^2m^6\alpha^2=k^2m^6\beta.$

Since

$$1=Index\, A=1/3\,P_1(A)\cdot[A],$$

we have

(4) $P_1(A)=3\beta.$

It follows from the formula $i^*P_1(M)=P_1(A)+P_1(\nu)$, (2), (3) and (4) that

$$lkm^2\beta=k^3m^6\beta+3\beta.$$

Thus we have

(5) $3=km^2(l-k^2m^4).$

It is not difficult to show that possible values of k, l and $|m|$ are $(k, l, |m|)=(3, 10, 1)$, $(-3, 8, 1)$, or $(-1, -2, 1)$.

Since l has the form $24j+4$ (see [6]), we have

(6) $(k, l, |m|)=(1, 4, 1)$, which implies

(7) $P_1(M)=4\gamma^2.$

From a result in [6], it following that M is diffeomorphic to CP^3.

Corollary 1-2. *If* $F(T^1, M)\underset{z}{\sim} CP^2+\{point\}$, *then M is diffeomorphic to* CP^3.

2. The Case (b)

In this section, we will study the case (b).

It is well known that M is the orbit space of a differentiable free S^1 action on a homotopy seven sphere Σ^7. Let π: $\Sigma^7 \longrightarrow M$ be the projection. We may assume that S^1 acts to the right on Σ^7 and T^1 acts to the left on M. By a theorem of [5], the left T^1 action on M lifts to a left T^1 action on Σ^7 which commutes with the right S^1 action.

In the case (b), it follows from the fact $F(Z_p, M) \supset F(T^1, M)$ for any subgroup Z_p of T^1 and a result in [1] (chap. VII 3–1), the action (T^1, M) is semifree. Then we can choose a lifting so that the action is semifree with the fixed point set $\pi^{-1}(F_0) \approx S^3$, where $F(T^1, M) = F_0 + F_1$. (It's proof is not difficult.)

THEOREM 2–1. *If $F(T^1, M) \underset{z}{\sim} S^2 + S^2$, then M is diffeomorphic to CP^3.*

PROOF. By the exact sequence [1] (chap. III, 10–5)

$$\cdots\cdots \xrightarrow{\mu^* - \delta^*} H^q(\Sigma^7/T^1, F(T^1, \Sigma^7)) \xrightarrow{q^*} H^q(\Sigma^7) \longrightarrow H^{q-1}(\Sigma^7/T^1, F(T^1, \Sigma^7))$$

$$\oplus H^q(F(T^1, \Sigma^7)) \xrightarrow{\mu^* - \delta^*} H^{q+1}(\Sigma^7/T^1, F(T^1, \Sigma^7)) \longrightarrow \cdots\cdots$$

we see that Σ^7/T^1 is a cohomology 6-sphere and the induced action $(S^1, \Sigma^7/T^1)$ is semifree with the fixed point set $\pi^{-1}(F_1)/T^1 \approx S^2$. It follows from above exact sequence, that $M/T^1 = (\Sigma^7/S^1)/T^1 = (\Sigma^7/T^1)/S^1$ is a cohomology 5-sphere. Since M is simply connected and T^1 is connected, M/T^1 is simply connected, and hence M/T^1 is diffeomorphic to S^5.

Let U be a T^1-invariant tubular neighborhood of F_0 in M. Clearly, U/T^1 is a tubular neighborhood of F_0 in $M/T^1 \approx S^5$. Hence, there is a diffeomorphism from U/T^1 to $S^2 \times \mathbb{D}^3$ such that F_0 corresponds to $S^2 \times \{0\}$.

Hence, $(M-U)/T^1 \approx S^5 - S^2 \times \mathbb{D}^3 \approx \mathbb{D}^3 \times S^2$

Therefore, we may take the following interpretation.

(9) $\qquad M/T^1 = S^2 \times \mathbb{D}^3 \cup \mathbb{D}^3 \times S^2,\ F_0 = S^2 \times \{0\}$ and $F_1 \subset \mathbb{D}^3 \times S^2$.

From the following exact sequences,

$$0 \longrightarrow H^2(\mathbb{D}^3 \times S^2, F_1) \longrightarrow H^2(M-U) \underset{\cong}{\xrightarrow{i^*}} H^2(F_1) \longrightarrow H^3(\mathbb{D}^3 \times S^2, F_1) \longrightarrow 0$$

(the sequence in [1] (chap. III. 10–5))

and

$$\cdots\cdots \longrightarrow H^2(\mathbb{D}^3 \times S^2, F_1) \longrightarrow H^2(\mathbb{D}^3 \times S^2) \xrightarrow{i^*} H^2(F_1) \longrightarrow H^3(\mathbb{D}^3 \times S^2, F_1) \longrightarrow \cdots\cdots$$

(cohomology exact sequence of pair $(\mathbb{D}^3 \times S^2, F_1)$).

We can prove

$$\text{(10)} \qquad i^*: H^2(\mathbb{D}^3\times S^2)\xrightarrow{\cong} H^2(F_1).$$

Therefore, we may assume that $F_1=\{0\}\times S^2\subset \mathbb{D}^3\times S^2$. (see [2])

Let X_0 be $a\times\mathbb{D}^3\subset S^2\times\mathbb{D}^3$ and X_1 a set in $\mathbb{D}^3\times S^2$ which is represented the mapping cylinder of $a\times S^2\longrightarrow\{0\}\times S^2=F_1$, where $a(\in S^2)$ is a point and the map is the restriction of the projection $\mathbb{D}^3\times S^2\longrightarrow\{0\}\times S^2$.

We will consider $p^{-1}(X_0\cup X_1)=N_0\cup N_1$, where p is the orbit projection of T^1 action on M and $N_i=P^{-1}(X_i)(i=0,1)$. Clearly, N_0 (N_1, respectively) is diffeomorphic to the mapping cylinder of $S^3\longrightarrow a\times\{0\}(\in F_0)(S^1\longrightarrow F_1$, respectively), where these maps are the restrictions of the normal bundle's projections of F_i in $M(i=0,1)$. Hence M containts a submanifold $N_0\cup N_1$ which is diffeomorphic to $\mathbb{C}P^2$, and hence M is $\mathbb{C}P^3$. This completes the proof of Theorem 2–1.

3. T² action on M

It is not difficult to prove the following.

LEMMA 3–1. *Let $T_1(T_2, D, D_{-1}$, respectively) be a subgroup of $T^1\times T^1$ such that $T_1=T^1\times\{1\}$ ($T_2=\{1\}\times T^1$, $D=\{(t,t)\in T^1\times T\}$, $D_{-1}=\{(t,t^{-1})\in T^1\times T^1\}$, respectively), and lel K be a one dimensional subtorus of $T^1\times T^1$.*

(i) *If $K\cap T_1=\{1\}$ and $K\cap T_2=\{1\}$, then $K=D$ or D_{-1}.*

(ii) *If $K\cap T_1=\{1\}$, $K\cap D=\{1\}$ and $K\cap D_{-1}=\{1\}$, then $K=T_2$.*

THEOREM 3–2. *If T^2 acts effectively on M, then M is diffeomorphic to $\mathbb{C}P^3$.*

PROOF. It is enough to prove the case in which $F(T^2, M)=\sum_{i=0}^{3}\{p_i\}$ $(p_i\in M)$ and dim $F(G,M)\leq 2$ for any subgroup G of T^2. In this case, we have subtori K_1, K_2 and K_3 of T^2 such that $K_i\cap K_j=\{1\}$ $(i\neq j)$ and each components of $F(K_i, M)$ containg P_0 are two dimensional. (Consider the slice representation of T^2 at $P_0\in M$.)
Now, we may assume that $T^2=K_1\times K_2$
By lemma 3–1, we have $K_3=D$ or D_{-1}

We may assume $F(K_1, M)=F_1+\{p_2\}+\{p_3\}$, in fact otherwise $M=\mathbb{C}P^3$. From the fact $F(K_2, M-F_1)$ is a cohomology sphere, it can be easily shoown that p_1 is not contained in the component F_2 of $F(K_2, M)$ containing p_0. Hence, we may assume that the component F_2 contains P_2. In this case, p_1 and p_2 are not contained in the component F_3 of $F(K_3, M)$ containing P_0. Hence, the component F_3 contains P_3.

By [1] (chap. VII, 3–2), it is known that $F(G, M)$ consists of exactly two components for any cyclic subgroup G of order 2 of T^2. Therefore, we may assume that $F(Z_2, M)=S_0^2+S_1^2$ for $Z_2\subset K_3$ such that P_0, $P_3\in S_0^2$ and P_1, $P_2\in S_1^2$. (clearly, one component of $F(K_3, M)$ is S_0^2).

Since S_1^2 is a T^2-invariant set, there is a subtorus T^1 of T^2 such that acts trivially on S_1^2. It is easy to see that $T^1 \cap K_1 = \{1\}$ and $T^1 \cap K_2 = \{1\}$, because $T^1 \cap K_i$ fixed $F_i \cup S_1^2$ which contains P_i. Hence, $T^1 = D$ or D_{-1}.
If $T^1 = K_3$, then $M = \mathbb{C}P^3$. Therefore, let $K_3 = D$ and $T^1 = D_{-1}$.
By the same argument as for K_2, we have a subtorus T_0^1 of T^2 such that $T_0^1 \cap K_1 = \{1\}$, $T_0^1 \cap D = \{1\}$ and $T_0^1 \cap D_{-1} = \{1\}$

It follows from Lemma 3-1 that T_0^1 is K_2. Therefore, we have $F(K_2, M) \underset{z}{\sim} S^2 + S^2$. Theorem 2-1 implies that $M = \mathbb{C}P^3$.

This completes the proof of Theorem 3-2.

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