SO(3)-action and 2-torus action on homotopy complex projective 3-spaces

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Introduction

In this note, we shall consider SO(3)-action and 2-torus action on homotopy complex projective 3-space with the view of proving that any exotic homotopy complex projective 3-space admits no effective SO(3)-action nor 2-torus action. In this direction the following results are known.

1. *A homotopy complex projective* 3-*space (abbreviated by* hCP_3*) which admits an effective* 3-*torus action is diffeomorphic to the standard complex projective* 3-*space* CP_3. ([8])
2. *An* hCP_3 *which admits an effective* 1-*torus action with fixed point set of* 2 *components is diffeomorphic to* CP_3. ([11])
3. *An* hCP_3 *which admits n-dimensional compact connected Lie group action* ($n \geqq 6$) *is diffeomorphic to* CP_3. ([4])

In this note we shall prove the following

THEOREM. *If an* hCP_3 *admits an effective SO(3)-action or* 2-*torus action, then it is diffeomorphic to* CP_3.

In the following all actions are assumed to be differentiable.

1. Statement of results

First we shall consider 2-torus action. Let G be a 2-torus and t_1, t_2 denote the standard complex 1-dimensional representations of G. Then it is well known that the complex representation ring $R(G)=Z[t_1, t_2, t_1^{-1}, t_2^{-1}]$.

LEMMA 1. *Let* ϕ_1 *and* ϕ_2 *be complex* 1-*dimensional representations of G. Put* $\phi_1=t_1^a t_2^b$ *and* $\phi_2=t_1^c t_2^d$, *where a, b, c, d, are integers. Then if* $\ker\phi_1 \cap \ker\phi_2=1$, *we have* $ad-bc=1$.

PROOF. Assume the contrary. Then $ad-bc=e\neq\pm1$ and e is not zero. It is easy to see that there are integers k, l such that at least one of $(kd-lb)/e$, $(la-kc)/e$ is not integer. Hence there exist real numbers θ and λ such that $a\theta+b\lambda\equiv 0\ (2\pi)$, $c\theta+\lambda d\equiv 0\ (2\pi)$

and θ or λ is not integral multiple of 2π. This contradicts to our assumption.

We have the following

COROLLARY. $R(G)=Z[\phi_1, \phi_2, \phi_1^{-1}, \phi_2^{-1}]$.

Let M be an hCP_3 and consider an effective action of G on M. Let F denote the fixed point set $F(G, M)$. It is clear that F contains no component of dimension 4. Then the following four cases can occur.

Case 1. $F=S^2\cup S^2$ (S^2=2-dimensional sphere)

Case 2. $F=S^2\cup\{x_1, x_2\}$

Case 3. $F=\{x_1, x_2, x_3, x_4\}$

Consider the case 1 and 2. Let φ_x denote the local representation of G at $x\in S^2$. φ_x may be written by $1+t_1^a t_2^b+t_1^c t_2^d$, where a, b, c and d are integers (Note that φ_x is considered as complex representation). The restricted action to ker $t_1^a t_2^b$ (=1-torus) has 4-dimensional fixed point set. It follows from the following proposition that M is diffeomorphic to CP_3.

PROPOSITION 1. *Assume that there exists an orientable 4-dimensional submanifold F of M such that $F\underset{Q}{\sim}CP_2$ and inclusion $i: F\longrightarrow M$ induces an isomorphism $i^*: H^2(M; Q)\cong H^2(F; Q)$, where Q is the field of rational numbers, for spaces X, Y, $X\underset{Q}{\sim}Y$ means that X and Y have isomorphic Q-cohomology. Then M is diffeomorphic to CP_3.*

We shall prove this proposition in section 2.

Consider case 3. Let $\rho_i=\varphi_{i1}+\varphi_{i2}+\varphi_{i3}$ be the local representation at isolated fixed point x_i (i=1, 2, 3, 4) (considered as complex representation). Assume that ker $\varphi_{ij}\cap$ ker $\varphi_{ik}\supsetneqq\{\text{identity}\}$ $(j\neq k)$. Then there exists a subgroup $Z_p\subset G$ (p; prime) such that $F(Z_p, M)$ is 4-dimensional. Denote F be the 4-dimensional component of $F(Z_p, M)$. Then F is Z_p-cohomological complex projective 2-space. Let T be a 1-torus containing Z_p. Since $F(T, M)=F(T/Z_p, F(Z_p, M))=F(T/Z_p, F)\cup\{\text{one point}\}$, and Euler characteristic of $F(T, M)$ is equal to that of M, we have $F\underset{Q}{\sim}CP_2$ and inclusion $i: F\longrightarrow M$ induces an isomorphism $i^*: H^2(M; Q)\cong H^2(F; Q)$ (see [2] chap. VII). Proposition 1 shows that M is CP_3. Thus we may assume that for any $i, j, k (j\neq k)$ ker $\varphi_{ij}\cap$ker $\varphi_{ik}=\{1\}$. Put $\varphi_1=\varphi_{11}$ and $\varphi_2=\varphi_{12}$. Then from lemma 1 it follows that $R(G)=Z[\varphi_1, \varphi_2, \varphi_1^{-1}, \varphi_2^{-1}]$. Since ker $\varphi_1\cap$ ker φ_{13}=ker $\varphi_2\cap$ ker $\varphi_{13}=\{1\}$, we may assume $\rho_1=\varphi_1+\varphi_2+\varphi_1\varphi_2$. In fact, let $\varphi_{13}=\varphi_1^a\varphi_2^b$. Then we have $|a|=|b|=1$. If $\varphi_{13}=\varphi_1\varphi_2^{-1}$, we have $\varphi_1=\varphi_2\varphi_{13}$ and hence we can take φ_{13} instead of φ_1. If $\varphi_{13}=\varphi_1^{-1}\varphi_2^{-1}$, no change is needed, because $\varphi_1\varphi_2$ and $\varphi_1^{-1}\varphi_2^{-1}$ determine the same real representation.

Put $\rho_i=\varphi_1^{a_i}\varphi_2^{a_i'}+\varphi_1^{b_i}\varphi_2^{b_i'}+\varphi_1^{c_i}\varphi_2^{c_i'}$ $(i=2, 3, 4)$, where a_i, a_i', b_i, b_i', c_i, c_i' are integers. Note that we may assume that $a_i+b_i=c_i$ and $a_i'+b_i'=c_i'$. Let K_i=ker φ_i $(i=1, 2)$. Consider the restricted action of K_i. We may assume $F(K_i, M)=S^2\cup\{\text{two points}\}$. Then the following two cases can occur. Write $F(K_1, M)=S_1^2\cup\{z_1, z_2\}$ and $F(K_2, M)=S_2^2\cup\{y_1, y_2\}$.

Case 1. $x_1, x_2 \in S_1^2$ $\quad x_1, x_2 \in S_2^2$

$\{x_3, x_4\} = \{z_1, z_2\}$ $\quad \{x_4, x_3\} = \{y_1, y_2\}$

Case 2. $x_1, x_2 \in S_1^2$ $\quad x_1, x_3 \in S_2^2$

$\{x_3, x_4\} = \{z_1, z_2\}$ $\quad \{x_2, x_4\} = \{y_1, y_2\}$.

Consider the case 1. It can be shown that $\rho_2 = \varphi_1 + \varphi_2 + \varphi_1\varphi_2^{\pm 1}$. In fact, since $\rho_2 = \varphi_1^{a_2}\varphi_2^{a_2'} + \varphi_1^{b_2}\varphi_2^{b_2'} + \varphi_1^{c_2}\varphi_2^{c_2'}$ and $x_2 \in S_1^2$, we have $\rho_2/K_1 = 1 + \varphi_2 + \varphi_2$ and hence we may assume $a_2' = 0$, $b_2' = \pm 1$ and $c_2' = \pm 1$. Moreover since ρ_2/K_2 is equivalent to $\rho_1|K_2$, we may assume $b_1 = 0$, $a_1 = \pm 1$ and $c_1 = \pm 1$. Thus we have $\rho_2 = \varphi_1 + \varphi_2 + \varphi_1\varphi_2^{\pm 1}$ (Note φ_1 and φ_1^{-1} determine the same real 2-dimensional representation of S^1). By similar arguments, we can show that in case 2 local representations at x_i are given as follows;

$$\rho_1 = \varphi_1 + \varphi_2 + \varphi_1\varphi_2$$
$$\rho_2 = \varphi_1 + \varphi_1^a\varphi_2 + \varphi_1^{a+1}\varphi_2$$
$$\rho_3 = \varphi_2 + \varphi_1\varphi_2^b + \varphi_1\varphi_2^{b+1}$$
$$\rho_4 = \varphi_1^{a_4}\varphi_2^{a_4'} + \varphi_1^{b_4}\varphi_2^{b_4'} + \varphi_1^{c_4}\varphi_2^{c_4'}$$

In section 3, we show that only possible case is case 1 with one exception and local representations are given;

$$\rho_1 = \varphi_1 + \varphi_2 + \varphi_1\varphi_2$$
$$\rho_2 = \varphi_1 + \varphi_2 + \varphi_1\varphi_2$$
$$\rho_i = \varphi_1^{a_i}\varphi_2^{a_i'} + \varphi_1^{b_i}\varphi_2^{b_i'} + \varphi_1^{c_i}\varphi_2^{c_i'} \qquad i = 3, 4$$

where $a_i' + a_i \neq 0$ and $b_i + b_i' \neq 0$.

In this case let D denote the subgroup of G defined by $\varphi_1 = \varphi_2$. Clearly D is 1-torus with $F(D, M) = F(G, M)$ and $R(D) = Z[\varphi, \varphi^{-1}]$, where $\varphi = \varphi_1 = \varphi_2$. Let η be the pull back of the Hopf bundle over CP_3 via a homotopy equivalence from M to CP_3. η may be considered a D-bundle over M. *Put* $\eta|x_i = \varphi^{\alpha_i}$. We may assume $\alpha_1 = 0$ (For these arguments, see Part II section 1 in [7]). The following results are proved in [7].

1. α_i's are all distinct integers.
2. Let $\varphi^{n_{i1}} + \varphi^{n_{i2}} + \varphi^{n_{i3}}$ be local representations of D at x_i ($i = 1, 2, 3, 4$). Then

$$\prod_{j \neq i} (\alpha_j - \alpha_i) = \varepsilon \prod_{j \neq i} n_{ij} \qquad (i = 1, 2, 3, 4)$$

where $\varepsilon = \pm 1$ is independent on i.

3. $\sum_{i=1}^{4} \prod_{j=1}^{3} (\varphi^{n_{ij}} + 1)(\varphi^{n_{ij}} - 1) = 0$

4. If $\prod_{j \neq i} \left(\varphi^{-\frac{\alpha_j - \alpha_i}{2}} - \varphi^{\frac{\alpha_j - \alpha_i}{2}}\right) \Big/ \prod_{j \neq i} \left(\varphi^{-\frac{n_{ij}}{2}} - \varphi^{\frac{n_{ij}}{2}}\right)$ is independent on i, then the first Pontrjagin class $p_1(M)$ of M is equal to $4a^2$, where $H^*(M; Z) = Z[a]/(a^4)$, and hence M is diffeomorphic to CP_3 ([12]).

Applying these results to our action of D, we have three non-zero all distinct integers $\alpha_1, \alpha_2, \alpha_3$ satisfying

(1) $\alpha_1 \alpha_2 \alpha_3 = \pm 2$

(2) $\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)=\pm 2$

(3) $\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)=\pm(a_3+a_3')(b_3+b_3')(c_3+c_3')$

(4) $\alpha_3(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)=\pm(a_4+a_4')(b_4+b_4')(c_4+c_4')$.

From (1) and (2) it follows that $\alpha_1=\pm 1$, $\alpha_2=\mp 1$ and $\alpha_3=\pm 2$ or $\alpha_1=\pm 1$, $\alpha_2=\pm 2$ and $\alpha_3=\mp 1$, respectively. Then the left hand sides of (3) and (4) are 6 or -6. Since a_3+a_3' and b_3+b_3' are relatively prime and $c_3+c_3'=(a_3+a_3')+(b_3+b_3')$, we may assume $a_3+a_3'=\pm 1$ and $b_3+b_3'=\pm 2$, respectively. Similarly we have $a_4+a_4'=\pm 1$ and $b_4+b_4'=\pm 2$, respectively. Then we can choose n_{ij} so that the condition of 4 is satisfied. Thus we have proved the following

Theorem A. *If an hCP_3 admits an effective 2–torus action, it is diffeomorphic to CP_3.*

Now consider SO(3)–action on hCP_3 M. Denote $G=SO(3)$. Note that there is a point of M whose isotropy subgroup is a maximal torus of G. In fact, assume the contrary. Then all orbits have the same Q–cohomology of 3–sphere or a point. Hence by Vietoris-Begle mapping thereom, the orbit map π: $M \longrightarrow M^*$ induces isomorphism π^*: $H^*(M^*; Q) \longrightarrow H^*(M; Q)$ for $* \leqq 3$. It is not difficult to see that this contradicts to the structure of $H^*(M; Q)$.

In section 4, we shall prove the following

Lemma 2. *If there is an element $g \neq 1$ (1 denotes the identity element of G) with 4–dimensional fixed point set, then M is diffeomorphic to CP_3.*

Since we are only interested in proving that M is diffeomorphic to CP_3, we may assume that any element of finite order has at most 2–dimensional fixed point set. It follows

Proposition 2. *Any principal isotropy subgroup consists of only identity.*

Let T be the standard maximal torus of G. We shall find possible types of $F(T, M)$. It follows from a result in [10] that it is impossible for $F(T, M)$ to have a 4–dimensional component. Hence we have following possible three cases;

1. $F(T, M)$=union of two 2–spheres
2. $F(T, M)$=union of 2–sphere and isolated two points
3. $F(T, M)$=union of isolated four points.

In case 1, M is diffeomorphic to CP_3. Hence we consider the cases 2 and 3.

We shall prove the following lemma in section 4.

Lemma 3. *In case 3, M is diffeomorphic to CP_3*

Let D_2 be the dihedral subgroup of order 4. We have

Lemma 4. $F(D_2, M) \neq \emptyset$.

Then by a result in [1] (chap. XIII. Th. 4.3), the dimension of $F(D_2, M)$ is given by

$$2\, dim\, F(D_2, M)=(\sum_H dim\, F(H, M))-6$$

where H is subgroups of D_2 of index 2. Let a and b be generators of D_2 such that $b \in T$ and $N = T \cup aT$ the normalizer of T. Then H is $\{a\}$, $\{b\}$ or $\{ab\}$. Since $dim\ F(a, M) = dim\ F(b, M) = dim\ F(ab, M) = 2$, we have $dim\ F(D_2, M) = 0$. It follows from a result in [1] (chap. XIII. Th. 3.6) that $F(D_2, M)$ consist of four points. Moreover $F(b, M)$ is union of two 2-spheres (see [2], chap. VIII) and hence $F(a, M)$ is also union of two 2-spheres. Let $F(b, M) = S_1^2 \cup S_2^2$. We may assume that $F(T, M) = S_1^2 \cup \{x_1, x_2\}$. Since $F(a, F(b, M)) = F(a, S_1^2) \cup F(a, S_2^2) \neq \emptyset$, $F(a, S_1^2)$ and $F(a, S_2^2)$ is not empty. It is known that the fixed point set of an involution on 2-sphere is 1-dimensional or two points (see [2] chap. VII) and hence $F(D_2, M) = \{y_1, y_2, y_3, y_4\}$ and $y_1, y_2 \in S_1^2 \subseteq F(T, M)$. Hence y_1, and $y_2 \in F(N, M)$. Possible types of $F(N, M)$ are following

(a) $F(N, M) = \{y_1, y_2\} \subseteq S_1^2$

(b) $F(N, M) = \{y_1, y_2, x_1, x_2\}$.

The following lemma is proved in [6] ([6] Th. (3. 2))

LEMMA 5. *Let $z \in F(D_2, M)$ and G_z is a normalizer of a 1-torus. Then $G(z) \cap F(D_2, M)$ consists of three points.*

Consider case (a). Suppose $y_1 \in F(N, M) - F(G, M)$. Then it follows from lemma 5 that $G(y_1) \cap F(D_2, M)$ consists of three points. Put $G(y_1) \cap F(D_2, M) = \{y_1, z_1, z_2\}$. It is clear that $y_2 = z_j$ for some j. Hence $\{z_1, z_2\} = \{y_3, y_4\}$, and one of y_1 and y_2 is a fixed point. Assume $y_1 \in F(G, M)$. Consider the action of G on S^5 (=the small unit sphere around y_1) induced by local representation of G at y_1. This action has S^2 as an orbit. It follows from a result in [9] ([9] section 2) that all 0-dimensional isotropy subgroup are $\{1\}$. Since 2-dimensional component of $F(a, M)$ containing y_1 intersects with 2-dimensional component of $F(b, M)$ containing y_1 at one point y_1, the action of G on S^5 must have a non-trivial finite isotropy subgroup, which contradicts to the above fact. Similarly we can show that case (b) can not occur. Thus under our assumption the case in which $F(T, M) = S^2 \cup \{x_1, x_2\}$ cannot occur. Hence we have proved the following

THEOREM B. *If an hCP_3 admits a non-trivial $SO(3)$-action, then it is diffeomorphic to CP_3.*

2. Proof of Proposition 1

In this section, we shall prove proposition 1. Let ν be the normal bundle of F in M. It follows from a result in [3] that the Euler class $\chi(\nu)$ is given by

$$\chi(\nu) = i^* D^{-1}(i_* [F]),$$

where $[F]$ denotes the fundamental class of F and D: $H^2(M, Z) \longrightarrow H_4(M; Z)$ Poincare duality. α denotes a generator of $H^2(F; Z)/Tor. \cong Z$ and a a generator of $H^*(M; Z) = Z[a]/(a^4)$. Put $i^*(a) = m\alpha + \beta$, *where* $\beta \in Tor\ H^2(F : Z)$. We ahve $i^*(a^2) = m^2 \alpha^2$.

Put $D^{-1} i_*[F] = ka$. Then we have

$$k=\langle[M], ka^3\rangle=\langle ka\cap[M], a^2\rangle$$
$$=\langle i_*[F], a^2\rangle=\langle[F], i^* a^2\rangle$$
$$=\langle[F], m^2 \alpha^2\rangle=m^2.$$

Hence we have $\chi(\nu)=i^*(m^2\ a)=m^2(m\alpha+\beta)$. This implies $p_1(\alpha)=(\chi(\nu))^2=m^6\alpha^2$. Put $p_1(M)=l\ a^2$. From $i^*p_1(M)=p_1(F)+p_1(\nu)$, it follows that $3=m^2(l-m^4)$. Since m and l are integers, we have $l=4$. Hence M is diffeomorphic to CP_3 (see [12]).

3. 2–torus action

In this section, we shall consider the remaining cases in section 2. We use the same notations as in section 2.

Case 1. One of a_i+a_i', b_i+b_i' ($i=3, 4$) is zero.

Without loss of generality, we may assume that $a_3+a_3'=0$. If $b_3+b_3'=0$, the fixed point set of D–action is 6–dimensional, which contradicts to the effectivity. Since $F(D, M)$ has 2–dimensional component S and x_1, x_2 as isolated fixed points, we have $x_4\in S$. We may assume that $a_4+a_4'=0$, because the local representations of D at x_3 and x_4 are equivalent. Choose new coordinates θ_1, θ_2 on G such that $\varphi_1=\theta_1$ and $\varphi_2=\theta_1\theta_2$. Then we have

$$\rho_i=\theta_1+\theta_1\theta_2+\theta_1^2\theta_2 \qquad i=1, 2$$

and
$$\rho_j=\theta_2^{a_i'}+\theta_1^{b_i+b_i'}\theta_2^{b_i'}+\theta_1^{c_i+c_i'}\theta_2^{c_i'} \qquad i=3, 4.$$

We have $a_i'=\pm1$ and $b_i+b_i'=\pm1$ and hence $\rho_j=\theta_2+\theta_1\theta_2^{b_j}+\theta_1\theta_2^{b_j\pm1}$ for $j=3, 4$. Moreover we choose coordinates ξ_1, ξ_2 on G such that

$$\rho_i=\xi_1+\xi_1^2\xi_2+\xi_1^4\xi_2 \qquad i=1, 2$$
$$\rho_j=\xi_1^2\xi_2+\xi_1^{b_j+1}\xi_2^{b_j}+\xi_1^{2(b_j\pm1)+1}\xi_2^{(b_j\pm1)} \qquad j=3, 4.$$

Consider the action restricted to $D'=\{\xi=\xi_2=\xi_1\}$. Then local representations of D' at x_i are as follows;

at x_i ($i=1, 2$); $\xi+\xi^4+\xi^5$

at x_i ($i=3, 4$); $\xi^3+\xi^{2b_j+1}+\xi^{3(b_j\pm1)+1}$

Clearly $F(D', M)=F(G, M)$. Then there exist three distinct integers α_1, α_2, α_3 satifying

(1) $\alpha_1(\alpha_2\alpha_3 \quad =\pm20$
(2) $\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)=\pm20$
(3) $\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)=\pm3(2b_3+1)(3(b_3\pm)+1)$
(4) $\alpha_3(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)=\pm3(2b_4+1)(3(b_4\pm1)+1)$.

From (1) and (2) it follows that possible pairs of $(\alpha_1, \alpha_2, \alpha_3)$ are followings

case	1	2	3	$\alpha_2(\alpha_1-\alpha_2)(\alpha_3-\alpha_2)$	$\alpha_3(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)$
1	1	−4	5	±180	±180
2	1	5	−4	±180	±180
3	−1	4	−5	±180	±180
4	−1	−5	4	±180	±180
5	4	−1	5	±130	± 30
6	4	5	−1	± 30	± 30
7	−4	1	−5	± 30	± 30
8	−4	−5	1	± 30	± 30
9	+5	1	4	± 12	± 12
10	5	4	1	± 12	± 12
11	−5	−4	−1	± 12	± 12
12	−5	−1	−4	± 12	± 12

Consider equations; $(2x+1)(3x+4)=\pm 60$, ± 10, or $(2x+1)(3x-2)=\pm 60$, ± 10. It is easily seen that these equations have no integral roots. Hence cases 1, 2,......, 8 cannot occur. Consider cases 9,......, 12. Since $(2x+1)(3x+4)=4$ has 0 as its integral roots, local representations of D' of following type may occur;

at x_i $(i=1, 2)$; $\xi+\xi^4+\xi^5$

at x_i $(i=3, 4)$; $\xi+\xi^3+\xi^4$

In [7], it is proved that the number of coonnected components of $F(Z_{p^r}, M)$ intercecting $F(D', M)$ is the number of distinct residue classes among the four integers 0, α_1, α_2, α_3. Let Z_5 be defined by $\xi^5=0$. Clearly the number of connected components of $F(Z_5, M)$ intercecting $F(D', M)$ is 3. Hence the number of distinct residue classes among 0, α_1, α_2, α_3 is 3. From relations

(1) $\alpha_1\alpha_2\alpha_3 = \pm 20$
(2) $\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3) = \pm 20$
(3) $\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3) = \pm 12$
(4) $\alpha_3(\alpha_3-\alpha_1)(\alpha_3-\alpha_2) = \pm 12$,

it follows that only possible residue classes are;

$\alpha_1 \equiv 0 \ (5)$

$\alpha_2 \not\equiv 0 \ (5)$

$\alpha_3 \not\equiv 0 \ (5)$

$\alpha_2 \not\equiv \alpha_3 \ (5)$.

Then clearly $\alpha_1=\pm 5$ and $\alpha_1=\alpha_2+\alpha_3$. Hence we have $\alpha_2=\pm 1$, $\alpha_3=\pm 4$, $\alpha_1-\alpha_2=\pm 4$, $\alpha_1-\alpha_3=\pm 1$ and $\alpha_2-\alpha_3=\pm 3$. It is not difficult to see that one can choose signs of n_{ij} so that condition of 4 in section 1 holds. Therefore M is diffeomorphic to CP_3.

Next we shall consider case 2. In this case we have shown that local representations at x_i are given as follows;

$$\rho_1=\varphi_1+\varphi_2+\varphi_1\varphi_2$$

$$\rho_2=\varphi_1+\varphi_1^a\varphi_2+\varphi_1^{a+1}\varphi_2$$

$$\rho_3=\varphi_2+\varphi_1\varphi_2^b+\varphi_1\varphi_2^{b+1}$$

$$\rho_4=\varphi_1^{a_4}\varphi_2^{a_4'}+\varphi_1^{b_4}\varphi_2^{b_4'}+\varphi_1^{c_4}\varphi_2^{c_4'}$$

where $c_4=a_4+b_4,\ c_4'=a_4'+b_4'$.

Since x_1 and x_2 are not contained in the same component of $F(K_2, M)$, a is neither 0 nor -1. Similarly we have $b\neq 0, 1$. Consider the action restricted to $D'=\{\varphi_1=\varphi_2^{-1}]$. Clearly $F(D', M)$ has 2-dimensional component which contains x_1. Then the following three cases can occur;

(1) $a=1$

(2) $a\neq 1$ and $b=1$

(3) $a\neq 1$, $b\neq 1$ and one of a_4-a_4', b_4-b_4' and c_4-c_4' is zero.

Case (1). Considered the action restricted to $H=\{\varphi_1=\varphi_2^{-1}\}$. Clearly x_2 is contained in a 2-dimensional component of $F(H, M)$. Since x_1 and x_3 are isolated fixed points of $F(H, M)$, just one of a_4-2a_4', b_4-2b_4' and c_4-2c_4' must be zero. Assume that $a_4=2a_4'$. It follows from the fact $\det\begin{pmatrix}a_4, & b_4 \\ a_4' & b_4'\end{pmatrix}=\pm 1$ that $a_4'=\pm 1$ and $2b_4'-b_4=\pm 1$. Hence we ahve $\rho_4=\varphi_1^{\pm 2}\varphi_2^{\pm 1}+\varphi_1^d\varphi_2^{d'}+\varphi_1^e\varphi_2^{e'}$ (Note $e=d\pm 2$, $e'=b'\pm 1$).

Since $2d'-d=\pm 1$, we have

$$\rho_4=\varphi_1^2\varphi_2+\varphi_1^{2d'\mp 1}\varphi_2^{d'}+\varphi_1^{2d'\mp 1+2}\varphi_2^{d'+1}$$

or

$$=\varphi_1^2\varphi_2+\varphi_1^{2d'\mp 1}\varphi_2^{d'}+\varphi^{2d'\mp 1-2}\varphi_2^{d'-1}$$

Clearly x_1, x_2 and x_3 are isolated fixed points of $H'=\{\varphi_2=\varphi_1^2\}$ and hence x_4 must be isolated fixed point. Thus we have the local representations of H'

at x_1; $\varphi+\varphi^2+\varphi^3$

at x_2; $\varphi+\varphi^3+\varphi^4$

at x_3; $\varphi^2+\varphi^{2b+1}+\varphi^{2b+3}$

at x_4; $\varphi^4+\varphi^{4d\mp 1}+\varphi^{4d\pm 4\mp 1}$

Hence there exist three distinct integers α_1, α_2, α_3 satisfying

(1) $\alpha_1\alpha_2\alpha_3 = \pm 6$

(2) $\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3) = \pm 12$

(3) $\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3) = \pm 2(2b+1)(2b+3)$

(4) $\alpha_3(\alpha_3-\alpha_1)(\alpha_3-\alpha_2) = \pm 4(4d'-1)(4d'\pm 41)$.

From (1) and (2) it follows the following three cases occur.

i) $\alpha_1=\pm 1 \quad \alpha_2=\mp 2 \quad \alpha_3=\mp 3$

ii) $\alpha_1=\pm 1 \quad \alpha_2=\mp 3 \quad \alpha_3=\mp 2$

iii) $\alpha_1=\pm 3 \quad \alpha_2=\mp 1 \quad \alpha_3=\pm 2,$

where the signs are corresponding respectively.

Case i). It follows from (3) and (4) that $b=-2$ and $d'=$ or $d'=-2$. If $d'=0$, then x_4 is contained in 2-dimensional component of $F(K_1, M)$, which contradicts to the assumption. If $d'=\pm 2$, we have $\rho_4=\varphi_1^2\varphi_2+\varphi_1^3\varphi_2^2+\varphi_1\varphi_2$, which implies that only x_3 is contained in a 2-dimensional component of the fixed point set of the subgroup $\{\varphi_1=\varphi_2\}$ of G, it is impossible. It is not difficult to see that case ii) and iii) are impossible.

Case $a\neq 1$ *and* $b=1$.
This case reduces to the case $a=1$.

Case $a\neq 1$, $b\neq 1$ *and one of* a_4-a_4', b_4-b_4' *and* c_4-c_4' *is zero.*

We may assume $a_3=a_3'$ without loss of generality. We have the following four possibilities of ρ_4;

$$\varphi_1\varphi_2+\varphi_1^c\varphi_2^{c+1}+\varphi_1^{c+1}\varphi_2^{c+2}$$

$$\varphi_1\varphi_2+\varphi_1^c\varphi_2^{c-1}+\varphi_1^{c+1}\varphi_2^{c}$$

$$\varphi_1\varphi_2+\varphi_1^c\varphi_2^{c+1}+\varphi_1^{c-1}\varphi_2^{c}$$

$$\varphi_1\varphi_2+\varphi_1^c\varphi_2^{c-1}+\varphi_1^{c-1}\varphi_2^{c-2}$$

If we put $\varphi_1=\theta_1$ and $\varphi_2=\theta_1^2\theta_2$, we have

$$\rho_1=\theta_1+\theta_1^2\theta_2+\theta_1^3\theta_2$$

$$\rho_2=\theta_1+\theta_1^{a+2}\theta_2+\theta_1^{a+3}\theta_2$$

$$\rho_3=\theta_1^2\theta_2+\theta_1^{2b+1}\theta_2^b+\theta_1^{2b+3}\theta_2^{b+1}$$

$$\rho_4=\theta_1^3\theta_2+\theta_1^{3c+2}\theta_2^{c+1}+\theta_1^{3c+5}\theta_2^{c+2}$$

$$\theta_1^3\theta_2+\theta_1^{3c-2}\theta_2^{c-1}+\theta_1^{3c+1}\theta_2^{c}$$

$$\theta_1^3\theta_2+\theta_1^{3c+2}\theta_2^{c-1}+\theta_1^{3c-1}\theta_2^{c}$$

$$\theta_1^3\theta_2+\theta_1^{3c-2}\theta_2^{c-1}+\theta_1^{3c-5}\theta_2^{c-2}.$$

The action restricted to $D=\{\theta_1=\theta_2\}$ has the same fixed point set as G-action. Then as above, there are three distinct integers α_1, α_2 and α_3 satisfiying

(1) $\alpha_1\alpha_2\alpha_3 = \pm 12$

(2) $\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3) = \pm(a-3)(a+4)$

(3) $\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3) = \pm 3(2b+1)(3b+4)$

(4) $\alpha_3(\alpha_3-\alpha_1)(\alpha_2-\alpha_2) = \pm 4(4c+3)(4c+7)$

$$=\pm 4(4c-3)(4c+1)$$
$$=\pm 4(4c+3)(4c-1)$$
$$=\pm 4(4c-3)(4c-7).$$

By direct computations we can show a contradiction.

4. SO(3)-action

In this section we shall consider actiona of $G=SO(3)$ on an hCP_3, M with the following property;

(*) Let T be the standard maximal torus of G. Then the fixed point set $F(T, M)$ is a union of one 2-sphere and two isolated points or a union of four isolated points.

First we shall prove lemma 2 in section 2. Assume there is an element g of order p (p; prime) such that dim $F(g, M)$ is greater than 2. If p is odd, dim $F(g, M)$ is 4 and $F(g, M)=F\cup\{pt.\}$, where $F\underset{Z_p}{\sim}CP_2$. Let S be a torus containing g. We have $F(S, M)=F(S/\{g\}, F(g, M))=F(S/\{g\}, F)\cup\{pt.\}$. Since the Euler characteristic of $F(T, M)$ is 4, we have the Euler characteristic of $F(S/\{g\}, F)=3$. Since $H^1(F; Z_p)=H^3(F; Z_p)=0$, we have $H^1(F; Q)=H^3(F; Q)=0$ and $H^2(F; Q)=Q$. In particular, $H^2(F; Z)/Tor.=Z$. It is easy to see that $F\underset{Q}{\sim}CP_2$. Since the inclusion $i: F\longrightarrow M$ induces an isomorphism $i^*{}_p$: $H^2(M; Z_p)\xrightarrow{\cong}H^2(F; Z_p)$, i^*: $H^2(M; Z)$——$H^2(F; Z)$ maps the generator of $H^*(M; Z)$ to a non-zero element and hence $i^*{}_Q$: $H^2(M; Q)\xrightarrow{\cong}H^2(F; Q)$ is an isomorphism. By proposition 1, M is diffeomorphic to CP_3. If p is even, dim $F(g, M)$ may be 3 (see [2], Chap. VII). In this case $F(g, M)$ is connected and $F(g, M)\underset{Z_2}{\sim}RP_3$, which contradicts to the fact the Euler characteristic of $F(S, M)$ is non-zero. In the case in which dim $F(g, M)=4$, the same argument as in the case in which p is odd show that M is diffeomorphic to CP_3. This completes the proof of lemma 2.

It follows from the assumption (*) that $GF(T, M)$ is at most 4-dimensional and hence there exists a point x in M whose isotropy subgroup is 0-dimensional. Let H be a principal isotropy subgroup and assume $H\neq\{1\}$. Since there is an element x with $G_x=T$, H is cyclic. Let g be an element of H whose order is p (p; prime). It is clear that $F(g, M)$ is at least 4-dimensional. It follows from lemma 2 that M is diffeomorphic to CP_3. This completes the proof of Proposition 2 in section 1.

Consider the case in which $F(T, M)=\{x_1, x_2, x_3, x_4\}$. Let a denote the element $\begin{bmatrix}1 & & \\ & -1 & \\ & & -1\end{bmatrix}$ and $b=\begin{bmatrix}-1 & & \\ & -1 & \\ & & 1\end{bmatrix}$. Then $N=T\cup aT$ and $D_2=\{1, a, b, ab\}$. Cleary $F(T, M)$ is a-invariant. Since $M_{(T)}$ is non-empty, there may occur following two case;

Case 1. $F(N, M)=\{x_1, x_2\}$

Case 2. $F(N, M)=\emptyset$.

Consider case 1. In this case $F(D_2, M)\neq\emptyset$. In section 1, we have noticed that $F(D_2,$

M) consists of isolated four points. We may put $F(D_2, M)=\{x_1, x_2, y_1, y_2\}$. Assume $F(G, M)=\emptyset$. It follows from lemma 5 in section 1 that $G(x_1)\cap F(D_2, M)$ and $G(x_2)\cap F(D_2, M)$ are consisting of 3 points and hence $G(x_1)$ and $G(x_2)$ must intercect, which is impossible. Thus $F(G, M)\neq\emptyset$. Assume $x_1\in F(G, M)$ and consider the $SO(3)$-action on S^5 induced by the slice representation of G at x_1. Clearly this action has only 0-dimensional isotropy subgroups and hence principal isotropy subgroups are icosahedral subgroup (see [9]). This is impossible, because $F(D_2, M)$ is 0-dimensional.

Consider case 2. Assume $S^2{}_i$ is a-invariant (Recall $F(b, M)=S_1^2\cup S_2^2$.) In this case the same arguments as in the proof of lemma 4 in section 1 show that $F(D_2, M)\neq\emptyset$. Let $F(D_2, M)\cap S_1^2=\{z_1, z_2\}$. Clearly G_{z_i} is 0-dimensional. By a result in [6] ([6], (3.7)) it follows that $F(D_2, M)\cap G(z_1)$ consists of six points which is impossible. Thus S_i is not a-invariant and hence we may assume that $ax_1=x_3$, $ax_2=x_4$. Consider the restricted action of T. We can decompose the tangent space at x_i into a direct sum $T_{x_i}(G(x_i))\oplus T_{x_i}(S_j^2)\oplus V_i$, where S_j^2 is the component of $F(b, M)$ containing x_i. Then local representation of T at x_i is given by $t+t^{2m_i}+t^{n_i}$, where $R(T)=Z[t, t^{-1}]$ and $(2m_i, n_i)=1$. Clearly $m_1=m_2$ and $m_3=m_4$. It is easy to see that $m_1=m_3$ and $n_1=n_3$, $n_2=n_4$. As noticed in section 1, there are distinct non-zero integers $\alpha_1, \alpha_2, \alpha_3$ satisfying

(1) $\alpha_1\alpha_2\alpha_3 = \pm 2mn$

(2) $\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3) = \pm 2mk$

(3) $\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3) = \pm 2mn$

(4) $\alpha_3(\alpha_3-\alpha_1)(\alpha_3-\alpha_2) = \pm 2mk$

It is not difficult to show that n and k are distinct. Hence there is a prime p such that n is divisible by p^s and k is not divisible by p^s. Clearly the number of components of $F(Z_{p^s}, M)$ intersecting $F(T, M)$ is 3 and hence the number of distinct residue classes among the four integers 0, α_1, α_2, α_3, is 3. Thus we have $\alpha_2\equiv 0\ (p^s)$, $\alpha_1\not\equiv 0\ (p^s)$, $\alpha_3\not\equiv 0\ (p^s)$ and $\alpha_1\not\equiv\alpha_3\ (p^s)$. Moreover $2m$ and n divide one of α_1, α_2, α_3, $\alpha_2-\alpha_1$ and $\alpha_2-\alpha_3$. Then n divides α_2. Similarly k divides $\alpha_1-\alpha_3$. Assume $2m$ divides α_2. Then by (1) we have $\alpha_1\alpha_3=\pm 1$, and hence $\alpha_1-\alpha_3=\pm 2$ and $k=\pm 2$, which contradicts to the fact $(2m, k)=1$. Thus we have $2m\,|\,\alpha_1$ or $2m\,|\,\alpha_3$. We may assume $\alpha_1=\pm 2m$ without loss of generality. Then $\alpha_3=\pm 1$. Thus the value of the formula in 4 in section 1 is constant on i and hence M is diffeomorphic to CP_3. This completes the proof of lemma 3.

We shall prove lemma 4. Let k be the largest integer such that $F(Z_{2^k}, M)\neq F(T, M)(Z_{2^k}\subseteq T)$. Since $F(b, M)\supsetneqq F(T, M)$, we have $k\geqq 1$. Assume $F(D_2, M)=\emptyset$. Let a_1 be a generator of Z_{2^k}. Clearly $N'=N/Z_{2^k}$ acts on $F(a_1, M)$. For $x\in F(a\ M)-F(T, M)$, N'_x is odd cyclic. In fact, assume G_x is cyclic. Then $N'_x=G_x/Z_{2^k}$. If order of N'_x is even, we have $F(Z_{2^{k+1}}, M)\neq F(T, M)$, which contradicts to the choise of k. Next assume G_x is not cyclic. Then $G_x=D_{2i+1}$ (dihedral subgroup), because $F(D_2, M)=\emptyset$. If $k\geqq 2$, then $G_x\supset Z_4$, which contradicts to the fact $G_x=D_{2i+1}$. Hence we have $k=1$. In this case $N'_x=\{1\}$. Consider the restriction of N'-action to a subgroup isomorphic to D_2. Since

$F(a_1, M)$ and $F(T, M)$ have a 2-dimensional component in common, $F(a_1, M)-F(T, M)$ is connected and has homotopy type of S^1. It follows from above arguments that the subgroup of N' acts on $F(a_1, M)-F(T, M)$ freely, which is impossible (see [2], Chap. II section 8). This completes the proof of lemma 4.

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References

[1] BOREL, A. et al.: Seminar on Transformation Groups. Ann. of Math. Studies 46. Princeton Univ. Press 1960.

[2] BREDON, G. E.: Introduction to Compact Transformation Groups. Academic Press 1972.

[3] HIRZEBRUCH, F.: Topological methods in algebraic geometry. Springer 1966.

[4] KAGA, T. and WATABE, T.: *Simply connected 6-manifolds of large degree of symmetry.* Science rep. of Niigata Univ. Ser. A. No. 12 (1975) 15–32.

[5] MONTGOMERY, D. and SAMELSON, H.: *On the action of SO(3) on S^n.* Pacific J. Math. 12 (1962) 649–659.

[6] MONTGOMERY, D. and YANG, C. T.: *A theorem on the action of SO(3).* Pacific J. Math. 12 (1962) 1385–1400.

[7] PETRIE, T.: *Smooth S^1-action on homotopy complex projective spaces and related topics.* Bull. Amer. Math. Soc. 78 (1972) 105–153.

[8] PETRIE, T.: *Torus action on homotopy complex projective spaces.* Inv. Math. 20 (1973) 139–146.

[9] RICHARDSON, R. W. Jr.: *Action of the rotational group on the 5-sphere.* Ann. of Math. 74 (1961) 414–423.

[10] UCHIDA, F.: *Compact transformation groups and fixed point set of restricted action to maximal torus* (to appear).

[11] YOSHIDA, T.: *On smooth semi-free S^1-action on cohomology complex projective* (to appear)

[12] MONTGOMERY, D and YANG, C. T.: *Free differentiable action on homotopy seven spheres II.* Proc. Conference on Transformation Groups (New Orleans, La., 1967) Springer (1968) 125–134.