Simply connected 6-manifolds of large degree of symmetry

By

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Introduction

For a compact connected differentiable manifold M, we define the degree of symmetry, denoted by $N(M)$, the maximum of dimensions of compact connected Lie groups which can act on M almost effectively.

In this note we shall determine simply connected 6-manifolds up to diffeomorphism whose degree of symmetry is greater than 5. Let M be a simply connected closed 6-manifold and G a compact connected Lie group acting almost effectively on M with dim $G=N(M)$. We may assume without loss of generality that G is a product $T^r\times G_1\times\cdots\times G_s$, where T^r is r-dimensional torus and G_i's are simply connected simple Lie groups. In [6], it is shown that if dim $G\geqq 12$, then G is transitive on M. In section 2, we shall determine simply connected 6-dimensional homogeneous spaces. Assume $N(M)\leqq 11$. Then we may consider only $Spin(5)$ $SU(3)$ and $SU(2)$ among the G_i's. It is shown that $r\leqq 5$. In section 3, we shall consider $Spin(5)$-actions, $SU(3)$-actions in section 4, $SU(2)\times SU(2)$-actions and $SU(2)\times SU(2)\times SU(2)$-actions in section 5 and $G\times T^r$-actions in section 6. We shall list the classification of simply conneted 6-manifolds by degree of symmetry in the last section.

Our initial aim was to find an exotic homotopy complex projective 3-space of large degree of symmetry. Our results show the following

Let M be a homotopy complex projective 3-*space. If $N(M)$ is greater than* 5, *then M is diffeomorphic to the standard complex projective* 3-*space.*

We can not determine degrees of symmetry of exotic homotopy complex projective 3-spaces.

1. Preliminaries

In this section we state some lemmas which are used in the sequel. Let (G, M) be a topological action. We denote by G_x the isotropy subgroup of G at $x\in M$, by $G(x)$ the orbit of x, $M^*=M/G$ the orbit space, by $F(G, M)$ the fixed point set and by $M_{(H)}$ the set of points x of M whose isotropy subgroup is conjugate to H.

LEMMA 1 ([1], Chap. XIV 2.4).

Let (G, M) be a topological action. Assume that on M there is a smallest class (U) and a biggest class (V) of isotropy subgroups and that an element of (U) is contained in exactly one element of (V). Then there is a continuous map $f: M \longrightarrow G/N(V, G)$ whose restriction to $M_{(V)}$ is the projection of the fibration $(M_{(V)}, G/N(V, G), F(V, M_{(V)}))$.

REMARK. In paticular, when V contains $N(T)$ (T is a maximal torus of G), we have $V=N(V)$ and hence we have $M_{(V)}=G/N(V)\times F(V, M_{(V)})$. Thus the above f has a cross section and hence $f^*: H^*(G/N(V); A) \longrightarrow H^*(M; A)$ is injective for any abelian group A.

COROLLARY. *Let M be a simply connected manifold on which $SU(2)$ acts almost effectively with a maximal torus T (we may assume T is the standard maximal torus) as a connected principal isotropy subgroup. Then there is no point x of M whose isotropy subgroup is conjugate to $N(T, SU(2))$.*

PROOF. Consider the case in which there is no fixed point. Assume $SU(2)_x$ is conjugate to $N=N(T, SU(2))$. It is easy to see that $N(N, SU(2))=N$. Hence it follows from lemma 1 that there is a map $f: M \longrightarrow RP_2$ such that $f^*: H^1(RP_2; Z_2) \longrightarrow H^1(M; Z_2)$ is injective, which is contradiction. Next consider the case in which there is at least one fixed point. Let F be the fixed point set. From a result in [7], it follows that principal isotropy subgroup is T.

Hence there are just two types of isotropy subgroups (T) and $SU(2)$. It is known that $\dim F \leqq \dim X-3$ and hence $H^1(X-F; Z_2)=0$. By applying the same arguments as above to the restricted action to $X-F$, we can show that there is no point whose isotropy subgroup is conjugate to N. q.e.d.

LEMMA 2. ([3] Corollary)

Let (G, M) be a topological action with orbits of uniform dimension. If $\pi_1(M)=0$ and G_x is of maximal rank for every $x \in M$, then $M=G/H\times M$, where H is a connected principal isotropy subgroup.

LEMMA 3 ([2] Chap. II 6. 1) *If M is a G-space, G a compact Lie group, such that M/G is homeomorphic to $I=[0, 1]$, then there is a global cross section for the orbit map $\pi: M \longrightarrow M/G$.*

LEMMA 4 ([2] Chap. I, 3. 4). *Let G be a compact group acting on spaces X and Y and $\sigma: X/G \longrightarrow X$ be a cross section for $\pi: X \longrightarrow X/G$. Let $\varphi: X/G \longrightarrow Y$ be a map such that $G\sigma(x^*) \leq G\varphi(x^*)$ for all x^* of X/G. Then there is a unique equivariant map $\overline{\varphi}: X \longrightarrow Y$ such that $\overline{\varphi}=\varphi_0 \pi$.*

REMARK. Let (G, X) and (G, Y) be two G-spaces such that their orbit spaces are homemorphic to $[0, 1]$. If there is a homeomorphism $h: X/G \longrightarrow Y/G$ preserving the orbit structures, then (G, X) and (G, Y) are equivariant homemorphic. For instance, if (G, X) and (G, Y) have principal orbits of codimension one, then their isotropy subgroups are (H_X), (K_X), (L_X) and (H_Y), (K_Y), (L_Y) respectively and their orbit spaces are homeomorphic to $[0, 1]$, where the orbit G/H_X, G/K_X and G/L_X coorespond to $(0, 1)$, $\{0\}$ and $\{1\}$

respectively, and analogous to (G, Y) (see the following lemma).

LEMMA 5 ([2] Chap. IV, 8. 2). *Let (G, M) be a locally smooth action with principal orbit G/H of codimension one. Then*

a. *If every orbit is pricipal, then M is a G/H-bundle over $M^* = S^1$ with structure group $N(H)/H$.*

b. *Otherwise, there are two non-principal orbits of type G/K and G/L with $K \geqq H$ and $L \geqq H$. Moreover, K and L may be chosen so that M is the union of the mapping cylinders M_K and M_L of the mappings $G/H \longrightarrow G/K$ and G/K and $G/H \longrightarrow G/L$ respectively.*

REMARK. From a result in [2], it follows that K/H and L/H are spheres or finite sets. If M is simply connected, then K/H and L/H are of positive dimension. In fact, from van Kampen's theorem we have $\pi_1(M) = \pi_1(M_K)*\pi_1(M_L)/\pi_1(G/H)$. Since $\pi_1(M) = 0$, it follows easily that $\pi_1(G/H) \longrightarrow \pi_1(G/L)$ are surjective and hence K/H and L/H are connected and positive dimensional.

LEMMA 6. *Let (T^k, M^n) be an effective action which is assumed to be differentiable. If the Euler characteristic of M is positive, then k is not greater than $n/2$.*

In fact, $F(T^k, M)$ is not empty because of $\chi(M) > 0$. Let φ_x be a local representation at $x \varepsilon F(T, M)$, then φ_x is faithful, so that we have $k \leqq n/2$.

LEMMA 7 ([4] Observ.). *Let (G, M) be a differentiable action. and K an equivariant differentiable transformation group. Suppose the $G \times K$- action on M is almost effective. Let K_0 be the ineffective kernel of the induced K-action on $M/G = X$. Then $G \times K_0$ acts naturally on the principal orbit G/H and the action is almost effective. Moreover K_0 is locally isormorphic to $N(H, G)/H$.*

LEMMA 8([5] Lemma 1). *Let $G = G_1 \times G_2$ act almost effectively on M. If G_1 acts transitively on M, then G_2 acts almost freely on M.*

2. Simply connected 6-dimensinal homogeneous spaces

Let M be a simply connected 6-dimensional manifold. In this section we intend to determine all pair (G, M), where G is a compact connected Lie group acting almost effectively and transitively. Without loss of generality we may assume that G is a product $T^r \times G_1 \times \cdots \times G_s$, where T^r is r-dimensional torus and G_i's are simply connected simple Lie groups. It is well known that dim $G \leqq 21$ and the eqality holds only in the case in which $G = SO(7)$ and $M = S^6$. Moreover it is known that dim $G \leqq 16$ when $M \neq S^6$. In the sequel, we denote by H a principal isotropy subgroup of G-action on M.

Case 1. dim G = 16

Subcase 1. G = T × Spin(6)

Lot $p: G \longrightarrow T$ be the projection. Since $p(H) = H/H \cap Spin(6)$ and dim $p(H) \leqq 1$, we have dim $H \cap Spin(6) = 10$ or 9. Consider the restricted $Spin(6)$-action on M, and put H_1

$=H\cap Spin(6)$. First consider the case in which dim $H_1=9$. From the table in [9] it follows that 9-dimensional subgroups of $Spin(6)$ are only of type $T\times A_2$ and hence we have $M=SO(6)/U(3)$. Since the Euler characteristic of M is positive, G can not act on M almost effectively.

Next, if dim $H_1=10$, it follows from lemma 5 that there are two non-principal isotropy subgroups K and L and $M=M_K\cup M_L$, where M_K and M_L are the mapping cylinders of the maps $Spin(6)/H\longrightarrow Spin(6)/K$ and $Spin(6)H\longrightarrow Spin(6)/L$ respectively. Since there is no other subgroup of dimension 10 of $Spin(6)$ than $Spin(5)$, we have $K=L=Spin(6)$ and hence $M=S^6$.

Subcase 2. G = SU(3)×SU(3)

Let $p_i: G\longrightarrow SU(3)$ be the projection to the i-th factor for $i=1, 2$. Since $p_2(H)=H/SU(3)\cap H$, we have dim $(H\cap SU(3))\geqq 2$. From the consideration of subgroups of $SU(3)$, it follows that dim $(H\cap SU(3))=2$. Thus we have $SU(3)/H\cap SU(3)=M$, and hence it follows from lemma 8 that $SU(3)$ acts almost freely on M which is a contradiction.

Subcase 3. G = SU(4)×T.

Subcase 4. G = Spin(5)×SU(2)×SU(2)

By similar arguments it is shown that these cases are all impossible. Note that it is sufficient to consider only the above 4 cases because T^5 can not act almost effectively on M.

Case 2. dim G=15

There are two well known 6-dimensional homogeneous spaces $SO(6)/U(3)=F_3$ and $SU(4)/U(3)=CP_3$. It is shown by the similar arguments that there is no other pair (G, M) with the required properties.

Case 3. dim G=14

It is shown in [6] that there is no 14-dimensional group acting on 6-dimensional manifold transitively.

Case 4. dim G=13

Subcase 1. G=Spin (5)×SU(2)

Let $p_2: G\longrightarrow SU(2)$ be the projection. Clearly dim $H_1\geqq 4$, where $H_1=H\cap Spin(5)$. First we assume that dim $H_1=4$. Since H_1 is of rank 2, H_1 is locally isomophic to $Spin(3)\times T$, so that $M=\mathrm{Spin}(5)/Spin(3)\times T=Q_3$(= complex quadric). Hence it follows from lemma 8 that $SU(2)$ acts almost freely on M, so that there is a fibration $SU(2)/N\longrightarrow M\longrightarrow M/(SU(2)/N)$, where N denotes the ineffective kernel. Thus it is easily shown that $M/(SU(2)/N)$ is a simply connected 3-dimensional manifold and hence it is homotopically equivalent to S^3. So, moreover, the homotopy exact sequence of the fibration implies that $N=\{e\}$, which contradicts to the fact $\pi_2(M)\neq 0$. Therefore we have dim $H_1=6$ and hence $H_1\sim SO(4)$ and $p_2(H)=T$. In fact if dim $H_1=0$, then dim $p_2(H)=7$. But this is impossible because there is no 7-dimensional subgroup of $Spin(5)$. Consequently $H\sim SO(4)\times T$ and $M=S^4\times S^2$. We can see in a similar way that there is no other (G, M) with dim $G=13$.

Case 5. dim G=12

Subcase 1. G = SU(2)×SU(2)×SU(2)×SU(2)

Denote $G=G_1\times G_2$, where $G_1\times G_2$, where G_1 and G_2 are the product $SU(2)\times SU(2)$ of former, and latter two factors, respectively, and let $p_i: G\longrightarrow G_i$ be the projection for $i=1$, 2. First when dim $H\cap G_1=0$, it is easy to show that $M=S^3\times S^3$. Secondly, if dim $H_1=1(H_1=H\cap G_1)$, we have dim M/G_1, which implies the restricted G_1-action on M has a principal orbit with codimension one, and dim $N(H_1, G_1)/H_1=3$. So, by lemma 7, a 3-dimmesional group acts on M/G_1 almost freely, but this contradicts to the fact dim $M/G_1=1$. Thirdly, when dim $H_1=2$, it is clear that $N(H_1, G_1)/H_1$ is a finite set, and hence G_2 acts on M/G_1 almost effectively. But this is a contradiction. Lastly, when dim $H_1=3$, we have dim $P_2(H)=3$ and hence the above arguments show that it is impossible that dim $H\cap G_2$ is smaller that 3. Hence dim $H\cap G_2=3$, so we have $H=(H\cap G_1)\times(H\cap G_2)$. Consequently we have $M=S^3\times S^3$.

Subcase 2. G = SU(3)×SU(2)×T

Consider the restricted $SU(3)\times SU(3)/T\times T$. Lemma 6 leads a comtradiction. Secondly assume dim $H\cap SU(3)=3$. Since $H\cap SU(3)$ is isomorphic to $SU(2)$, it is not difficult to see that the Euler characteristic of M is non-zero and hence G cannot act on M almost effectively.

Case 6. dim G=11.

Subcase 1. G = Spin(5)×T

Let $H_1=H\cap Spin(5)$. It is clear that dim $H_1=4$, and hence we have $H_1=SO(3)\times T$, which means that $M=Q_3$. Since T acts almost freely on M there is a fibration $T\longrightarrow Q_3$ Q_3/T. It follows that Q_3/T is a simply connected 5-manifold. From the Gysin's exact sequence of the fibration it follows immeadiately a contradiction.

Subcase 2. G = SU(3)×SU(2)

It is clear that dim $H_1(H_1=H\cap SU(3))=2$. When dim $H_1=2$ we have $H_1=T\times T$ and hence $M=SU(3)/T\times T$. So $SU(2)$ acts on M almost freely, but this is impossible. If dim $H_1=3$, then $H_1=SU(2)$ and hence $M/SU(3)$ is one dimensional. On the other hand $SU(2)$

$/N(\dim N \leqq 1)$ acts almost effectively on $M/SU(3)$, which is a conradiction. Lastly, when $\dim H_1 = 4$, we have $H_1 = N(SU(2), SU(3))$ and hence it follows from lemma 2 that $M = CP_2 \times M^*$, where M^* is a simply connected 2-manifold. Consequently we have $M = CP_2 \times S^2$. By the same arguments we can show that all other cases are impossible.

Case 7. dim G=10. $\mathbf{M=Q_3}$.

Case 8. dim G=9. $\mathbf{M=S^2 \times S^2 \times S^2}$

Case 9. dim G=8 $\mathbf{M=SU(3)/T \times T}$

Case 10. dim G=7 There is no group acting transitively on 6-dimensional manifold. We omit the proof of above results because they are not difficult.

3. The 6-dimensional manifolds on which Spin(5) acts almost effectively

From now on, let M be a simply connected 6-dimensional manifold. Suppose $Spin(5)$ act on M almost effectively with H as a principal isotropy subgroup. Since we may assume that $\dim Spin(5)/H = 5$, we have $\dim H \geqq 5$. By checking subgroups of $Spin(5)$, we know that H is locally isomorphic to $Spin(4)$ and hence the $Spin(5)$-action indunes an $SO(5)$-action with $SO(4)$ as its connected principal isotropy subgroup. Suppose $H = N(SO(4), SO(5))$. Since there is no linear action of $SO(5)$ with $N(SO(4), SO(5))$ as a principal isotropy subgroup, we have the fixed point set $F = F(SO(5), M) = \phi$, and hence there is a unique orbit type. Then we have $M = RP_4 \times M^*$, which contradicts to the fact $H_1(M; Z_2) = 0$. Thus we have prove that any principal isotropy subgroup is conjugate to $SO(4)$.

Case 1. $\mathbf{F = F(SO(5), M) = \phi}$

We have a fibration $S^4 \longrightarrow M \longrightarrow M/SO(5) = S^2$, so it follows from the fact $N(SO(4), SO(5))/SO(4) = Z_2$ that $M = S^4 \times S^2$.

Case 2. $\mathbf{F \neq \phi}$

In this case $M^* = M/SO(5)$ is a simply connected 2-manifold with boundary. Since $H^i(G(x) : Q) = 0$ for $0 < i < 4$, the Vietoris- Begle s theorem implies that $H^i(M^*; Q)$ is siomorphic to $H^i(M; Q)$ for $i \leqq 3$, and hence $H^i(M; Q) = 0$ for $0 < i \leqq 3$. Therefore we have $M = S^6$.

4. The 6-dimensional manifolds on which SU(3) acts almost effectively

Suppose $SU(3)$ act on M with H as a principal isotropy subgroup. We may assume $\dim SU(3)/H \leqq 5$.

Case 1. dim H=4

We may assume $H = N(SU(2), SU(3))$. Sinace H is maximal subgroup of $SU(3)$

possible isotropy subgroups are only H and $SU(3)$. If $F(SU(3), M)=\phi$, it follows from lemma 2 that $M=SU(3)/H\times M/SU(3)=CP_2\times S^2$. If $F(SU(3), M)\neq\phi$, since there are presicely two orbits types (H) and $SU(3)$, $SU(3)/H$ must be a sphere, which is impossible.

Case 2. dim H=3

It follows from lemma 5 and the remark following it, that $M^*=M/SU(3)$ is homeomor phic to [0, 1], there are non-principal isotropy subgroups K and L such that M is the union of the mapping cylinders M_K and M_L of mappings $SU(3)/H\longrightarrow SU(3)/K$ and $SU(3)/H\longrightarrow SU(3)/L$ respectively. It also follows that K and L are singular isotropy subgroups.

Subcase 1. dim K=4 and dim L=8. It is easy to show that M is CP_3.

Subcase 2. dim K=dim L=8. It is clear that M is S^6.

Subcase 3. dim K=dim L=4. It is not difficult to see that M is $(CP_3)\#(\pm CP_3)$.

For the degree of symmetry of $(CP_3)\#(\pm CP_3)$, we have the following

Proposition. $N(CP_3)\#(\pm CP_3))=9$.

Proof. We shall prove the proposition only for $M=CP_3\# CP_3$. Since the Euler characteristic of M is 6, lemma 6 implies that there is no compact connected Lie group with rank $\geqq 4$ acting almost effectively on M. Assume $N(M)\geqq 9$, in other words, there is a group G acting almost effectively on M with dim $G\geqq 9$ and rank $G\leqq 3$. Since $SO(5)$ and $SU(2)\times SU(2)\times SU(2)$ cannot act on M almost effectively (see section 5), $SU(3)\times T$ is only one possible one. It is easy to see that $N(SU(3), SU(4))$ acts on M almost effectively. This shows that $N(M)=9$. Q.E.D.

5. The 6-dimensional manifolds on which SU(2)×SU(2) or SU(2)×SU(2)×SU(2) acts almost effectively

Suppose $G=SU(2)\times SU(2)$ acts on M with H as a principal isotropy subgroup. Then it is shown that dim $H\leqq 3$. Assume dim $H\geqq 4$. Put $H^0=H_1\times H_2$, it follows from dim $H\geqq 4$ and rank $H=2$ that at least one of the H_1's must be $G_i=SU(2)$, which contradicts to almost effectivity. Hence dim $H\leqq 3$.

Case A. dim H=3.

It is easy to show that G acts on G/H^0 almost effectively. Clearly G/H^0 is a simply connected 3-dimensional manifold and hence it is a 3-dimensional homotopy sphere. So it is well known that G/N is just $SO(4)$ where N is the ineffective kernel of the G-action on G/H^0.

Thus we need only consider the $SO(4)$-action on M with H as a principal isotropy subgroup, dim $H=3$. The inclusion $H^0=SO(3)\longrightarrow SO(4)$ is a faithful 4-dimensional real respresentation of $SO(3)$. But such a respresentation is only $\rho\oplus\theta$ so, that $N(SO(3), SO(4))/SO(3)=Z_2$. So our action has no exceptional orbit.

If there is no singular orbit, the $SO(4)$-action has only one orbit (H) and hence there is a fibration $SO(4)/H \longrightarrow M \longrightarrow M^*$. Hence we have $M=S^3\times S^3$.

If there are singular orbits, the $SO(4)$-action has $(SO(3))$ and $SO(4)$ as orbit types and the orbit space M^* is a simply connected 3-dimensional manifold with boundary $bM^* = M^G$. From consideration of a local representation at a fixed point it follows that dim $M^G=2$. Moreover, since there is a cross section for the map $M-M^G \longrightarrow M^*-M^G$ (lemma 2), we have a cross section for the orbit map $M \longrightarrow M^G$ ([2], 1. 3. 2)). On the other hand the standard $SO(4)$-action on S^6 has $(SO(3))$ and $SO(4)$ as orbit types and the orbit space $S^6/SO(4)=D^3$. Consequently it follows from lemma 4 that M is homeomorphic to S^6, and hence M is diffeomorphic to S^6.

Case B. dim H=2.

We may put $H^0=T\times T$. Since the principal orbit is of codimension two, it is known ([2], IV 8. 6) that if there are singular orbits, there is no exceptional orbit and M^* is a 2-disk with boundary $bM^*=B^*=B/G$ where B is the union of all singular orbits.

First we consider the case that our action has no singular orbit. Since the action has uniform dimensional orbit, it follow from lemma 2 that M is homeomorphic to $S^2\times S^2\times S^2$, Thus M is diffeomorphic to $S^2\times S^2\times S^2$.

From now on, we assume that the action has singular orbits. Let G_x be a singular isotropy subgroup. Because of rank $G_x=2$, we can show that dim G_x is larger than 3, that is, G_x is either of the form $G_1\times N$ or $N\times G_2$ where N is possitive dimensional. Let K be any singular isotropy subgroup. Then there is a fibration $G/K \longrightarrow B_{(K)}^*$ where $B_{(K)} = \{x \in B \mid (G_x)=(K)\}$ and $B_{(K)}^*=B_{(K)}/G$, so that we have dim $B_{(K)} \leqq 3$ since dim $G/K \leqq 2$ and dim $B^*_{(K)} \leqq 1$. There we have dim $B \leqq 3$ and hence $\pi_1(M-B)=0$. From consideration of the fibration $G/H \longrightarrow M-B \longrightarrow M^*-B^*$, it follows that $\pi_1(G/H)=0$ and hence H is connected i.e. $H=T\times T$. On the other hand, the singular isotropy subgroups are either of the form $G_1\times N$ or $N\times G_2$ where dim $N=1$ and N/T is contained in Z_2. Since the restricted G_i-action $(i=1, 2)$ has no special exceptional orbit ([2], IV. 12), $N/T=\{e\}$, i.e. $N=T$. Thus it has shown that the possible orbit types are $(T\times T)=(H)$, $(G_1\times T)=(K)$, $(T\times G_2)=(L)$ and $G_1\times G_2=G$.

Consequently there are possible five cases as follows;

i) (H) and (K), or (H) and (L), ii) (H), (K) and G, or (H), (L) and G,
iii) (H), (K) and (L), iv) (H) and G
v) (H), (K), (L) and G.

Subcase. i) The action has (H) and (K) as orbit types

Since the restricted G_2-action on M has unique orbit type (T), it follows from lemma 2 that M is homeomorphic to $S^2\times M/G^2$. Moreover the G_1-action on M/G_2 has (T) and G_1 as orbit types, and hence it follows from the following lemma that M/G_2 is homeomerphic to S^4. Thus M is diffeomorphic to $S^2\times S^4$.

LEMM. *Let N be a simply connected, closed 4-dimensional manifold. If $SU(2)$ acts almost effectively on N with a principal isotropy subgroup T and the non-empty fixed point set, then N is homeomorphic to S^4.*

PROOF. There is a cross section for the orbit map $N \longrightarrow N^* = N/SU(2)$ and N^* is 2-disk with boundary $F(SU(2), N)$. On the other hand, let $SU(2)$ act on S^4 regarding $SU(2)$ as the subgroup $SO(3)$ of $SO(5)$. Then the orbit space is 2-disk whose boundary is the fixed point set. Thus it follows from the remark following lemma 4 that N is homeomorphic to S^4.

Subcase ii). The action has (H), (K) and G as orbit types.

Put $F = F(G, M) \neq \phi$. It is known ([2], IV. 3. 8) that dim $F \leqq 1$. If dim $F = 1$, then $F = bM^*$ and honce $B_{(K)} = \phi$ which contradicts to the assumption. Thus F is a finite set. Let V be a slice at $x \epsilon F$ in M and S unit sphere in V. Since the restricted G-action on S has a principal isotropy subgroup H, it follows from lemma 5 that S^5/G is homeomorphic to [0, 1], G/K is coresponding to {0} and {1}, and S^5 is homeomorphic to the union of two copies of the mapping cylinder of $G/H = S^2 \times S^2 \longrightarrow G/K = S^2$. But we can easily deduce a contradiction.

Subcase iii). The action has (H), (K) and G as orbit tyces.

In this case $bM^* = M_{(K)}{}^* \cup M_{(L)}{}^*$. But M^* is connected and hance $M_{(K)}{}^* \cap M_{(L)}{}^*$ is not empty which is impossible.

Subcase iv). The action has (H) and G as orbit types.

Since there are only two orbits around $x \in F$, it is shown in the remark following lemma 5 that G/H must be sphere, this is impossible.

Subcase v). The action has (H), (K), (L) and G as orbit types.

First of all we show that there is a cross section for the orbit map $M \longrightarrow M^* = M/G$. In fact, consider the resricted G_1-action on M, it is easily shown from dim $M^{G_1} \leqq 3$ that $M - M^{G_1}$ and hence $M^*_L - M^{G_1}$ is simply connected where $M_1{}^* = M_1/G_1$. Since $M - M^{G_1} \longrightarrow M^*_1 - M^{G_1}$ is a fiber bundle with fibre $G_1/H_1 = S^2$ and the structure group $\Gamma_{H_1} = N(T, SU(2))/T = Z_2$ where $H_1 = H \cap G_1 = T$, it follows that the associated principal $N(T, SU(2))/T = Z_2$ where $H_1 = H \cap G_1 = T$, the associated principal bundle $\Gamma_{H_1} \longrightarrow M^{H_1} - M^{G_1} \longrightarrow M_1{}^* - M^{G_1}$ is trivial, and hence there is a cross section $C' \subset M^{H_1} - M^{G_1}$. Hence it follows from [2] (1. 3. 2) that there is a cross section for the orbit map $M \longrightarrow M^*$. Next consider the G_2-action on $M_1{}^*$, then similar arguments show that the orbit map $M_1{}^* \longrightarrow M_1{}^* \longrightarrow M_1{}^*/G_2 = M^*$ has also a cross section. Composing these sections, we have the required one.

The same arguments as in subcase ii) show that M^G is a finite set. Put $M^G=\{x_1^*, x_2^*, \cdots, x_r^*\}$. Decompose $M_{(K)}^*$ and $M_{(L)}^*$ into connected components; $M_{(K)}^*=\bigcup_{i=1}^{s} A_i$ and $M_{(L)}^*=\bigcup_{i=1}^{t} B_i$. We first claim that r is even. It is clear that $\overline{A_i}\cap\overline{A_j}(i\neq j)\overline{A_i}\cap\overline{B_j}$ and $\overline{B_i}\cap\overline{B_j}(\neq j)$ are in M^G. To prove our assertion it is sufficient to show that for any two components A_i and A_j (or B_i and B_j) of $M_{(K)}^*$ (or $M_{(L)}^*$) $\overline{A_i}\cap\overline{A_j}$ (or $\overline{B_i}\cap\overline{B_j}$) is empty. Suppose the contrary, i.e. $\overline{A_i}\cap\overline{A_j}$ $(i\neq j)=\{x_k^*\}$. Let V be a slice at x_k in M and S a unit sphere in V. Then G acts on V, and hence on S with a principal isotropy subgroup H and only one singular orbit (K). The arguments similar to subcase ii) show that this is impossible. Put $r=2k$.

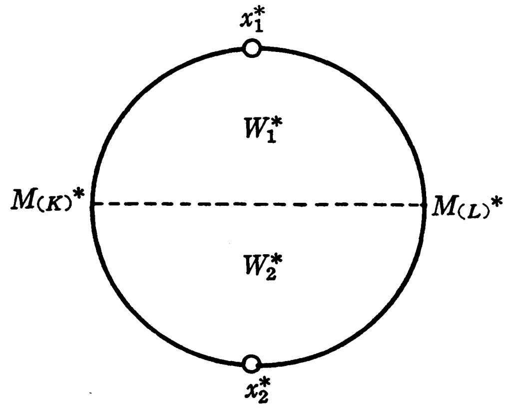

Fig. 1.

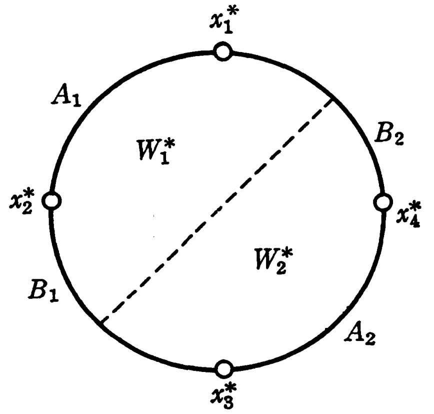

Fig. 2.

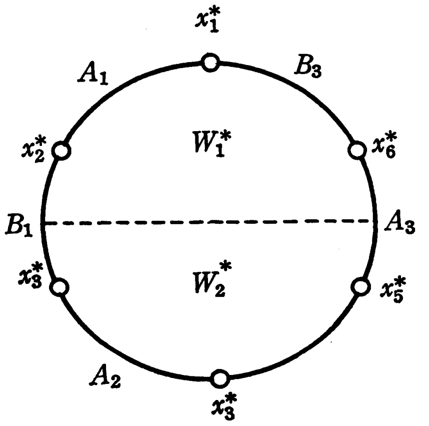

Fig. 3.

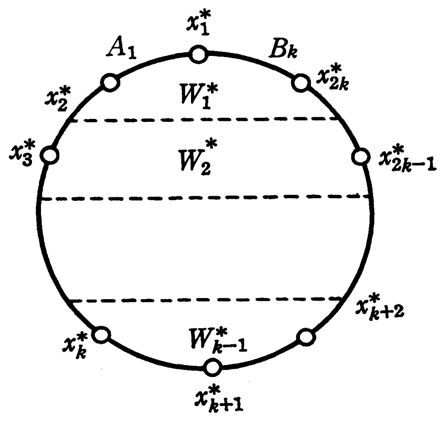

Fig. 4.

(a) The case χ ($\mathbf{M}^G$)=2.

In this case, M^* is illustrated as Fig. 1. Let W_i^* denote the subset as in Fig. 1, and W_i the inverse image of W_i^* by the orbit map. We define a G-action on D^6 by $(g_1, g_2)(x, y)=(g_1x, g_2y)$ for (g_1, g_2) G and $(x, y)\in D^3\times D^3\approx D^6$ where g_i acts on D^3 as an element of $SO(3)$. Then the orbit space D^6/G is homeomorphic to W_i^* by a homeomorphism preserving orbit structures. Then it follows from lemma 4 that W_i is homeomorphic to D^6 and hence M is homeomorphic to S^6. Thus M si diffeomorphic to S^6.

(b) The case $\chi(M^G)=4$.

In this case, M^* is illustrated as Fig. 2. Let W_i^* and W_i be as above. We define a G-action on $S^3\times D^3$ by $(g_1, g_2y)(x, y)=(g_1x, g_2y)$ for $(g_1, g_2)\in G$ and $(x, y)\in S^3\times D^3$ where g_1 acts on S^3 as an element of $SO(4)$ and g_2 acts on D^3as an element of $SO(3)$. Then the orbit space $S^3\times D^3/G$ is homeomorphic to W_i^* by a homeomorphism preserving the orbit structures. The same arguments as above show that M is diffeomorphic to $S^3\times S^3$.

(c) The case $(\chi)M^G=6$.

In this case, M^* is illustrated as Fig. 3. From (a) (b) it is easily shown that W_i is homeomorphic to $S^3\times S^3-\text{Int } D^6$ and hence M is homeomorphic to $S^3\times S^3\# S^3\times S^3$. Thus M is diffeomorphic to $S^3\times S^3\# S^3\times S^3$. We remark that in this case $SU(2)\times SU(2)$ acts on M in a standard way.

(d) The case $\chi(M^G)=2k(k\geqq 4)$.

In this case, M^* is illustrated as Fig. 4. It is easily shown as above that W_1 and W_{k-1} are homeomorphic to $S^3\times S^3-\text{Int } D^6$, and W_1 is homeomorphic to $S^3\times S^3-\text{Int } D^6-\text{Int } D^6$ for $i=2, \cdots k-2$. Hence M is diffeomorfic to the connected sum of $(k-1)$ copies of $S^3\times S^3$.

Case C. dim H=1.

First of all we prove the following;

PROPOSITION. $H_*(G/H; Q)=H_*(S^2\times S^3; Q)$.

PROOF. Let H^0 be the identity component of H and S a maximal torus of G containing $H^0=T^1$. Consider the fibration $S/H^0=T^1\longrightarrow G/H^0\longrightarrow G/S$. It follows from the fact the second Stiefel-Whitney class $w_2(G/S)$ of G/S is zero, that $w_2(G/H^0)=0$. Since G/H^0 is a simply connected 5-manifold with the second Betti number $b_2(G/H^0)=1$, G/H^0 is diffeomorphic to $S^2\times S^3$ (see [8]). Since G/H^0 is a finite covering space of G/H, we have $H_*(G/H; Q)=H_*(S^2\times S^3; Q)$. Q.E.D.

For the case in which dim $H=1$, it follows from lemma 5 and the remark following it that there are two types (K) and (L) of singular isotropy subgroups and M is the union of mapping cylinders M_K and M_L.

We claim that dim K and dim L are smaller than 5. In fact, suppose dim $K\geqq 5$. Then it is easy to see that $K=G$. Choose a fixed point x. Since M is the union of mapping cylinders M_K and M_L, any small neighborhood of x is homeomorphic to a cone over G/H, which contradicts to the fact M is a manifold at x.

Consequently there are possible six cases as follows;

1) dim K=dim $L=2$, 2) dim $K=3$, dim $L=2$, 3) dim K=dim $L=3$,
4) dim $K=4$, dim $L=2$, 5) dim $K=4$, dim $L=3$, 6) dim K=dim $L=4$.

Subcase 1. dim K=dim L=2.

Since $K^0=T\times T\subseteqq K\subseteqq N(T, G_1)\times N(T, G_2)$, we have $K/K^0\subseteqq Z_2\oplus Z_2$. Similarly we have $L/L^0\subseteqq Z_2\oplus Z_2$. Without loss of generality we may assume dim $G_1\cap H=0$. It is not difficult to see that the induced action of G_2 on M/G_1 has uniform dimensional orbits and every isotropy subgroup has maximal rank. Hence it follows from lemma 2 that G_2 acts on M/G_1 with unique orbit G_2/T. This means that K/K^0 is either $Z_2 \oplus 0$ or 0. Similarly L/L^0 is also either $Z_2 \oplus 0$ or 0. Consequently only the following three cases are possible; i) $K=L=N(T, G_1)\times T$, ii) $K=N(T, G_1)\times T$, $L=T\times T$, iii) $K=L=T\times T$.

In subcases i) and ii), the restricted G_1-action on M has only two types of isotropy subgroups, $(G_1\cap H)$ and $(N(T))$. This contradicts to Corollary to lemma 1.

Subcase iii). From the Gysin's sequence of the fibre bundle $S^1 \longrightarrow G/H \longrightarrow G/L$ it follows easily that H is connected. Hence we have $G/H=S^2\times S^3$. By considering the fibre bundle $G_1/H\cap G_1 \longrightarrow G/H \longrightarrow G_2/P_2(H)$, we have $\pi_1(G_1/H\cap G_1)=0$ and Hence $H\cap G_1=\{e\}$. Thus it is easily shown that the above fibre bundle is $(S^3\times S^2, S^2, S^3, pr_1)$. From the following commutative diagram of fiberings;

$$\begin{array}{ccccc}
K_1/H\cap K_1=S^1 & \longrightarrow & K/H=S^1 & \longrightarrow & K_2/P_2(H)\cap K_2=\text{a point} \\
\downarrow & & \downarrow & & \downarrow \\
G_1/H\cap G_1=S^3 & \longrightarrow & G/H=S^3\times S^2 & \longrightarrow & G_2/P_2(H) \quad =S^2 \\
\downarrow \text{Hopf map} & & \downarrow & & \downarrow \\
G_1/K\cap G_1=S^2 & \longrightarrow & G/K=S^2\times S^2 & \longrightarrow & G_2/P_2(K) \quad =S^2
\end{array}$$

it follows that the projection of the fibre bundle $K/H \longrightarrow G/H \longrightarrow G/K$ is $h\times \mathrm{id}$ where $h: S^3 \longrightarrow S^2$ is the Hopf map, so that the mapping cylinder M_K is $(CP_2-\mathrm{Int}\ D^4)\times S^2$. Similarly M_L is also $(CP_2-\mathrm{Int}\ D_4)\times S_2$. Consequently it follows from lemma 5 that M is homeomorphic to $(CP_2 \# CP_2)\times S^2$. Thus M is diffeomorphic to $(CP_2 \# CP_2)\times S^2$.

Subcase 2. dim K=3 and dim L=2.

We may assume dim $p_2(H)=1$. Clearly $p_2(K)=SU(2)=G_2$. It is easy to show that M/G_1 is a simply connected 3-dimensional manifold and the induced action of G_2 on M/G_1 has $(p_2(H))$, $((p_2\ (L))$ and G_2 as orbit types. From $\dim(N/G)^{G_2}=0$ ([2], IV 3. 8) it follows that $M/G_1-(M/G_1)^{G_2}$ is simply connected. Since G_2 acts on $M/G_1-(M/G_1)^{G_2}$ with uniform dimensional orbits, it follow from lemma 2 that $p_2(H)=p_2(L)=T$ and hence L is either $N(T, SU(2))\times T$ or $T\times T$.

The restricted G_1-action on M has $(H\cap G_1)$, $(K\cap G_1)$ and $(L\cap G_1)$ as orbit types. But from the relations $H/H\cap G_1=p_2(H)=T$, $K/K\cap G_1=p_2(H)=T$, $K/K\cap G_1=p_2(K)=S^3$ and $K/H=S^2$([2]), we have $\#\pi_0(K\cap G_1)\leq \#\pi_0(H\cap G_1)$ and hence $H\cap G_1=K\cap G_1$.

So the arguments similar to in subcase 1 show that the case $L-N(T, SU(2))\times T$ is impossible. Hence $L=T\times T$.

Also the same arguments as subcase 1 show that H is connected, $H\cap G_1=K\cap G_1=\{e\}$ and the mapping cylinder M_L is $(CP_2-\mathrm{Int}\ D^4)\times S^2$. As to K, from the following commutative diagram;

$$\begin{array}{ccccc} K/L=S^2 & \simeq K\cap G_1/H\cap G_1\times P_2(K)/P_2(H) & \simeq & \{*\}\times S^2 \\ \downarrow & & & \downarrow \{*\}\times \mathrm{id} \\ G/H=S^3\times S^2 \simeq & G_1/H\cap G_1\times G_2/P_2(H) & \simeq & S^3\times S^2 \\ \downarrow & & & \downarrow \mathrm{id}\times c \\ G/K=S^3 \quad \simeq & G_1/K\cap G_1\times G_2/P_2(K) & \simeq & S^3\times\{*\} \end{array}$$

it follows that the projection of the fibration $K/H \longrightarrow G/H \longrightarrow G/K$ is $\mathrm{id}\times c$ where c is a constant map, and hence the mapping cylinder M_K is $S^3\times D^3$. Consequently $M=((CP_2 - \mathrm{Int}\ D^4)\times S^2)\cup S^3\times S^2 S^3\times D^3$. This manifold is clearly obtained from $CP_2\times S^2$ by surgery based on the homotopy class of the embedding $S^2 \longrightarrow \{*\}\times S_2 \subset CP_2\times S_2$.

Subcase 3. dim K=dim L=3.

Consider the spectral sequence of the fibration $K/K^0 \longrightarrow G/K^0 \longrightarrow G/K$. Then we $H_2(G/K; Q)$ because of $G/K_0=S_3$. Similarly $H_2(G/L; Q)=0$. Therefore the Mayer-Vietoris's exact sequence implies that $H_2(M; Q)=H_4(M; Q)=0$ and $H_3(M; Q)=2Q$, and hence M is homeomorphic to $S^3\times S^3$. Thus M is diffeomorphic to $S^3\times S^3$([8].

Subcase 4. dim K=4 and dim L=2.

We may assume that $K=G_1\times K_2$ where $\dim K_2=1$ and $L^0=T\times T$.

(a) The case $\dim p_2(H)=1$.

Since M/G_1 is a simply connented 3-dimensional manifold and G_2 acts on M/G_1 with uniform dimensional orbits, it follows from a result in [3] that the isotropy subgroups are connected, i.e. $K_2=T$. Also we can show that H is connected because there is a fibration $K/H \longrightarrow G/H \longrightarrow G/K$ where $K/H=S^3$ and $G/K=S^2$. Similarly L is also connected, i.e. $L=T\times T$.

Next we show $\dim H\cap G_2=0$. Then the restricted G_2-action on M/G has two orbit types (Z_m), (T), and the G_1-action on M/G_2 has (T) and G as orbit types. From $(M/G_2)/G_1=MG=[0, 1]$ and lemma 3 it follows that there is a cross section for $M/G_2 \longrightarrow (M/G_2)/G_1$. Since $SU(2)$ acts naturally on D^3 with the principal orbit type (T) and fixed points, the orbit space $D^3/SU(2)$ is $[0, 1]$, the remark following lemma 4 shows that M/G_2 is equiviarantly homeomorphic to D^3 where $bD^3=M_{(T)}*=M_{(T)}/G_2$. Consequently we have $M_{(T)}\approx S^2\times S^2$ since there is a fibre bundle $SU(2)/T=S^2 \longrightarrow M_{(T)} \longrightarrow M_{(T)}*=S^2$ with the structure group $N(T, SU(2))/T=Z_2$. On the other hand the Mayer-Vietoris's sequence shows that $H^2(M; Q)=H^4(M; Q)=2Q$ and $H^3(M; Q)=0$. Moreover from the fibration $G_2/Z_m \longrightarrow M-M_{(T)} \longrightarrow D^3-bD^3$ it follows that $H_*(M-M_{(T)}; Q)=H_*(G_2/Z_m; Q)$. Thus from consideration of the cohomology exact sequence of pair $(M, M_{(T)})$ we can easily deduce a contradiction.

Therefore we have $\dim H\cap G_2=1$ and hence $H=\{e\}\times(H\cap G_2)$, so that $G/H=G_1\times G_2/H\cap G_2=S^3\times S^2$. Since it is easily shown that $M_K=D^4\times S^2$ and $M_L=(CP_2-\mathrm{Int}\ D^4)\times S^2$, and hance M is diffeomorphic to $CP_2\times S^2$.

(b) The case $\dim p_2(H)=0$.

In this case $H\cap G_1$ is one dimensional and $H^0=(H\cap G_1)^0\times\{e\}$. Since $K=G_1\times T$, K/H is not S^3 which contradicts to the remark following lemma 6.

Suboase 5. dim K=4 and dim L=3.

We may assume $K=G_1\times K_2$ where $\dim K_2=1$. It is easily shown that G/K is either RP_2 or S^2 and G/L is a rational homology 3-dimensional sphere. If $G/K=RP_2$, the Mayer-Vietoris's sequence implies that M is a 6-dimensional 2-connected manifold and $\chi(M)=1$. which contradicts to the fact that $\chi(M)$ must be even because of $M=bW^7$ for some W.

Thus $G/K=S^2$. From the fibration $K/H=S^3\longrightarrow G/H\longrightarrow G/K$ it follows immediatel that H is connected and G/H is homeomorphic to $S^2\times S^3$.

(a) The case $\dim p_2(H)=1$.

In this case we have $p_2(H)=T$ and hence $H\cap G_1=\{e\}$. More it is easy to show that $p_2(L)-G_2$ and $L\cap G_1=\{e\}$. Thus the induced action of G_2 on M/G_1 has (T) and G_2 as orbit types. By similar arguments to in subcase 4 we can regard M/G_1 as D^3, the G_2-action on $M/G_1=D_3$ as the sandard one and $M^{G_1}=S^2$. Moreover there is a cross section for $M\longrightarrow M/G_1$ because there is one for $M\longrightarrow M/G$ by lemma 3. Consequently M is diffeomorphic to S^6 on which $SU(2)$ acts in the standard way.

(b) The case $\dim p_2(H)=0$.

This is impossibe as (b) in subcase 4.

Subcase 6. dim K=dim L=4.

(a) The case $K=G_1\times K_2$ and $L=G_1\times L_2$ where $\dim K_2$ and $\dim L_2=1$.

As above the case $\dim p_2(H)=0$ is shown to be impossible. So we have $\dim p_2(H)=1$. Moreover we may assume as subcase 5 that $K_2=L_2=T$, H is connected and $H\cap G_1=\{e\}$. Thus G_1 acts on M with orbit types $(\{e\})$ and G_1, and G_2 acts on M/G_1 with unique orbit type (T). Hence it follows from lemma 5 that $M/G_1\longrightarrow(M/G_1)/G_2=[0,1]$ is a S^2-bundle with the structure group Z_2, and consequently $M/G_1=S^2\times[0,1]$ and $M^{G_1}=S^2\times S^2$. Note that there is a cross section for the orbit map $M\longrightarrow M/G_1$.

We define a $SU(2)$-action on $S^2\times S^4$ by $g(x,y)=(x,gy)$ for $g\in SU(2)$ and $(x,y)\in S^2\times S^4$ where gy is induced from the $SU(2)$-action on R^5 defined by $\rho\oplus\theta$. Then this action has the same orbit space and the same set of fixed points as our G_1-action on M. Consequently M is diffeomorphic to $S^2\times S^4$.

(b) The case $K=G_1\times K_2$ and $L=L_1\times G_2$ where $\dim K_2=\dim L_1=1$.

First assume $\dim p_2(H)=1$. As above $K_2=T$. It is shown that M/G_1 is simply connected and the induced G_2-action on M/G_1 has 0-dimensional set of fixed points. Thus, since G_2 acts on the simply connected manifold $M/G_1-(M/G_2)^{G_2}$ with uniform dimensional orbitist follows from lemma 2 that $p_2(H)$ is connected and hence H is connected, i.e. $H=\{e\}\times T$. Hence L/H is not S^3, which is a contradiction.

The case dim $p_2(2)=0$ is also impossible, because in this case dim $p_1(H)=1$.

(c) The case $K=K_1\times G_2$ and $L=L_1\times G_2$ where dim $K_1=$dim $L_1=i$.

The same arguments as in (a) show that M is diffeomorphic to $S^4\times S^2$.

The remainder of this section will be devoted to studying $SU(2)\times SU(2)\times SU(2)$-actions.

Suppose $G=SU(2)\times SU(2)$ acts on M with H as a principal isotropy subgroup. We may assume dim $H\geqq 4$. Consider the restricted $G_1\times G_2$-action on M (denote by G_i the i-th facter of G). Since we have already determined completely all M s when dim $H\cap(G_1\times G_2)\geqq 2$, we may assume dim $H\cap(G_1\times G_2)=1$. So we need only consider the case dim $H=4$, because of $p_3(H)=H/H\cap(G_1\times G_2)$.

Since the G-action has a principal orbit of codimension one, lemma 5 shows that it has two non-principal orbi types (K) and (L), and hence the restricted $G_1\times G_2$-action on M has $(H\cap(G_1\times G_2))$, $(K\cap(G_1\times G_2)$ and $(L\cap(G_1\times G_2))$ as orbit types. Moreover we need only consider the case dim $K\cap(G_1\times G_2)$ is either 2 or 3 and so is dim $L\cap(G_1\times G_2)$. Thus we consider only the three cases as follows;

i) dim $K=$dim $L=5$. ii) dim $K=6$ and dim $L=5$. iii) dim $K=$dim $L=6$.

But among the isotropy subgroups of our action there is no 5-dimensional subgroup, and hence the cases *i*) and ii) are impossible. Assume dim $K=5$. Because of $K/H=S^1$, we have rank $K>$rank $H=2$. Put $K^0=K_1\times K_2\times K_3$ it follows from $K^0\cap(G_1\times G_2)=T\times T$ that $K_3=SU(2)$. From the fibration $K_3/H^0\cap K^3\longrightarrow K^0/H^0=S^1\longrightarrow p(K^0)/p(H^0)=S^1$, where p; $G\longrightarrow G_1\times G_2$ is the projection, we have dim $K_3/H^0\cap K_3=0$ and hence $H^0\cap K_3=SU(2)$. Thus $H\cap G_3=SU(2)$ which contradicts to almost effectivity.

Consider the case iii). Then dim $K\cap(G_1\times G_2)=$dim $L\cap(G_1\times G_2)=3$. So M has already determined to be $S^3\times S^3$.

6. The 6-dimensional manifolds on which $\mathbf{G'\times T^r}$ acts almost effectively

Put $G=G'\times T^r$. Then we may assume $6\leqq$dim $G\leqq 11$.

The case G is either of the form $SU(2)\times SU(2)\times SU(2)\times T^r(r\geqq 1)$ or $SU(2)\times SU(2)\times T^r(r\geqq 2)$, is impossible. In fact, consider the restricted $SU(2)\times SU(2)$-action on M, $\chi(M)$ is easily shown to be positive. Because of rank $G\geqq 4$, lemma 6 shows that the restricted action of a maximal torus of G on M is not almost effective, this is a contradiction. Similary the case $G=SU(3)\times T^r$ $(r\geqq 2)$ is impossible.

Case 1. $\mathbf{G=SU(2)\times SU(2)\times T}$.

We need only conside the subcases i) and ii) of the case C in the restricted $SU(2)\times SU(2)$-action on M. From the relations dim $H\cap(G_1\times G_2)=1$, dim $H\geqq 2$ and $H/H\cap(G_1\times G_2)p_3(H)$(denote by H a principal isotropy subgroup of G-action on M has two non-principal orbit types (K) and (L) wıth dim $K\geqq 3$ and dim $L\geqq 3$. Since dim $K\cap(G_1\times G_2)$ is 2 or 3 in our situation we have dim $K<5$. Similarly dim $L>5$.

If dim $K=3$, from consideration of the fibration $K/H^0\cap K \longrightarrow K^0/H^0 \longrightarrow p(K^0)/p(H^0)$ where $p: G_1 \longrightarrow G_1\times G_2$ is the projection, and $K^0/H^0=S^1$, we have $K_3=H^0\cap K_3$ which contradicts to almost effectivity. Similarly dim $L\neq 3$.

In the case dim $K=$dim $L=4$, we have $K/H=S^2$ and rank $L=2$, so that dim $K\cap(G_1\times G_2)=$dim $L\cap(G_1\times G_2)=3$. Hence M is diffeomorphic to $S^3\times S^3$.

Case 2. $\mathbf{G=SU(2)\times T^3}$.

Suppose G acts on M with H as a principal isotropy subgroup. Then dim $H=1$. In fact, since the restricted T^3-action is almost effective, we have dim $H\cap T^3=0$ so that we have dim $H=$ dim $p_1(H)=1$ or 3 from $H/H\cap T^3=p_1(H)$. If dim $H=3$, $p_1(H)=SU(2)$. This is impossible.

Subcase 1. The case dim $\mathbf{H\cap SU(2)=0}$.

In this case it is known ([2] IV. 4. 7) that $M^*=M/SU(2)$ is a simply connected 3-dimensional manifold with or without boundary. Moreover the ineffective kernel N of the induced action of T^3 on M^* is of dimension $\leqq 1$, since N is a subgroup of $N(H\cap SU(2); SU(2))/H\cap SU(2)$. If dim $N=0$, that is, T^3 acts on M^* almost effectively, then T^3 is a principal orbit and hence M^* must be T^3 which is impossible.

Thus $T^2=T^3/N$ acts effectively on M^* and it follows lemma 5 that $M^*/T^2=[0, 1]$.

If $bM^*\neq\phi$, then we have $bM^*=S^1\times S^1$ from $M^*/T^2=[0, 1]$, and hence M is homeomorphic to $S^1\times D^2$ which contradicts to simply connctedness of M^*.

If $bM^*=\phi$, M^* must be homeomorphic to S^3 and the restricted $SU(2)$-action on M has neither singular nor exceptional orbit. So from applying the Vietoris-Begle's theorem to $M\longrightarrow M^*$, it follows that $H^2(M; Q)=0$. Therefore, since $\chi(M)=0$, it follows from a result in [8] that M is diffeomorphic to $S^3\times S^3$.

Subcase 2. The case dim $\mathbf{H\cap SU(2)=1}$.

In this case $H\cap SU(2)$ is either T or $N(T; SU(2))$. But it follows from corollary of lemma 1 that the latter case is impossible.

So $H\cap SU(2)=T$. Hence it follows from lemma 2 that M is homeomorphic to $S^2\times M^*$ where M^* is a simply connected 4-dimensional manifold with or without boundary. Then the induced action of T^3 on M^* is also almost effective, and the orbit space M^*/T^3 is [0, 1]. From this, we can show that, if $bM^*=\phi$, M^* is homeomorphic to either $S^1\times S^1\times S^2$ or $S^1\times S^3$, and if $bM^*\neq\phi$, then bM^* is homeomorphic to $S^1\times S^1\times S^1$ and hence M^* is homeomorphic to $S^1\times S^1\times D^2$. But they are not simply connected. So this case is also impossible.

Remark. From this, it follows that all the cases $G=SU(2)\times T^r (r\geqq 4)$ are impossible.

7. Classification

Summing up the results in the preceding sections, we have the following table.

In this table $N(M)$ denote the degree of symmetry of M, G compact connected. Lie group which can act almost effectively on M and φ the action of G on M.

$N(M)$.	M	G	φ
21	S^6	$SO(7)$	transitive
15	F_3	$SO(6)$	〃
	CP_3	$SU(4)$	〃
13	$S^4\times S^2$	$SO(5)\times SU(2)$	〃
12	$S^3\times S^3$	$SO(4)\times SO(4)$	〃
11	$CP_2\times S^2$	$SU(3)\times SO(3)$	〃
10	Q_3	$SO(5)$	〃
9	$S^2\times S^2\times S^2$	$SO(3)^3$	〃
	$CP_3\# CP_3$ $CP_3\#(-CP_3)$	$SU(3)\times T$	union of $N(SU(3); SU(4))$-action on $CP_3-\text{Int}\, D^6$
8	$SU(3)/T\times T$	$SU(3)$	transitive
6	$k(S^3\times S^3)(k\geqq 2)$	$SU(2)\times SU(2)$	union of $SU(2)\times SU(2)$-action on $S^3\times S^3-\text{Int}\, D^6$
	$(CP_2\# CP_2)\times S^2$ $S^3\times D^3\, S(CP_2-\text{Int}\, D^4)\times S^2$	〃 〃	union of $SU(2)\times SU(2)$-action on $(CP_2-\text{Int}\, D^4)\times S^2$

Note that no manifold other than the above has the degree of symmetry $\leqq 5$.

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