On the degree of symmetry of complex quadric and homotopy complex projective space

By

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Introduction

Let M be a compact connected differentiable manifold of dimension $2n$. Following [8], we define $N(M)$, *the degree of symmetry of* M, the maximum of dimension of isometry groups of all possible Riemannian structures on M. Of course, $N(M)$ is the maximum of dimensions of compact connected Lie groups which can act almost effectively on M.

In this note, we shall consider the degree of complex quadric $Q_n=SO(n+2)/SO(2)\times SO(n)$ and homotopy complex projective space CP_n.

In [8], W. Y. Hsiang has proved the following

THEOREM. $N(CP_n)=dim\ SU(n+1)=n^2+2n$.

We have the following

THEOREM A. *Let M be a closed differentiable manifold of dimension $2n$ which is homotopy equivalent to CP_n. Assume that $n\geqq 13$. If $N(M)\geqq(1/2)(n^2+3n+2)$, then M is diffeomorphic to CP_n.*

As a corollary of this we have the following

THEOREM B. *The degree of symmetry of an exiotic homotopy complex projective space of dimension of $2n$ is less than* $(1/2)(n^2+3n+2)$. $(n\geqq 13)$

For a complex quadric Q_n, we have the following

THEOREM C. $N(Q_n)=dim\ SO(n+2)=(1/2)(n^2+3n+2)(n\geqq 13)$.

In section 1, we state the results and prove Theorem A and C modulo lemmas and propositions which are proved in later sections.

In this note all actions are differentiable.

1. Statement of results

A closed differentiable manifold M^{2n} is said to be *homologically kählerian* if there exists an element $a\in H^2(M;\ Q)$ ($Q=$rationals) such that the multiplication by a^{n-s} ($s=0,$

1,……, n) is an isomorphism of $H^s(M; Q)$ onto $H^{2n-s}(M; Q)$.

In the following M^{2n} denotes a homologically kählerian manifold with $N(M) \geqq (1/2)(n^2+3n+2)(n \geqq 13)$.

For example CP_n and Q_n are homologically kählerian. Let M be a simply connected homologically kählerian manifold with the second Betti number $b_2(M)=1$. Let G be a compact connected Lie group acting almost effectively on M with dim $G=N(M)$. We may assume that G is a product $T^r \times G_1 \times \cdots \times G_s$ of a torus and simple compact connected Lie group G_i's.

First consider the case where G acts transitively on M. Since $\pi_1(M)=0$ the restricted action of a maximal torus of G has at least one fixed point (see [3] chap. XII.). Then the unique isotropy subgroup H is of maximal rank and connected. Hence we have $M=G/H=G_1/H_1 \times \cdots G_s/H_s$, where H_i is a subgroup of G_i of maximal rank. The following lemma implies that $M=G/H$, where G is a simple compact connected Lie group and H is a subgroup of maximal rank.

LEMMA 1. 1 *Let $X=X_1 \times X_2$ be a simply connected homologically kählerian manifold with $b_2(X)=b_2(X_1)=1$. Then X_2 is a point.*

We have the following

PROPOSITION 1. *Let G/H be a simply connected homogeneous space of a simple compact connected Lie group G with $b_2(G/H)=1$. Assume G/H is homologically kählerian and dim $G \geqq (1/2)(n^2+3n+2)$ ($2n=$dim G/H). Then possible pair (G, H) is $(SO(n+2), SO(2) \times SO(n))$, $(SU(n+1), N(SU(n), SU(n+1))$ or $Sp((n+1)/2), T \times Sp((n-1)/2))$.*

Next consider the case where G acts non-transitively on M. Then we have dim $G/H \leqq 2n-1$, where H denotes a principal isotropy subgroup and hence we have dim $G \geqq 1(/2)(n^2+3n+2) > (1/8)(2n+7)$ dim G/H. By a result in [8], there exists a simple normal subgroup, say G_1 of G satifying

$$\text{dim } G_1 + \text{dim } N(H_1, G_1)/H_1 > (1/8)(2n+7) \text{ dim } G_1/H_1 \tag{1. 2}$$

and

$$\text{dim } H_1 > ((2n-9)/(2n-1)) \text{ dim } G_1, \tag{1. 3}$$

where $H_1=(H \cap G_1)^0$ and $N(H_1, G_1)$ is the normalizer of H_1 in G_1.

We have the following

PROPOSITION 2. *Possible pairs (G_1, H_1) satifying* (1. 2), (1. 3) *and dim $G_1/H_1 \leqq 2n-1$ are followings*;

(*i*) $(Sp(m)\ Sp(m-1))\ (n<2m-1)$

(*ii*) $(Sp(m), Sp(m-1) \times T)(n<2m-1)$

(*iii*) $(Sp(m), Sp(m-1) \times Sp(1))(n<2m)$

(*Iv*) $(SO(m), SO(m-1))(n<2m)$

(*v*) $(SU(m), N(SU(m-1), SU(m)))\ (n \leqq 2m-2)$

(*vi*) $(SU(m), SU(m-1))(n<2m-2)$.

We consider the following six cases.

Case 1. $(Sp(m), Sp(m-1))$.

By assumption, we have that dim $G_1/H_1 \leqq 2n-1$ and hence $2m<n$, which contradicts to the fact $n<2m-1$.

Case 2. $(Sp(m), Sp(m-1)\times T)$.

By the same arguments as in case 1, it is verified that this case is impossible.

Case 3. $(Sp(m), Sp(m-1)\times Sp(1))$.

Since the resticted action of G_1 on M has principal isotropy subgroup which is locally isomorphic to $Sp(m-1)\times Sp(1)$ and this is a maximal subgroup of G_1, all orbits $G_1(x)$ have cohomology groups $H^i(G_1(x); Q)=0$ for $0<i<4$. Hence it follows from the Vietoris-Begle theorem that $\pi^*: H^i(M/G_1; Q) \longrightarrow H^i(M; Q)$ is isomorphic for $i\leqq 3$, where π: $M \longrightarrow M/G_1$ is the orbit map. Thus the generator a of $H^2(M; Q)$ is in the image of π^*, i. e. $a=\pi^* b$, $b\in H^2(M/G_1: Q)$. Since dim $M/G_1=$dim $M-$dim $G_1/H_1<2n$, we have $b^n=0$ and hence we have $a^n=0$, which is a contradiction.

Case 4. $(SO(m), SO(m-1))$.

The same arguments as in case 3 show that this case is impossible.

Case 5. $(SU(m), N(SU(m-1), SU(m)))$.

In this case there is no fixed point for the restricted action of G_1. In fact assume that there is a fixed point. Then a result in [3] ([3], chap. XIV) and the fact that the normalizer of $N(SU(m-1), SU(m))$ in $SU(m)$ is $N(SU(m-1), SU(m))$ show that G_1/H_1 is a sphere, which is clearly impossible. Thus G_1 acts on M with only one type of orbit CP_{m-1}, and hence $M=CP_{m-1}\times M/G_1$. By lemma (1. 1), M/G_1 is a point, which contradicts to our assumption.

Case 6. $(SU(m), SU(m-1))$.

Subcase 1. *There is no orbit of type* CP_{m-1}.

In this case possible orbits are rational homology spheres or points. The same argument as in case 3 shows that this case is impossible.

Subcase 2. *There is at least one orbit of type* CP_{m-1} *and no fixed point.*

Put $N=N(SU(m-1), SU(m))$. Since there is no fixed point, there is a biggest conjugate class (N) of isotropy subgroups. Here we mean "biggest" in the following sense: the conjugate class (U) is smaller than (V) if every element of (U) is contained in some element of (V). It is not difficult to see that if $gSU(m-1)g^{-1}\leqq N$, then $g \in N$. Let H_1 be a prinicipal isotropy subgroup. This implies that every element of (H_1) is contained in exactly one element of (N). Since (H_1) is the smallest class of conjugate class of isotropy subgroups, the subspace $F=F(H_1, M)$ meets every orbit. In fact, for any point $x \in M$, $(G_{1,x})\geqq(H_1)$, i. e. $H_1\subseteqq gG_{1,x}g^{-1}$ for some $g \in G_1$. This implies that $gx \in F\cap G_1(x)$. [Let $x \in M$, $x_0\in F\cap G_1(x)$ and g be such that $gx_0=x$. We show that such a g is uniquely determined modolo N. In fact it is sufficient to show that for any $x \in M$, if g_1x,

$g_2x \varepsilon F$, then $g_1^{-1}g_2 \varepsilon N$: in other words, if $y, z \varepsilon F$, $y=gz$, then $g \varepsilon N$. But this is clear because every element of (H_1) is contained in exactly one element of (N). Now we define a map $f: M \longrightarrow G_1/N = CP_{m-1}$ by $f(x)=go$, where o is the coset of e in G_1/N. It is not diffiicult to see that f is continuous and equivariant. Let $M_{(N)}=\{x \in M; G_{1,x} \in (N)\}$. The normalizer on N in $SU(m)$ being N itself, we have $M_{(N)}=G_1/N \times F(N, M_{(N)})$. Since the restriction of f to $M_{(N)}$ is clearly the projection $M_{(N)} \longrightarrow G_1/N$, the homomorphism $f^*: H^*(CP_{m-1}: Z) \longrightarrow H^*(M; Z)$ is injective. Let b be a generator of $H^2(CP_{m-1}; Z)$ such that $f^*b=a$. Since dim $CP_{m-1}<$dim M, we have $b^n=0$ and hence we have $a^n=0$, which is a contradiction.

The above arguments are valid for any homologically kählerian manifolds.

Subcase 3. *There is at least one orbit of type CP_{m-1} and fixed point.*

First let M be a homotopy complex quadric. Let F be the fixed point set of G_1-action. Since the action of G_1 on $M-F$ has $SU(m-1)$ as a principal isotropy subgroup and no fixed point, the argument as in subcase 2 shows that there exists a map $f: M-F \longrightarrow CP_{m-1}$ such that $f^*: H^i(CP_{m-1}; Z) \longrightarrow H^i(M-F; Z)$ is injective. Since dim $F \leqq 2n-2m$, we have $H^i(M. M-F; Z)=0$ for $i<2m$. Let T be a maximal torus of G_1. From a result in [3] (chap. XII), the fixed point set $F(T, M)$ has torsion free cohomology and vanihsing odd Betti numbers. Since any component of F is a component of $F(T, M)$, $H^i(M, M-F: Z)=0$ for odd i. Thus we have isomorphism $H^{2i}(M; Z) \approx H^{2i}(M-F; Z)$ for $i=0, \ldots, m-1$. Let $j: M-F \longrightarrow M$, $i: CP_{m-1} \longrightarrow M-F$ be inclusions. Then $i=j \cdot i_1$. Let a be a generator of $H^2(M; Z)$. We may assume $i^*a=b$ is a generator of $H^2(CP_{m-1}; Z)$. It is not difficult to see that $i^*: H^k(M; Z) \longrightarrow H^k(CP_{m-1}; Z)$ is surjective for $k<2m$. Recall that the cohomology ring of M is given as follows; for $H^*(M; Z)$, there can be chosen an additive basis $\{e_0, e_1, \ldots, e_n\}$ in the case $n=2k+1$ and $\{e_0, \ldots e_n, e_k'\}$ in the case $n=2k$ so that

i) for $n=2k+1$, $e_i \varepsilon H^{2i}(M; Z)$ and $H^*(M; Z)=Ze_0+\cdots\cdots+Ze_n$
for $n=2k$, $e_i, e_i' \varepsilon H^{2i}(M; Z)$ and $H^*(M; Z)=Ze_0+\cdots\cdots+Ze_k+Ze_k'+\cdots\cdots+Ze_n$.

ii) $e_i \cup e_{n-i}=e_n$ if $n=2k+1$ and $n=2k$, $i \neq k$.

iii) if $n=2k$, $e_k \cup e_k = e_k' \cup e_k' = \begin{cases} e_n & k; \text{ even} \\ 0 & k; \text{ odd} \end{cases}$

$$e_k \cup e_k' = \begin{cases} 0 & k; \text{ even} \\ e_n & k; \text{ odd} \end{cases}$$

iv) if $n=2k+1$, $e_1^r = \begin{cases} e_r & r \leqq k \\ 2e_r & r>k \end{cases}$

if $n=2k$ $\quad e_1^r = \begin{cases} e_r & r<k \\ e_k+e_k' & r=k \\ 2e_r & r>k. \end{cases}$

First consider the case when $n=2k+1$. Since $n<2m-2$, we have $k \leqq m-2$. Put $i^*(e_{k+1}) = Ab^{k+1} (A \in Z)$. Since $i^*(a)=b$, we have $i^*(a^{k+1})=b^{k+1}$ and hence $b^{k+1}=2i^*(e_{k+1})=2Ab^{k+1}$., which is clearly impossible. By similar arguments, we can deduce a contradiction when

$n=2k$.

Thus we have the following

PROPOSITION 3. *Let M be a closed differentiable manifold of dimension* $2n$ *which is homotopy equivalent to* Q_n. *If* $N(M)\geqq 1/2(n^2+3n+2)$, $(n\geqslant 13)$, *then any compact connected Lie group G which acts almost effectively on M with dim* $G=N(M)$ *acts transitively on M.*

From proposition 1, it follows the following

THEOREM 1. $N(Q_n)=1/2(n^2+3n+2)=dim\ SO(n+2)$.

Next let M be a homotopy complex projective space. We have the following

PROPOSITION 4. *Let M be a closed differentiable manifold which is homotopy equivalent to complex projective space* CP_n. *Assume* $SU(m)(2m-2>n)$ *act on M with* $F(SU(m), M)\neq\phi$, $M_{(N)}\neq\phi$ *and* $SU(m-1)$ *as identity component of principal isotropy subgroup, where N denotes the normalizer of* $SU(m-1)$ *in* $SU(m)$. *Then M is diffeomorphic to* CP_n.

Thus we have the following

THEOREM 2. *Let M be a closed differentiable manifold of dim.* $2n$ *which is homotopy equivalent to* CP_n. *If* $N(M)\geqq 1/2(n^2+3n+2)$, $(n\geqslant 13)$, *then M is diffeomorphic to* CP_n.

2. Some properties of homologically kählerian manifolds

In this section, X denotes a simply connected homologically kählerian manifold of dimension $2n$ and a an element of $H^2(X; Q)$ such that the multiplication by $a^{n-s}(s=0, 1,\ldots, n)$ is an isomorphism of $H^s(X; Q)$.

PROOF of lemma 1. 1. Let X_1 be $2m$- manifold. Since $b_2(X_2)=0$, a is written as $a_1\otimes 1$, $a_1\in H^2(X_1; Q)$. If dim $X_2>0$, $a^n=0$ and hence $a^n=0$, which is a contradiction. *Q.E.D.*

Let G/H be a simply connected homogeneous space of simple compact connetced Lie group G. Assume that G/H is homologically kählerian with $b_2(G/H)=1$ and dim $G\geqq(1/2)(n^2+3n+2)(2n=\dim G/H)$. Let T be a maximal torus of G. From a result in [3] ([3], chap. XII), it follows that the restricted action of T has at least one fixed point. Thus H has the same rank as G. Since dim $G\geqq(1/2)(n^2+3n+2)$, we have dim $G>(1/4)(n+3)$ dim G/H. From this, it follows that $(\dim G)^2-2(\dim G)(\dim H)+(\dim H)^2-2\dim G-6\dim H<0$ and hence we have

$$\text{(2. 1)} \qquad \dim G+3-\sqrt{8\dim G+9}<\dim H<\dim G+3+\sqrt{8\dim G+9}.$$

Let U be the maximal subgroup of G which contains H. We consider the following two cases.

Case 1. $H=U$.

In this case, the pair (G, H) may be divided into following three cases ([1]).

(1) H is the connected centralizer of an element of order 2, which generates its center.

(2) H is the centralizer of a one dimensional torus S, and S is the identity component of the center of H.

(3) H is the connected centralizer of an element of order 3 or 5 which generates its center.

Since $H^2(G/H; Q)\neq 0$, H is not semi-simple. Hence H is of the case (2) and the coset space G/H is an irreducible hermitian symmetric space. It is not difficult to see that the irreducible hermitian space satisfying (2. 1) is $SU(n+1)/S(U(1)\times U(n))=CP_n$ and $SO(n+2)/SO(2)\times SO(n)=Q_n$.

Case 2. $H\subsetneqq U$.

Consider the fibration $U/H\longrightarrow G/H\longrightarrow G/U$. Since odd Betti number of U/H and G/U are all zero, it follows that $b_2(G/U)=1$ and $b_2(U/H)=0$ or $b_2(G/U)=0$ and $b_2(U/H)=1$. Consider the first case. Let b be a generator of $H^2(G/U: Q)$. Then we may assume that $\pi^*b=a$. Since dim $G/U<$dim G/H, we have $a^n=0$, which is a contradiction. Next consider the second case. It is well known than U is semi-simple (see [1]).

Subcase 1. $G=A_m$.

All maximal subgroups of G with maximal rank are not semi-simple. ([2])

Subcase 2. $G=B_m$.

From (2. 1), it follows that dim $H>2m^2-3m+1$. From the table in [2] ([2]. p. 219), only possibility for U is D_m. Since $b_2(U/H)=1$, H is of the form $T\times H_1$, where dim $H_1<$dim D_{m-1}, which does not satisfy (2. 1).

Subcase 3. $G=C_m$.

Among maximal subgroups $C_i\times C_{m-i}$ with maximal rank, $C_1\times C_{m-1}$ is the only subgroup which satifies (2. 1). Hence H is locally isomorphic to $H_1\times H_2$, where H_1 or H_2 is a subgroup of C_1 or C_{m-1} respectively of maximal rank. It is easy to see that H must be $T\times C_{m-1}$. From the Gysin sequence of the fibration $T\longrightarrow Sp(m)/Sp(m-1)\longrightarrow Sp(m)/H$, it follows that $Sp(m)/H$ is a homologically kählerian manifold. It is clear that $Sp(m)/H$ satisfies (2. 1).

Subcase 4. $G=D_m$.

Since $U=D_i\times D_{m-i}$ $(i=2, 3,\dots, m-1)$, no semi-simple maximal subgroup U doen not satisfy (2. 1).

Subcase 5. $G=$*exceptional.*

From dimensional considerations, it follows immeadiately that there exists no subgroup H which satifies (2. 1).

Thus we have completed the proof of Proposition 1 in section 1.

3. Large subgroups of simple Lie groups

In this section, we shall find subgroup H of simple Lie group G which satifies the following conditions

(3. 1) $\quad$ dim $G/H\leqq 2n-1$

(3. 2) $\dim H \geqq ((2n-9)/(2n-1)) \dim G (n \geqq 13)$

and

(3. 3) $\dim G + \dim N(H, G)/H > (1/8)(2n+7) \dim G/H.$

We consider the following two cases.

Case 1. *G is exceptional.*

(i) $G=G_2$. From (3. 2), it follows that $\dim H>9$. There exists no subgroup H with $\dim H \geqq 10$.

(ii) $G=F_4$. Only possibility for H which satifies (3. 2) is *Spin* (9). Since *Spin* (9) is maximal, $\dim N(H, G)/H=0$ and hence (3. 3) implies that $52=\dim G 1/8(2n+7) \dim G/H \geqq 66$, which is impossible.

(iii) $G=E_6, E_7, E_8$. In this case, it is not difficult to see that there is no subgroup satisfying above conditions.

Case 2. *G is classical.*

Let $G=CL(m)$, where CL denotes SU, SO, Sp. From a result in [8] and the fact $\dim G/H < 1/3 \dim G$, it follows that there exists a normal subgroup H_1 of H which is conjugate to a standardly embedded $CL(k)$ with $k > m/2$. Moreover in cases of $G=SO(m)$ and $Sp(m)$, H is conjugate to $CL(k) \times K \subseteqq CL(k) \times CL(m-k)$, where $K \subseteqq CL(m-k)$. One needs restrictions on m: $m \geqq 9$, $m \geqq 11$ and $m \geqq 8$ accordingly to $CL=SU$, SO and S_p respectively.

LEMMA 3. 4. *$\dim N(H, G)/H \leqq \dim CL(m-k)+d$, where $d=0, 1, 3$ according to $CL=SO$, SU, Sp respectively.*

PROOF. We shall prove only the case of $G=SU(m)$. We may assume that $SU(k) \subseteqq H$. Since the identity component $N_0(SU(k), SU(m))$ of the normalizer of $SU(k)$ in $SU(m)$ is $S(U(k) \times U(m-k))$, $SU(k) \subseteqq H \subseteqq S(U(k) \times U(m-k))$. It follows that $N(H, SU(m)) \subseteqq S(U(k) \times U(m-k))$. Hence we have $\dim N(H, SU(m)) \leqq \dim SU(k) + \dim SU(m-k)+1$. Thus we have $\dim N(H, SU(m))/H \leqq \dim SU(m-k)+1$. Q.E.D.

We consider the case in which $G=SU(m)$ and $H \subsetneqq S(U(k) \times U(m-k))$, $G=SO(m)$ or $G=Sp(m)$. In this case, since $\dim G/H \geqq \dim CL(m) - \dim CL(k) - \dim CL(m-k)$, we have

(3. 5) $$\dim CL(m) + \dim CL(m-k) + d > 1/8(2n+7)(\dim CL(m) - \dim CL(k) - \dim CL(m-k)).$$

It follows from (3. 1) that

(3. 6) $$2n \geqq \dim CL(m) - \dim CL(k) - \dim CL(m-k) + 1.$$

Thus we have

(3. 7) $$\dim CL(m) + \dim CL(m-k) + d > 1/8(\dim CL(m) - \dim CL(k) -$$

$\dim CL(m-k)+8)(\dim CL(m)-\dim CL(k)-\dim CL(m-k))$.

We shall show that $k=m-1$. Put $A=$the left hand side of (3. 7) and $B=8$ (the right hand side of (3. 7)) and $F(k)=B-8A$. It is to show that $F(k)<0$ holds only when $k=m-1$. Since the computations for three cases of $G=SU$, SO and Sp are parallel, we consider only the case of $G=SO(m)(m\geqq 11)$. In this case we have $F(k)=k^4-2mk^3+k^2(m^2-12)+k(16m-2)-8m^2+8m$. It is not difficult to see that $F(k)<0$ holds only when $k=m-1$ (Note $k<m$). It is also easy to see that the same result holds when $G=SU(m)$ and $H=S(U(k)\times U(m-k))$. By dimensional considerations, we can show that when m is smaller than 11, the same result holds.

From (3. 1) and (3. 3), it follows immeadiately that the inequalities between m and n must hold. Thus we have completed the proof of Proposition 1 in section 2.

4. Proof of Proposition 4

Let M be a closed differentiable manifold of dimension $2n$ which is homotopy equivalent to CP_n. Assume that $SU(m)$ acts on M in the following way: the identity component of any principal isotropy subgroup H is conjugate to $SU(m-1)$, there exists at least one fixed point and an orbit of type CP_{m-1}.

Put $F=F(SU(m), M)$ and $N=N(SU(m-1), SU(m))$. Let T be a maximal torus of $SU(m)$ such that $T\subset N$. From the fact that there is a fixed point, it follows that any pricipal isotropy subgroup is conjugate to $SU(m-1)$

LEMMA 4. 1. $F(T, M)\cap M_{(N)}=(N(T, SU(m))/N(T, SU(m))\cap N)\times F(N, M_{(N)})$.

PROOF. It is well known that $M_{(N)}=SU(m)/N\times F(N, M_{(N)})$. It is clear that

$$F(T, M)\cap M_{(N)}=\{gy\in M_{(N)};\ g\in SU(m),\ y\in F(N, M_{(N)}),\ g^{-1}Tg\subset N\}$$
$$=\{gy\in M_{(N)};\ y\in F(N, M_{(N)}),\ n^{-1}g\in N \text{ for some } n\in N(T, SU(m))\}.$$

Hence we have

$F(T, M)\cap M_{(N)}=\{nN\in SU(m)/N;\ n\in N(T, SU(m))\}\times F(N, M_{(N)})$. This proves the lemma.

Remark. $N(T, SU(m))/N(T, SU(m))\cap N$ consists exactly m elements.

LEMMA 4. 2. *There exists only one orbit of type CP_{m-1} and the fixed point set F is connected.*

PROOF. Since there exists an equivariant closed neighborhood of U of F such that int $U\cap M_{(N)}=\phi$, any component of F is a component of $F(T, M)$. Thus we have $F(T, M)=F\cap(F(T, M)\cap M_{(N)})$. From lemma (4. 1), it follows that the euler characteristic of $F(T, M)\cap M_{(N)})$ is $me(F(N, M_{(N)}))$. If $\dim F(N, M_{(N)})>0$ or $F(N, M_{(N)})$ is disconnected, then $e(F(N, M_{(N)}))\geqq 2$. Hence we have $n+1=e(M)=e(F(T, M))=e(F)+me(F(N, M_{(N)}))>2m>n+2$, which is a contradiction.

Thus dim $F(N, M_{(N)})=0$ and $F(N, M_{(N)})$ is connected; in other words $F(N, M_{(N)})$ is point. This means that there exists only one orbit of type CP_{m-1}. Considering the local representation at any fixed point, it can be shown that any component of F has dimension $2n-2m$. From results in [6] (chap. VII), it follows that any component of F has integral cohomology ring of CP_{n-m}, and hence $e(F)\geqq n-m+1$. Then F must be connected. Q. E. D.

Remark. The inclusion $i: F\longrightarrow M$ induces isomorphism $i^*: H^k(M; Z)\longrightarrow H^k(F; Z)$ for $k\leqq 2n-2m$.

Let U be a closed equivariant tubular neighborhood of F in M and $P=F(SU(m-1), M$ int $U)$. From the similar arguments in [9] (see [9], section 3), it follows the following

LEMMA 4. 3. *$M=S^{2m-1}\underset{S^1}{\times} P\cup D^{2m}\underset{S^1}{\times} bP$, identified along $S^{2m-1}\underset{S^1}{\times} bP$ and the orbit space can be given by $M/SU(m)=P/S^1\cup[0, 1]\times bP/S^1$ with $\{1\}\times bP/S^1\subset[0, 1]\times bP/S^1$ attached to $bP/S^1\subset P/S^1$.*

Remark. The orbit CP_{m-1} corresponds to a point in $F(S^1, P)$ and hence $F(S^1, P)$ consists of a single point.

LEMMA 4. 4. *The inclusion $j: CP_{m-1}\longrightarrow M-int\, U$ is a homotopy eqivalence.*

PROOF. Since both of CP_{m-1} and $M-$int U are simply connected, it is sufficient to show that j induces isomorphisms of cohomology groups. By the arguments as in subcase 2 of case 6 in section 1, there exists a map $f: M-$int $U\longrightarrow CP_{m-1}$ such that $f\circ j=id$. Hence $j^*: H^k(M-\text{int } U; Z)\longrightarrow H^k(CP_{m-1}; Z)$ is surjective for all k and in particular j^* is isomorphism for $k\leqq 2m-2$. We shall show that $H^k(M-\text{int } U; Z)=0$ for $k\geqq 2m-1$. Consider the cohomology exact sequence of the pair (M, U);

$$\begin{array}{ccccccc} \longrightarrow H^k(M, U; Z) & \longrightarrow & H^k(M; Z) & \longrightarrow & H^k(M, U; Z) & \longrightarrow \\ & i^* \swarrow & & \searrow \approx & & \\ & & H^k(F; Z). & & & \end{array}$$

Since i^* is isomorphism for $k\leqq 2n-2m$ and $H^{odd}(M; Z)=0$, $H^k(M, U; Z)=0$ for $k\leqq 2n-2m+1$ and hence $H_k(M; U; Z)=0$ for $k\leqq 2n-2m+1$. From the isomorphsm $H^k(M-\text{int } U; Z)=H^k(M-U; Z)=H_{2n-k}(M, U; Z)$, it follows that $H^k(M-\text{int } U; Z)=0$ for $k\geqq 2m-1$. Q. E. D.

LEMMA 4. 5. *P is acyclic over integers.*

PROOF. Consider the action of $SU(m-1)$ on $M-$int U. It is not difficult to see that principal isotropy subgroup of the action is conjugate to $SU(m-2)$. Put $N'=N(SU(m-2), SU(m-1))$. We show that $(M-\text{int } U)_{(N')}\neq\phi$. Assume the contrary. Then all orbits K of the $SU(m-1)$-action have cohomology groups $H^k(K; Q)=0$ for $0<k<2m-3$. and hence the Vietoris-Begle theorem implies that the homomorphism $p^*: H^k(M-\text{int } U/SU(m-1); Q)\longrightarrow H^k(M-\text{int } U; Q)$ is isomorphism for $k<2m-3$, where $P: M-\text{int } U\longrightarrow M-\text{int } U/SU(m-1)$ is the orbit map. Therefore a generator a of $H^2(M-int\, U; Q)$

is in the image of p^*, i.e. $a=p^*b$, $b\in H^2(M-\text{int }U/SU(m-1);Q)$. Since dim $(M-\text{int }U/SU(m-1))=2n-2m+3<2m-2$, $a^{m-1}=0$, which is a contradiction. Since $M-\text{int }U$ has the cohomology ring of CP_{m-1}, the same arguments as in lemma 4. 2 show that P has the cohomology ring of a point. This proves the lemma.

Since $F(S^1, P)$ consists of a single point, the results in [6] (7. 2 and 7. 3 in chap. IV) imply that S^1 acts semi-freely on P.

LEMMA 4. 6. *P is contractible.*

PROOF. It is sufficient to show that P ia simply connected. Let W be a disk nbhd. of $x_0\in F(S^1, P)$ *in* P. Clearly P/S^1 is homotopy equivalent to $M/SU(m)$ and hence simply connected. From van Kampen theorem, it follows that $P/S^1-\text{int }W/S^1$ is simply connected. Note that W/S^1 is simply connected (see [6], chap. II (6. 2)). Consider the homotopy exact sequence of the fibration $S^1\longrightarrow P-\text{int }W\longrightarrow(P-\text{int }W)/S^1$;

$$\longrightarrow\pi_2(P-\text{int }W)\longrightarrow\pi_2(P-\text{int }W)/S^1)\longrightarrow\pi_1(S^1)\longrightarrow\rightarrow\pi_1(P-\text{int }W)\longrightarrow 0.$$

From this, it follows that $\pi_1(P\text{-int }W)$ is an abelian group. Since $H_1(P\text{-int }W; Z)=0$, we have $\pi_1(P\text{-int }W)=0$. Q. E. D.

Since $M=b(D^{2m}\times P)/S^1$, proposition 4 follows from the following

PROPOSITION 5. *Let (S^1, X) be a differentiable circle action on a contractible manifold X of dimension $(2n+2)$ with $bX\neq\phi$. Assume (S^1, bX) is free. Then bX/S^1 is a manifold having the same integral cohomology as CP_n. Moreover bX is simply connected and $n\geqq 3$, then bX/S^1 is diffeomorphic to CP_n.*

We omit the proof since it is completely analoguous to the proof of the result (2. 4) in [9].

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